

Intuitionistic trapezoidal fuzzy prioritized operators and their application to multiple attribute group decision making

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ABSTRACT

Aims: The aim of this paper is to propose some intuitionistic trapezoidal fuzzy prioritized operators and interval-valued intuitionistic trapezoidal fuzzy prioritized operators for handling multiple attribute group decision making problems in which the attributes and decision makers are in different priority level.

Study design: First, we present two intuitionistic trapezoidal fuzzy prioritized aggregation operators called the intuitionistic trapezoidal fuzzy prioritized weighted average (ITFPWA) operator and the intuitionistic trapezoidal fuzzy prioritized weighted geometric (ITFPWG) operator.

Place and Duration of Study: These two operators can capture the prioritization phenomenon among the aggregated arguments.

Methodology: Then, some desirable properties of two new operators are examined in detail. Next, we utilize the proposed operators to develop an approach to multiple attribute group decision making under intuitionistic trapezoidal fuzzy environment in which the attributes and decision makers are in different priority level, and a practical example is provided to illustrate the practicality and effectiveness of the developed approach.

Results: Furthermore, the intuitionistic trapezoidal fuzzy prioritized aggregation operators are extended to interval-valued intuitionistic trapezoidal fuzzy environment, and two interval-valued intuitionistic trapezoidal fuzzy prioritized aggregation operators are introduced. Subsequently, an approach to multiple attribute group decision making with interval-valued intuitionistic trapezoidal fuzzy information is developed.

Conclusion: Finally, an application of the proposed approach to group decision making problem with interval-valued intuitionistic trapezoidal fuzzy information is shown with the demonstration of a numerical example.

Keywords: Multiple attribute group decision making; Intuitionistic fuzzy sets; Intuitionistic trapezoidal fuzzy numbers; Intuitionistic trapezoidal fuzzy prioritized operators; Interval-valued intuitionistic trapezoidal fuzzy prioritized operators.

1. INTRODUCTION

The concept of an intuitionistic fuzzy set, originally proposed by Atanassov [1,2], is an important tool for dealing with imperfect and imprecise information. Compared with Zadeh's fuzzy sets [22], an intuitionistic fuzzy set gives the membership and non-membership degree to which an element belongs to a set. Hence, coping with imperfect and imprecise information is more flexible and effective for intuitionistic fuzzy sets. However, it should be noted that the domains of the intuitionistic fuzzy sets are discrete sets, which are also the same as fuzzy sets, and intuitionistic fuzzy sets are used to indicate the extent to which the criterion does or does not belong to some fuzzy concepts. Shu, Cheng, and Chang introduced [4] the intuitionistic triangular fuzzy numbers and their operations. Intuitionistic

