

Throughout this article, $\mathbb{R}$ denoted the set of real numbers; $I = [0, \infty)$, $\mathbb{R}_+^* = (0, \infty)$, $\mathbb{R}_1 = [1, \infty)$ and $'$ denotes the derivative. $\mathcal{C}(I, I)$ denotes the set of all nonnegative real-valued continuous functions from I into I and $\mathcal{C}^1(I, I)$ denotes the set of all nonnegative real-valued continuously differentiable functions from I into I .

Theorem 2.1. Let $u(t)$, $g(t)$, $f(t) \in \mathcal{C}(I, \mathbb{R}_+)$, $\alpha \in \mathcal{C}^1(I, I)$ be nondecreasing with $\alpha(t) \leq t$ on I . If the inequality

$$u^p(t) \leq u_0 + \int_0^{\alpha(t)} f(s) \left[u^p(s) + \int_0^s g(\lambda) u(\lambda) d\lambda \right] ds, \quad \forall t \in I, \quad (2.1)$$

holds for all $t \in I$, where u_0 and p are positive constants. Then

$$u(t) \leq u_0^{\frac{1}{p}} \left[1 + \int_0^{\alpha(t)} f(s) \exp \left(\int_0^s [f(\lambda) + g(\lambda)] d\lambda \right) ds \right]^{\frac{1}{p}}, \quad \forall t \in I. \quad (2.2)$$

Proof. Let $J^p(t)$ equal the right hand side in (2.1), we have $J(0) = u_0^{\frac{1}{p}}$ and

$$u(t) \leq J(t), u(\alpha(t)) \leq J(\alpha(t)) \leq J(t), \quad \forall t \in I. \quad (2.3)$$

Differentiating $J^p(t)$ with respect to t , and using (2.3) leads to

$$pJ^{p-1}(t) \frac{dJ(t)}{dt} \leq \alpha'(t) f(\alpha(t)) Y(t), \quad \forall t \in I, \quad (2.4)$$

where $Y(t) = J^p(t) + \int_0^{\alpha(t)} g(s) J(s) ds$, thus we have $Y(0) = J^p(0) = u_0$ and

$$J(t) \leq Y(t), \quad \forall t \in I. \quad (2.5)$$

Differentiating $Y(t)$ and using (2.4) and (2.5), produces

$$\frac{dY(t)}{dt} \leq \alpha'(t) [f(t) + g(t)] Y(t), \quad \forall t \in I,$$

By taking $t = s$ in the above inequality and Integrating from 0 to t , we get

$$Y(t) \leq u_0 \exp \left(\int_0^{\alpha(t)} [f(s) + g(s)] ds \right), \quad \forall t \in I. \quad (2.6)$$

Substituting from (2.6) in (2.4) we have

$$J^{p-1}(t) \frac{dJ(t)}{dt} \leq \frac{1}{p} u_0 \alpha'(t) f(\alpha(t)) \exp \left(\int_0^{\alpha(t)} [f(s) + g(s)] ds \right), \quad \forall t \in I.$$

The above inequality implies an estimation for $J(t)$ as follows

$$J(t) \leq u_0^{\frac{1}{p}} \left[1 + \int_0^{\alpha(t)} f(s) \exp \left(\int_0^s [f(\lambda) + g(\lambda)] d\lambda \right) ds \right]^{\frac{1}{p}}, \quad \forall t \in I. \quad (2.7)$$

Using (2.7) in (2.3), we get the required inequality in (2.2). The proof is complete. $\square$

Remark 2.1. If we put $\alpha(t) = t$, then Theorem 2.1 reduces Theorem 2.4 in [1].

Theorem 2.2. Let $u(t)$, $g(t)$, $h(t) \in \mathcal{C}(I, I)$, $f \in \mathcal{C}(I, \mathbb{R}_+^*)$, $\alpha \in \mathcal{C}^1(I, I)$ be nondecreasing with $\alpha(t) \leq t$ on I . If the inequality

$$u^p(t) \leq u_0 + \int_0^{\alpha(t)} f(s) \left[u^q(s) + \int_0^s g(\lambda) u(\lambda) d\lambda \right] ds, \quad \forall t \in I, \quad (2.8)$$

where $u_0 > 0$, $p > q \geq 1$ are be constants. Then

$$u(t) \leq \left[u_0 + \int_0^{\alpha(t)} f(s) k(\alpha^{-1}(s)) ds \right]^{\frac{1}{p}}, \quad \forall t \in I, \quad (2.9)$$

where

$$k(t) = \exp \left(\int_0^{\alpha(t)} g(\lambda) d\lambda \right) \left[u_0^{\frac{q(p-q)}{p}} + \left[\frac{q[p-q]}{p} \right] \int_0^{\alpha(t)} f(s) \exp \left(-[p-q] \int_0^s g(\lambda) d\lambda \right) ds \right]^{\frac{1}{p-q}}, \quad (2.10)$$

for all $t \in I$.

