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COEFFICIENTS BOUNDS FOR CERTAIN CLASSES OF ANALYTIC FUNCTIONS OF COMPLEX ORDER

Abstract. In this paper, we determine coefficients bounds for functions in certain subclasses of analytic functions of complex order, which are introduced here by means of the nonhomogeneous Cauchy-Euler differential equation of order m. Our main result contain some corollaries as special cases.

1. Introduction and definitions

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic and univalent in the open disk $\mathcal{U} = \{z : |z| < 1\}$. A function $f(z) \in \mathcal{A}$ is said to be starlike of complex order $\gamma(\gamma \in \mathbb{C}^* := \mathbb{C}\diagdown\{0\})$ and type $\beta(0 \leq \beta < 1)$, that is $f(z) \in \mathcal{S}^*_\gamma(\beta)$, if and only if

$$\mathrm{Re}\left\{1 + \frac{1}{\gamma}\left(\frac{zf'(z)}{f(z)} - 1\right)\right\} > \beta \qquad (z \in \mathcal{U}; \gamma \in \mathbb{C}^*), \tag{1.2}$$

and is said to be convex of complex order $\gamma(\gamma \in \mathbb{C}^*)$ and type $\beta(0 \leq \beta < 1)$, denoted by $\mathcal{C}_\gamma(\beta)$ if and only if

$$\mathrm{Re}\left\{1 + \frac{1}{\gamma}\frac{zf''(z)}{f'(z)}\right\} > \beta \qquad (z \in \mathcal{U}; \gamma \in \mathbb{C}^*). \tag{1.3}$$

The classes $\mathcal{S}^*_\gamma(\beta)$ and $\mathcal{C}_\gamma(\beta)$ were introduced by the first author in [3]. Note that $\mathcal{S}^*_\gamma(0) = \mathcal{S}^*_\gamma$ and $\mathcal{C}_\gamma(0) = \mathcal{C}_\gamma$ the classes considered earlier by Nasr and Aouf [4] and Wiatrowski [5]. Also, $\mathcal{S}^*_1(\beta) = \mathcal{S}^*(\beta)$ and $\mathcal{C}_1(\beta) = \mathcal{C}(\beta)$ which are, respectively, the familiar classes of starlike functions of order $\beta(0 \leq \beta < 1)$ and convex functions of order $\beta(0 \leq \beta < 1)$.

Let $\mathcal{Q}(\gamma, \lambda, \mu, \beta)$ denote the subclass of $\mathcal{A}$ consisting of functions $f(z)$ which satisfy the following condition

$$\mathrm{Re}\left[1 + \frac{1}{\gamma}\left(\frac{z[\lambda\mu z^2 f''(z) + (\lambda - \mu)zf'(z) + (1 - \lambda + \mu)f(z)]'}{\lambda\mu z^2 f''(z) + (\lambda - \mu)zf'(z) + (1 - \lambda + \mu)f(z)} - 1\right)\right] > \beta \tag{1.4}$$

where $0 \leq \mu \leq \lambda \leq 1; 0 \leq \beta < 1; \gamma \in \mathbb{C}^*$ and $z \in \mathcal{U}$.

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For $\mu = 0$, the class $\mathcal{Q}(\gamma, \lambda, \mu, \beta)$ reduces to the class $\mathcal{SC}(\gamma, \lambda, \beta)$ introduced by Altintaş et al. [1]. Clearly, we have $\mathcal{Q}(\gamma, 0, 0, \beta) = \mathcal{S}^*_\gamma(\beta)$ and $\mathcal{Q}(\gamma, 1, 0, \beta) = \mathcal{C}_\gamma(\beta)$.

In the present paper, we propose to derive some coefficient bounds for the class $\mathcal{Q}(\gamma, \lambda, \mu, \beta)$ and also for functions in the subclass $\mathcal{H}(\gamma, \lambda, \mu, \beta, m; \zeta)$ of $\mathcal{A}$, which consists of functions $f(z) \in \mathcal{A}$ satisfying the following nonhomogeneous Cauchy-Euler differential equation of order m :

$$z^m \frac{d^m w}{dz^m} + \binom{m}{1}(\zeta+m-1)z^{m-1}\frac{d^{m-1}w}{dz^{m-1}} + \cdots + \binom{m}{m} w \prod_{j=0}^{m-1}(\zeta+j) = g(z)\prod_{j=0}^{m-1}(\zeta+j+1) \tag{1.5}$$

$$(w = f(z); g(z) \in \mathcal{Q}(\gamma, \lambda, \mu, \beta),\ \zeta \in \mathbb{R}\backslash(-\infty, -1]; m \in \mathbb{N}^* := \mathbb{N}\backslash\{1\} = \{2, 3, ...\}).$$

2. Coefficient estimates

We begin by obtaining coefficient bounds for functions in the class $\mathcal{Q}(\gamma, \lambda, \mu, \beta)$.