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trapezoidal fuzzy numbers are introduced in [9], which are the extending of intuitionistic triangular fuzzy numbers. Intuitionistic triangular fuzzy numbers and intuitionistic trapezoidal fuzzy numbers are the extending of intuitionistic fuzzy sets in another way, which extends discrete set to continuous set, and they are the extending of fuzzy numbers. Compared to intuitionistic fuzzy sets, intuitionistic trapezoidal fuzzy numbers are defined by using trapezoidal fuzzy numbers to express their membership and non-membership functions, which makes the membership degrees and the non-membership degrees no longer relative to a fuzzy concept “Excellent” or “Good”, but relative to the trapezoidal fuzzy numbers. Therefore, intuitionistic trapezoidal fuzzy numbers may better reflect the information of decision problems and express the information in different dimensions than intuitionistic fuzzy sets [12]. Considering that there exist many multiple attribute group decision making (MAGDM) problems in which the attribute values are given in the form of intuitionistic trapezoidal fuzzy numbers, it is necessary and interesting to study how to aggregate intuitionistic trapezoidal fuzzy numbers. In order to do this, a variety of intuitionistic trapezoidal fuzzy operators have been developed recently. For example, Wang and Zhang [11] developed the Hamming distance of intuitionistic trapezoidal fuzzy numbers and the intuitionistic trapezoidal fuzzy weighted arithmetic averaging (ITFWAA) operator. Wang and Zhang [10] gave the definition of expected values of intuitionistic trapezoidal fuzzy number and proposed the programming method of multi-criteria decision-making based on intuitionistic trapezoidal fuzzy number with incomplete certain information. Wang and Zhang [12] proposed some aggregation operators, including intuitionistic trapezoidal fuzzy weighted arithmetic averaging operator and weighted geometric averaging operator, and defined expected values of intuitionistic trapezoidal fuzzy number. Wei [13] proposed some aggregation operators, including intuitionistic trapezoidal fuzzy ordered weighted averaging (ITFOWA) operator and intuitionistic trapezoidal fuzzy hybrid aggregation (ITFHA) operator, and developed an approach to multiple attribute group decision making (MAGDM) with intuitionistic trapezoidal fuzzy information. Wu and Cao [15] presented the intuitionistic trapezoidal fuzzy weighted geometric (ITFWG) operator, the intuitionistic trapezoidal fuzzy ordered weighted geometric (ITFOWG) operator, the induced ordered weighted geometric (I-ITFOWG) operator, and the hybrid geometric (ITFHG) operator. Wan and Dong [8] defined the expectation and expectant score of intuitionistic trapezoidal fuzzy numbers from the geometric angle and also proposed trapezoidal intuitionistic fuzzy ordered weighted averaging operator and trapezoidal intuitionistic fuzzy hybrid aggregation operator. Du and Liu [3] proposed an extended fuzzy VIKOR method with respect to multiple attribute decision making problems in which the attribute value takes the form of intuitionistic trapezoidal fuzzy number, and the attribute weight is unknown. Zhang, Jin, and Liu [23] proposed a grey relational projection method for multi-attribute decision making based on intuitionistic trapezoidal fuzzy number. Wan [7] developed power average operators of trapezoidal intuitionistic fuzzy numbers and examined their application to multi-attribute group decision making.

The aforementioned linguistic aggregation operators are usually used to deal with MAGDMs where the attribute and the decision makers are at the same priority level respectively. In these MAGDMs, we have the ability to trade off between attributes. For instance, if C_i and C_j are two attributes with the weights w_i and w_j respectively, then we

can compensate for a decrease of θ in satisfaction to attribute C_i by gain $\frac{w_j}{w_i}\theta$ in

satisfaction to attribute C_j . However, in many MAGDM problems, this kind of compensation between attributes is infeasible. For example, consider the situation where we are buying a car based on the safety and cost of cars. We should not allow a benefit with respect to cost to compensate for a loss in safety. In this situation we have a typical kind of prioritization of the attributes, i.e., safety has a higher priority than cost. In addition, decision making in a university, president have a higher priority than vice president. Yager [17] first investigated

80 this kind of problem by introducing the prioritized “and” and “or” operators. Then, Yager
 81 [18,19] gave deeper insights into on this issue. Motivated by the ideas of Yager [17,18,19],
 82 Wei [14] generalized prioritized aggregation operators to hesitant fuzzy environment and
 83 developed some hesitant fuzzy prioritized aggregation operators. Yu and Xu [21]
 84 investigated the prioritization relationship of attributes in multi-attribute decision making with
 85 intuitionistic fuzzy information and developed some prioritized intuitionistic fuzzy aggregation
 86 operators by extending the prioritized aggregation operators. Yu et al. [20] proposed some
 87 interval-valued intuitionistic fuzzy prioritized aggregation operators and investigated the
 88 application of these operators in the group decision making under interval-valued
 89 intuitionistic fuzzy environment in which the attributes and experts are in different priority
 90 level. Zhao, Lin, and Wei [24] developed some prioritized aggregation operators for
 91 aggregating triangular fuzzy information and then applied them to develop some models for
 92 triangular fuzzy multiple attribute group decision making (MAGDM) problems in which the
 93 attributes and experts are in different priority level.