Proof. Let $J_1^p(t)$ equal the right hand side in (2.8), we have $J_1(0) = u_0^{\frac{1}{p}}$ and

$$u(t) \leq J_1(t), u(\alpha(t)) \leq J_1(\alpha(t)) \leq J_1(t), \quad \forall t \in I. \quad (2.11)$$

Differentiating $J_1^p(t)$, with respect to t , using (2.11) we get

$$p J_1^{p-1}(t) \frac{dJ_1(t)}{dt} \leq \alpha'(t) f(\alpha(t)) Y_1(t), \quad \forall t \in I, \quad (2.12)$$

where $Y_1(t) = J_1^q(t) + \int_0^{\alpha(t)} g(s) J_1(s) ds$, thus we have $Y_1(0) = J_1^q(0) = u_0^{\frac{q}{p}}$, and $J_1^q \leq Y_1(t)$, but $q \geq 1$, thus

$$J_1(t) \leq Y_1(t), \quad \forall t \in I. \quad (2.13)$$

Differentiating $Y_1(t)$ then using (2.12) and (2.13), produces

$$\frac{dY_1(t)}{dt} \leq \frac{q}{p} \alpha'(t) f(\alpha(t)) Y_1^{[q-p+1]}(t) + \alpha'(t) g(\alpha(t)) Y_1(t), \quad \forall t \in I,$$

but $Y_1(t) > 0$, thus we have

$$Y_1^{[p-q-1]}(t) \frac{dY_1(t)}{dt} - \alpha'(t) g(\alpha(t)) Y_1^{[p-q]}(t) \leq \frac{q}{p} \alpha'(t) f(\alpha(t)), \quad \forall t \in I. \quad (2.14)$$

Now if we put

$$Z(t) = Y_1^{[p-q]}(t), \quad \forall t \in I, \quad (2.15)$$

then we have $Z(0) = Y_1^{[p-q]}(0) = u_0^{\frac{q(p-q)}{p}}$ and $Y_1^{[p-q-1]}(t) \frac{dY_1(t)}{dt} = \left[\frac{1}{p-q} \right] \frac{dZ(t)}{dt}$, thus from (2.14) we obtain

$$\frac{dZ(t)}{dt} - [p-q] \alpha'(t) g(\alpha(t)) Z(t) \leq \left[\frac{q[p-q]}{p} \right] \alpha'(t) f(\alpha(t)), \quad \forall t \in I. \quad (2.16)$$

The above inequality implies the following estimations for $Z(t)$, such that

$$Z(t) \leq k^{[p-q]}(t), \quad \forall t \in I,$$

where $k(t)$ is as defined in (2.10). From (2.8) and the above inequality we have

$$Y_1(t) \leq k(t), \quad \forall t \in I,$$

thus from (2.12) and the above inequality we have

$$pJ_1^{[p-1]}(t) \frac{dJ_1(t)}{dt} \leq \alpha'(t)f(\alpha(t))k(t), \quad \forall t \in I.$$

By taking $t = s$ in the above inequality and integrating from 0 to t , gives

$$J_1(t) \leq \left[u_0 + \int_0^{\alpha(t)} f(s)k(\alpha^{-1}(s)) \right]^{\frac{1}{p}}, \quad \forall t \in I, \quad (2.17)$$

Using (2.17) in (2.11) leads to the required inequality in (2.9). The proof is complete. $\square$

Remark 2.2. If we put $\alpha(t) = t$, then Theorem 2.2 reduces Theorem 2.3 in [1].

Theorem 2.3. Let $u(t)$, $g(t)$, $h(t) \in \mathcal{C}(I, I)$, $f \in \mathcal{C}(I, \mathbb{R}_+^*)$, $\alpha \in \mathcal{C}^1(I, I)$ be nondecreasing with $\alpha(t) \leq t$ on I . If the inequality

$$u(t) \leq u_0 + \int_0^{\alpha(t)} f(s)u(s) \left[u^q(s) + \int_0^s g(\lambda)u(\lambda)d\lambda \right]^p ds, \quad \forall t \in I \quad (2.18)$$

for all $t \in I$, where $u_0 > 0$, $p > 0$ and $q > 0$ are be constants such that $p+q > 1$. Then

$$u(t) \leq u_0 \exp \left(\int_0^{\alpha(t)} f(s)k_1(\alpha^{-1}(s))ds \right), \quad \forall t \in I, \quad (2.19)$$

where

$$k_1(t) = \frac{u_0^{pq} \exp \left(p \int_0^{\alpha(t)} g(s)ds \right)}{\left[1 - q[p+q-1]u_0^{q[p+q-1]} \int_0^{\alpha(t)} f(s) \exp([p+q-1] \int_0^s g(\lambda)d\lambda)ds \right]^{\frac{1}{p+q-1}}}, \quad (2.20)$$

for all $t \in I$, such that

$$q[p+q-1]u_0^{q[p+q-1]} \int_0^{\alpha(t)} f(s) \exp([p+q-1] \int_0^s g(\lambda)d\lambda)ds < 1, \quad \text{for all } t \in I.$$