Theorem 2.1. *Let the function $f(z) \in \mathcal{A}$ be given by (1.1). If $f(z) \in \mathcal{Q}(\gamma, \lambda, \mu, \beta)$, then*

$$|a_n| \leq \frac{\prod_{j=0}^{n-2}[j + 2|\gamma|(1-\beta)]}{(n-1)![1 + (\lambda\mu n + \lambda - \mu)(n-1)]} \qquad (n \in \mathbb{N}^*), \tag{2.1}$$

where $0 \leq \mu \leq \lambda \leq 1; 0 \leq \beta < 1$, and $\gamma \in C^*$.

Proof. Let the function $F(z)$ be defined by

$$F(z) = \lambda\mu z^2 f''(z) + (\lambda - \mu)zf'(z) + (1 - \lambda + \mu)f(z) \quad (f \in \mathcal{A}; z \in \mathcal{U}). \tag{2.2}$$

Then $F(z)$ is analytic in $\mathcal{U}$ with $F(0) = F'(0) - 1 = 0$. From (1.1) and (2.2) it is easily seen that

$$F(z) = z + \sum_{k=2}^{\infty} A_k z^k \qquad (z \in \mathcal{U}).$$

where

$$A_k := [1 + (\lambda\mu k + \lambda - \mu)(k-1)]a_k \qquad (k \in \mathbb{N}^*). \tag{2.3}$$

Define the function $p(z)$ by

$$p(z) = \frac{1 + \frac{1}{\gamma}\left(\frac{zF'(z)}{F(z)} - 1\right) - \beta}{1 - \beta}$$

or, equivalently,

$$zF'(z) - F(z) = \gamma(1-\beta)(p(z) - 1)F(z) \tag{2.4}$$

then $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$ is analytic in $\mathcal{U}$ and $\mathrm{Re}\{p(z)\} > 0$. Therefore, we have $|c_n| \leq 2 \quad (n \in \mathbb{N})$. From (2.4), it follows that

$$(n-1)A_n = \gamma(1-\beta)(c_{n-1} + c_{n-2}A_2 + \cdots + c_1 A_{n-1}).$$

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In particular, for $n = 2, 3, 4$, we have

$$\begin{aligned} |A_2| &\leq 2\,|\gamma|\,(1-\beta), \\ |A_3| &\leq \frac{2\,|\gamma|\,(1-\beta)[1+2\,|\gamma|\,(1-\beta)]}{2!}, \end{aligned}$$

and

$$|A_4| \leq \frac{2\,|\gamma|\,(1-\beta)[1+2\,|\gamma|\,(1-\beta)][2+2\,|\gamma|\,(1-\beta)]}{3!},$$

respectively. Thus, by using the principle of mathematical induction, we obtain

$$|A_n| \leq \frac{\prod_{j=0}^{n-2}[j+2\,|\gamma|\,(1-\beta)]}{(n-1)!} \qquad (n \in \mathbb{N}^*). \tag{2.5}$$

From (2.3) it is clear that

$$A_n = [1+(\lambda\mu n+\lambda-\mu)(n-1)]a_n \qquad (n \in \mathbb{N}^*). \tag{2.6}$$

Now the inequality (2.1) follows immediately from (2.5) and (2.6). This completes the proof of Theorem 2.1. □

Putting $\mu = \lambda = 1$ in Theorem 2.1, we get the following corollary.

Corollary 2.2. *Let the function $f(z) \in \mathcal{A}$ be given by (1.1)and satisfies the condition*

$$Re\left[1+\frac{1}{\gamma}\left(\frac{z[z^2f''(z)+f(z)]'}{z^2f''(z)+f(z)}-1\right)\right] > \beta \tag{2.7}$$

then

$$|a_n| \leq \frac{\prod_{j=0}^{n-2}[j+2\,|\gamma|\,(1-\beta)]}{(n^2-n+1)(n-1)!} \qquad (n \in \mathbb{N}^*), \tag{2.8}$$

where $0 \leq \beta < 1$, and $\gamma \in C^*$.