94 However, it is worthwhile to mention that the existing prioritized aggregation operators can
 95 not be used to accommodate the situations in which the input arguments are in the form of
 96 intuitionistic trapezoidal fuzzy numbers. Though the existing intuitionistic trapezoidal fuzzy
 97 operators can accommodate intuitionistic trapezoidal fuzzy information, they have difficulty in
 98 dealing with the MAGDM in which the attributes and decision makers are in different priority
 99 level. To overcome this drawback, in this paper, we extend the prioritized aggregation
 100 operators to intuitionistic trapezoidal fuzzy environments, i.e., propose some intuitionistic
 101 trapezoidal fuzzy prioritized operators. The prominent characteristic of these operators is
 102 that they not only accommodate the situations in which the input arguments are intuitionistic
 103 trapezoidal fuzzy numbers but also take into account the prioritization phenomenon among
 104 the input arguments. Then, based on these operators, we develop an approach to multiple
 105 attribute group decision making with intuitionistic trapezoidal fuzzy information. The
 106 advantages of the developed approach are mainly twofold. First, they can accommodate
 107 intuitionistic trapezoidal fuzzy environments. Second, they can consider the prioritization
 108 phenomenon among the attributes and the decision makers. Finally, we apply the developed
 109 approach to a practical problem and show the practicality and effectiveness of the proposed
 110 approach. Furthermore, we extend the intuitionistic trapezoidal fuzzy prioritized aggregation
 111 operators to interval-valued intuitionistic trapezoidal fuzzy environment [25] and introduce
 112 two interval-valued intuitionistic trapezoidal fuzzy prioritized aggregation operators. An
 113 approach to multiple attribute group decision making with interval-valued intuitionistic
 114 trapezoidal fuzzy information is developed based on these operators. Finally, an application
 115 of the proposed approach to group decision making problem with interval-valued intuitionistic
 116 trapezoidal fuzzy information is shown with the demonstration of a numerical example.

117 This paper is organized as follows. In Section 2, we briefly review the intuitionistic
 118 trapezoidal fuzzy number and the prioritized average operator. Section 3 develops the
 119 intuitionistic trapezoidal fuzzy prioritized weighted average (ITFPWA) operator and the
 120 intuitionistic trapezoidal fuzzy prioritized weighted geometric (ITFPWG) operator for
 121 aggregating intuitionistic trapezoidal fuzzy numbers. Furthermore, we develop a method for
 122 MAGDM with intuitionistic trapezoidal fuzzy information. Meanwhile, a practical example
 123 involving talent introduction is given to demonstrate the practicality and effectiveness of the
 124 developed approach. In Section 4, we extend the discussions given in Section 3 to the
 125 interval-valued intuitionistic trapezoidal fuzzy environment. Concretely, we develop the
 126 interval-valued intuitionistic trapezoidal fuzzy prioritized weighted average (IVITFPWA)
 127 operator and the interval-valued intuitionistic trapezoidal fuzzy prioritized weighted geometric
 128 (IVITFPWG) operator. Then, based on the developed operators, we propose an approach to
 129 MAGDM with interval-valued intuitionistic trapezoidal fuzzy information. Finally, a numerical
 130 example is provided to illustrate the interval-valued intuitionistic trapezoidal fuzzy MAGDM
 131 process. Section 5 gives some concluding remarks.

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2. PRELIMINARIES

In this section, we shall introduce some basic concepts related to intuitionistic fuzzy sets, intuitionistic trapezoidal fuzzy numbers, and the prioritized average (PA) operator. Let R be a set of all real numbers.

2.1. Intuitionistic fuzzy sets and intuitionistic trapezoidal fuzzy numbers

Definition 2.1 [1,2]. Let X be an ordinary nonempty set. An intuitionistic fuzzy set A in X is an object that has the form

$$A = \{ \langle x, \mu_A(x), \nu_A(x) \rangle \mid x \in X \}, \quad (1)$$

where $\mu_A : X \rightarrow [0,1]$ and $\nu_A : X \rightarrow [0,1]$ satisfy $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for all $x \in X$.

$\mu_A(x)$ is called the degree of membership of element $x \in X$ to A , $\nu_A(x)$ is called the degree of nonmembership of $x \in X$ to A , and $1 - \mu_A(x) - \nu_A(x)$ is called the hesitancy degree or intuitionistic index of $x \in X$ to A .

If $\mu_A(x) + \nu_A(x) = 1$, then the intuitionistic fuzzy set A reduces to a fuzzy set [1,2].