Proof. Let $J_2(t)$ equal the right hand side of (2.18), which is a positive and nondecreasing function on I with $J_2(0) = u_0$. Then (2.18) reduces to

$$u(t) \leq J_2(t), u(\alpha(t)) \leq J_2(\alpha(t)) \leq J_2(t), \quad \forall t \in I. \quad (2.21)$$

Remark 2.4. If we put $\alpha(t) = t$, then Theorem 2.3 reduces Theorem 2.7 in [1].

Remark 2.5. If we put $q = 1$ and $\alpha(t) = t$, then Theorem 2.3 reduces Theorem 3.2 in [6].

Theorem 2.4. Let $u(t)$, $g(t)$, $h(t) \in C(I, I)$, $f \in C(I, \mathbb{R}_+^*)$, $\alpha \in C^1(I, I)$ be nondecreasing with $\alpha(t) \leq t$ on I , and let $n(t) \in C(I, I)$ be nondecreasing. If the inequality

$$u(t) \leq n(t) + \int_0^{\alpha(t)} f(s) \left[u(s) + \int_0^s g(\lambda) u(\lambda) d\lambda \right]^p ds, \quad (2.28)$$

for all $t \in I$, where $p \in (0, 1)$. Then

$$u(t) \leq n(t) \left[1 + \int_0^{\alpha(t)} f(s) n^{[p-1]}(s) k_2(\alpha^{-1}(s)) ds \right], \quad \forall t \in I, \quad (2.29)$$

where

$$\begin{aligned} k_2(t) &\leq \exp \left(p(1-p) \int_0^{\alpha(t)} g(s) ds \right) \left[1 + (1-p) \int_0^{\alpha(t)} f(s) n^{[p-1]}(s) \right. \\ &\quad \times \exp \left(-(1-p) \int_0^s g(\lambda) d\lambda \right) ds \left. \right]^{\frac{1}{1-p}}, \quad \forall t \in I. \end{aligned} \quad (2.30)$$

Proof. Since $n(t)$ is a positive monotonic nondecreasing function, we observe from the inequality (2.28)

$$\left[\frac{u(t)}{n(t)} \right] \leq 1 + \int_0^{\alpha(t)} f(s) n^{[p-1]}(s) \left[\left(\frac{u(s)}{n(s)} \right) + \int_0^{\alpha(s)} g(\lambda) \left(\frac{u(\lambda)}{n(\lambda)} \right) d\lambda \right]^p ds, \quad \forall t \in I.$$

Let

$$m(t) = \frac{u(t)}{n(t)}, \quad m(0) \leq 1, \quad \forall t \in I. \quad (2.31)$$

Hence

$$m(t) \leq 1 + \int_0^{\alpha(t)} f(s) n^{[p-1]}(s) \left[m(s) + \int_0^s g(\lambda) m(\lambda) d\lambda \right]^p, \quad \forall t \in I. \quad (2.32)$$

Let $J_3(t)$ equal the right hand side of (2.32), which is a positive and nondecreasing function on I with $J_3(0) = 1$. Then (2.32) reduces to

$$m(t) \leq J_3(t), \quad m(\alpha(t)) \leq J_3(\alpha(t)) \leq J_3(t) \quad \forall t \in I. \quad (2.33)$$

Differentiating $J_3(t)$ with respect to t , using (2.33), we get

$$\frac{dJ_3(t)}{dt} \leq \alpha'(t) f(\alpha(t)) n^{[p-1]}(\alpha(t)) Y_3^p(t), \quad \forall t \in I, \quad (2.34)$$

where $Y_3(t) = J_3(t) + \int_0^{\alpha(t)} g(s)J_3(s)ds$, $Y_3(0) = J_3(0) = 1$, and

$$J_3(t) \leq Y_3(t), \quad \forall t \in I. \quad (2.35)$$

Differentiating $Y_3(t)$ with respect to t and using (2.34) and (2.35) we get

$$\begin{aligned} \frac{dY_3(t)}{dt} &= \frac{dJ_3(t)}{dt} + \alpha'(t)g(\alpha(t))J_3(t) \\ &\leq \alpha'(t)f(\alpha(t))n^{[p-1]}(\alpha(t))Y_3^p(t) + \alpha'(t)g(\alpha(t))Y_3(t), \forall t \in I. \end{aligned}$$