Putting $\mu = 0$ in Theorem 2.1, we get the following result obtained by Altintaş et al. [2].

Corollary 2.3. *Let the function $f(z) \in \mathcal{A}$ be given by (1.1). If $f(z) \in \mathcal{SC}(\gamma, \lambda, \beta)$, then*

$$|a_n| \leq \frac{\prod_{j=0}^{n-2}[j+2\,|\gamma|\,(1-\beta)]}{(n-1)![1+\lambda(n-1)]} \qquad (n \in \mathbb{N}^*), \tag{2.9}$$

where $0 \leq \lambda \leq 1; 0 \leq \beta < 1$, and $\gamma \in C^*$.

Finally, we prove the following theorem.

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Theorem 2.4. *Let the function $f(z) \in \mathcal{A}$ be given by (1.1). If $f(z) \in \mathcal{H}(\gamma, \lambda, \mu, \beta, m; \zeta)$, then*

$$|a_n| \leq \frac{\prod_{j=0}^{n-2}[j + 2|\gamma|(1-\beta)] \prod_{j=0}^{m-1}(\zeta + j + 1)}{(n-1)![1 + (\lambda\mu n + \lambda - \mu)(n-1)] \prod_{j=0}^{m-1}(\zeta + j + n)} \qquad (m, n \in \mathbb{N}^*), \tag{2.10}$$

where $0 \leq \mu \leq \lambda \leq 1; 0 \leq \beta < 1; \gamma \in C^*$ and $\zeta \in \mathbb{R}\backslash(-\infty, -1]$.

Proof. Let the function $f(z) \in \mathcal{A}$ be given by (1.1). Also let

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k \in \mathcal{Q}(\gamma, \lambda, \mu, \beta).$$

Then from (1.5), we get

$$a_n = \left(\frac{\prod_{j=0}^{m-1}(\zeta + j + 1)}{\prod_{j=0}^{m-1}(\zeta + j + n)} \right) b_n \qquad (n \in \mathbb{N}^*; \zeta \in \ R\backslash(-\infty, -1]).$$

Thus, by using Theorem 2.4 , we readily obtain the inequality (2.10) . □

Putting $\mu = \lambda = 1$ in Theorem 2.4, we get the following corollary.

Corollary 2.5. *Let the function $f(z) \in \mathcal{A}$ be given by (1.1). If $f(z)$ satisfies the equation (1.5) and $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$ satisfies the condition (2.7), then*

$$|a_n| \leq \frac{\prod_{j=0}^{n-2}[j + 2|\gamma|(1-\beta)] \prod_{j=0}^{m-1}(\zeta + j + 1)}{(n^2 - n + 1)(n-1)! \prod_{j=0}^{m-1}(\zeta + j + n)} \qquad (m, n \in \mathbb{N}^*), \tag{2.11}$$

where $0 \leq \beta < 1; \gamma \in C^*$ and $\zeta \in \mathbb{R}\backslash(-\infty, -1]$.

Putting $\mu = 0$ and $m = 2$ in Theorem 2.4, we get the following result obtained by Altintaş et al. [2].

Corollary 2.6. *Let the function $f(z) \in \mathcal{A}$ be given by (1.1). If $f(z)$ satisfies the nonhomogeneous Cauchy-Euler differential equation of order 2, given by (1.5) and $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$ satisfies the condition (2.7), then*

$$|a_n| \leq \frac{(\zeta + 1)(\zeta + 2) \prod_{j=0}^{n-2}[j + 2|\gamma|(1-\beta)]}{(n-1)![1 + (\lambda\mu n + \lambda - \mu)(n-1)](\zeta + n)(\zeta + n + 1)} \qquad (n \in \mathbb{N}^*), \tag{2.12}$$

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where $0 \leq \lambda \leq 1; 0 \leq \beta < 1; \gamma \in C^*$ *and* $\zeta \in \mathbb{R}\backslash(-\infty, -1]$.

A similar work can also be referred to Eker et al.[6]. In this article they studied the Dziok-Srivastava operator.

Remark: Is it possible to solve problems related to the Fekete-Szegö theorem as given in [7]? It is yet to be solved.

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