Definition 2.2 [9]. Let $\tilde{\alpha}$ be an intuitionistic fuzzy set in R , whose membership function and non-membership function are defined as follows:

$$\mu_{\tilde{\alpha}}(x) = \begin{cases} \frac{x-a}{b-a} \mu_{\tilde{\alpha}}, & a \leq x < b, \\ \mu_{\tilde{\alpha}}, & b \leq x \leq c, \\ \frac{d-x}{d-c} \mu_{\tilde{\alpha}}, & c < x \leq d, \\ 0, & \text{otherwise,} \end{cases} \quad (2)$$

and

$$\nu_{\tilde{\alpha}}(x) = \begin{cases} \frac{(b-x) + \nu_{\tilde{\alpha}}(x-a_1)}{b-a_1}, & a_1 \leq x < b, \\ \nu_{\tilde{\alpha}}, & b \leq x \leq c, \\ \frac{(x-c) + \nu_{\tilde{\alpha}}(d_1-x)}{d_1-c}, & c < x \leq d_1, \\ 0, & \text{otherwise,} \end{cases} \quad (3)$$

respectively, where $0 \leq \mu_{\tilde{\alpha}} \leq 1$, $0 \leq \nu_{\tilde{\alpha}} \leq 1$, $0 \leq \mu_{\tilde{\alpha}} + \nu_{\tilde{\alpha}} \leq 1$, and $a, a_1, b, c, d, d_1 \in R$. Then, $\tilde{\alpha}$ is called an intuitionistic trapezoidal fuzzy number. For convenience, $\tilde{\alpha}$ is denoted by $\tilde{\alpha} = \langle ([a, b, c, d]; \mu_{\tilde{\alpha}}), ([a_1, b, c, d_1]; \nu_{\tilde{\alpha}}) \rangle$.

The values $\mu_{\tilde{\alpha}}$ and $\nu_{\tilde{\alpha}}$ represent the maximum degree of membership and the minimum degree of non-membership, respectively. The function $\pi_{\tilde{\alpha}}(x) = 1 - \mu_{\tilde{\alpha}}(x) - \nu_{\tilde{\alpha}}(x)$ denotes the hesitation of $\tilde{\alpha}$. The smaller $\pi_{\tilde{\alpha}}(x)$, the more certain $\tilde{\alpha}$.

Let Ω be the set of all intuitionistic trapezoidal fuzzy numbers.

If $[a, b, c, d] = [a_1, b, c, d_1]$, then the intuitionistic trapezoidal fuzzy number $\tilde{\alpha}$ can be simply denoted by $\tilde{\alpha} = ([a, b, c, d]; \mu_{\tilde{\alpha}}, \nu_{\tilde{\alpha}})$. If $b = c$, then an intuitionistic trapezoidal fuzzy number reduces to an intuitionistic triangle fuzzy number. If $\mu_{\tilde{\alpha}} = 1$ and $\nu_{\tilde{\alpha}} = 0$, then an intuitionistic trapezoidal fuzzy number reduces to a trapezoidal fuzzy number.

Let $\tilde{\alpha} = ([a, b, c, d]; \mu_{\tilde{\alpha}}, \nu_{\tilde{\alpha}})$ be an intuitionistic trapezoidal fuzzy number. If $a, b, c, d \in [0, 1]$, then $\tilde{\alpha}$ is called a standardized intuitionistic trapezoidal fuzzy number. Fig. 1 shows an intuitionistic trapezoidal fuzzy number $\tilde{\alpha} = ([a, b, c, d]; \mu_{\tilde{\alpha}}, \nu_{\tilde{\alpha}})$.