The above inequality implies the following estimation for $Y_3(t)$, such that

$$Y_3^p(t) \leq k_2(t), \quad \forall t \in I,$$

where $k_2(t)$ is as given in (2.30). From the above inequality in (2.34), we have

$$\frac{dJ_3(t)}{dt} \leq \alpha'(t)f(\alpha(t))n^{[p-1]}(t)k_2(t), \quad \forall t \in I,$$

the above inequality implies the following estimation for $J_3(t)$ such that

$$J_3(t) \leq 1 + \int_0^{\alpha(t)} f(s)k_2(\alpha^{-1}(s))n^{[p-1]}ds, \quad \forall t \in I.$$

Using the above inequality in (2.33), we get

$$m(t) \leq 1 + \int_0^{\alpha(t)} f(s)k_2(\alpha^{-1}(t))n^{[p-1]}ds, \quad \forall t \in I. \quad (2.36)$$

We get the desired bound in (2.29) from (2.31) and (2.36), the proof is complete. $\square$

Remark 2.6. If we put $\alpha(t) = t$, then Theorem 2.4 reduces Theorem 2.9 in [1].

Theorem 2.5. Let $u(t)$, $g(t)$, $f(t) \in C(I, I)$, be nondecreasing and we suppose that $\psi_1(t)$, $\psi_2(t)$ and $\alpha \in C^1(I, I)$ are increasing functions with $\alpha(t) \leq t$, $\alpha(0) = 0$, $\psi_i(t) > 0$, $\forall t > 0$, $i = 1, 2$. If the inequality

$$u(t) \leq u_0 + \int_0^{\alpha(t)} f(s)\psi_1(u(s)) \left[u^q(s) + \int_0^s g(\lambda)\psi_2(u(\lambda))d\lambda \right]^p ds, \forall t \in I, \quad (2.37)$$

where $u_0 > 0$, $p > 0$ and $q \geq 1$, are constants such that $p + q > 1$. Then

$$u(t) \leq \Psi_1^{-1} \left[\Psi_2^{-1} \left(\Psi_2 \left(\Psi_1(u_0^q) + \int_0^{\alpha(t)} g(s)ds \right) + \int_0^{\alpha(t)} f(s)ds \right) \right], \quad (2.38)$$

for all $t \in [0, T_1]$, where

$$\Psi_1(r) = \int_1^r \frac{dt}{\psi_2(t)}, r > 0, \quad (2.39)$$

$$\Psi_2(r) = \int_1^r \frac{\Psi_1(\Psi_1^{-1}(s))ds}{\Psi_1(\Psi_1^{-1}(s))(\Psi_1^{-1}(s))^{p+q-1}}, r > 0, \quad (3.12)$$

and T_2 is the largest number such that

$$\Psi_2 \left(\Psi_1(|u_0|^q) + \int_0^{\alpha(t)} \frac{g(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))} ds \right) + \int_0^{\alpha(t)} \frac{f(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))} ds \leq \frac{\Psi_1(\Psi_1^{-1}(s))ds}{\Psi_1(\Psi_1^{-1}(s))(\Psi_1^{-1}(s))^{p+q-1}},$$

$$\Psi_2^{-1} \left(\Psi_2 \left(\Psi_1(|u_0|^q) + \int_0^{\alpha(t)} \frac{g(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))} ds \right) + \int_0^{\alpha(t)} \frac{f(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))} ds \right) \leq \frac{dt}{\Psi_2(t)}.$$

Proof. Integrating both side of the Eq. (3.7) from 0 to t , we have

$$u(t) = u_0 + \int_0^t H(s, u(\alpha(s)), \int_0^s K(\lambda, u(\alpha(\lambda)))d\lambda)ds, \forall t \in I, \quad (3.13)$$

using the conditions (3.8) and (3.9), from (3.13) we get

$$|u(t)| \leq |u_0| + \int_0^t f(s)\varphi(|u(\alpha(s))|) \left[|\alpha'(s)| + \int_0^s g(\lambda)\varphi(|u(\lambda)|)d\lambda \right]^p ds, \quad (3.14)$$

the inequality (3.14) can be written in the form

$$|u(t)| \leq |u_0| + \int_0^{\alpha(t)} \frac{f(\alpha^{-1}(s))}{\alpha'(\alpha^{-1}(s))} \varphi(|u(\alpha(s))|) \left[|u^q(s)| + \int_0^s \frac{g(\alpha^{-1}(\lambda))}{\alpha'(\alpha^{-1}(\lambda))} \varphi(|u(\lambda)|)d\lambda \right]^p ds, \quad (3.15)$$

for all $t \in I$. Applying Theorem 2.5 to (3.15), we get the required inequality in (3.10). The proof is complete. $\square$

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