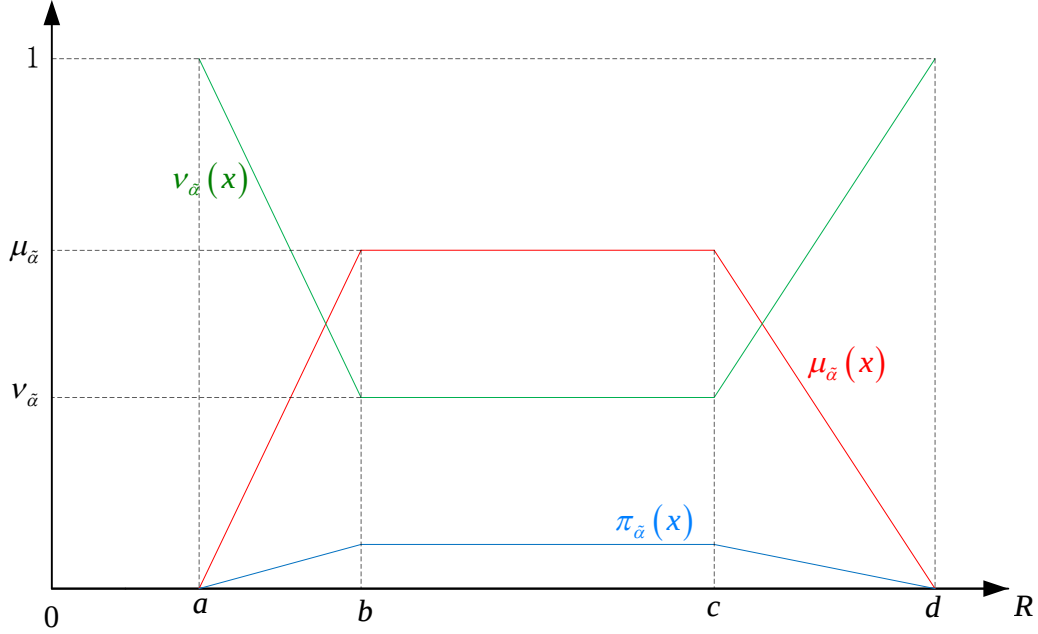


Fig. 1. An intuitionistic trapezoidal fuzzy number $\tilde{\alpha} = ([a, b, c, d]; \mu_{\tilde{\alpha}}, \nu_{\tilde{\alpha}})$.

Definition 2.3 [11]. Let $\tilde{\alpha}_1 = ([a_1, b_1, c_1, d_1]; \mu_{\tilde{\alpha}_1}, \nu_{\tilde{\alpha}_1})$, $\tilde{\alpha}_2 = ([a_2, b_2, c_2, d_2]; \mu_{\tilde{\alpha}_2}, \nu_{\tilde{\alpha}_2})$ be two intuitionistic trapezoidal fuzzy numbers, and $\lambda \geq 0$. Then, the operation laws for intuitionistic trapezoidal fuzzy numbers (ITFNs) are defined as follows:

(1) $\tilde{\alpha}_1 \oplus \tilde{\alpha}_2 = ([a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2]; \mu_{\tilde{\alpha}_1} + \mu_{\tilde{\alpha}_2} - \mu_{\tilde{\alpha}_1} \mu_{\tilde{\alpha}_2}, \nu_{\tilde{\alpha}_1} \nu_{\tilde{\alpha}_2})$;

(2) $\tilde{\alpha}_1 \otimes \tilde{\alpha}_2 = ([a_1 a_2, b_1 b_2, c_1 c_2, d_1 d_2]; \mu_{\tilde{\alpha}_1} \mu_{\tilde{\alpha}_2}, \nu_{\tilde{\alpha}_1} + \nu_{\tilde{\alpha}_2} - \nu_{\tilde{\alpha}_1} \nu_{\tilde{\alpha}_2})$;

(3) $\lambda \tilde{\alpha}_1 = ([\lambda a_1, \lambda b_1, \lambda c_1, \lambda d_1]; 1 - (1 - \mu_{\tilde{\alpha}_1})^\lambda, \nu_{\tilde{\alpha}_1}^\lambda)$;

(4) $\tilde{\alpha}_1^\lambda = ([a_1^\lambda, b_1^\lambda, c_1^\lambda, d_1^\lambda]; \mu_{\tilde{\alpha}_1}^\lambda, 1 - (1 - \nu_{\tilde{\alpha}_1})^\lambda)$.

Theorem 2.1. Let $\tilde{\alpha}_1 = ([a_1, b_1, c_1, d_1]; \mu_{\tilde{\alpha}_1}, \nu_{\tilde{\alpha}_1})$, $\tilde{\alpha}_2 = ([a_2, b_2, c_2, d_2]; \mu_{\tilde{\alpha}_2}, \nu_{\tilde{\alpha}_2})$ be two intuitionistic trapezoidal fuzzy numbers, and $\lambda \geq 0$. Then, $\tilde{\alpha}_1 \oplus \tilde{\alpha}_2$, $\tilde{\alpha}_1 \otimes \tilde{\alpha}_2$, $\lambda \tilde{\alpha}_1$, and $\tilde{\alpha}_1^\lambda$ are also intuitionistic trapezoidal fuzzy numbers.

Proof. According to Theorem 1 given in [15], $\tilde{\alpha}_1 \otimes \tilde{\alpha}_2$ and $\tilde{\alpha}_1^\lambda$ are intuitionistic trapezoidal fuzzy numbers. In the following, we prove that $\tilde{\alpha}_1 \oplus \tilde{\alpha}_2$ and $\lambda \tilde{\alpha}_1$ are intuitionistic trapezoidal fuzzy numbers.

Since $\tilde{\alpha}_1 = ([a_1, b_1, c_1, d_1]; \mu_{\tilde{\alpha}_1}, \nu_{\tilde{\alpha}_1})$ and $\tilde{\alpha}_2 = ([a_2, b_2, c_2, d_2]; \mu_{\tilde{\alpha}_2}, \nu_{\tilde{\alpha}_2})$ are two intuitionistic trapezoidal fuzzy numbers, we have $0 \leq \mu_{\tilde{\alpha}_1}, \nu_{\tilde{\alpha}_1}, \mu_{\tilde{\alpha}_2}, \nu_{\tilde{\alpha}_2} \leq 1$, $0 \leq \mu_{\tilde{\alpha}_1} + \nu_{\tilde{\alpha}_1} \leq 1$, and $0 \leq \mu_{\tilde{\alpha}_2} + \nu_{\tilde{\alpha}_2} \leq 1$. Then, we can obtain

$$0 \leq \mu_{\tilde{\alpha}_1} + \mu_{\tilde{\alpha}_2} - \mu_{\tilde{\alpha}_1} \mu_{\tilde{\alpha}_2} = 1 - (1 - \mu_{\tilde{\alpha}_1})(1 - \mu_{\tilde{\alpha}_2}) \leq 1, \quad 0 \leq \nu_{\tilde{\alpha}_1} \nu_{\tilde{\alpha}_2} \leq 1, \text{ and}$$

$$0 \leq \mu_{\tilde{\alpha}_1} + \mu_{\tilde{\alpha}_2} - \mu_{\tilde{\alpha}_1} \mu_{\tilde{\alpha}_2} + \nu_{\tilde{\alpha}_1} \nu_{\tilde{\alpha}_2} \leq \mu_{\tilde{\alpha}_1} + \mu_{\tilde{\alpha}_2} - \mu_{\tilde{\alpha}_1} \mu_{\tilde{\alpha}_2} + (1 - \mu_{\tilde{\alpha}_1})(1 - \mu_{\tilde{\alpha}_2}) = 1.$$

Thus, $\tilde{\alpha}_1 \oplus \tilde{\alpha}_2$ is an intuitionistic trapezoidal fuzzy number.

Since $\tilde{\alpha}_1 = ([a_1, b_1, c_1, d_1]; \mu_{\tilde{\alpha}_1}, \nu_{\tilde{\alpha}_1})$ is an intuitionistic trapezoidal fuzzy number, we have

$$0 \leq 1 - (1 - \mu_{\tilde{\alpha}_1})^\lambda \leq 1, \quad 0 \leq \nu_{\tilde{\alpha}_1}^\lambda \leq 1, \text{ and } 0 \leq 1 - (1 - \mu_{\tilde{\alpha}_1})^\lambda + \nu_{\tilde{\alpha}_1}^\lambda \leq 1 - (1 - \mu_{\tilde{\alpha}_1})^\lambda + (1 - \mu_{\tilde{\alpha}_1})^\lambda = 1.$$

Thus, $\lambda \tilde{\alpha}_1$ is also an intuitionistic trapezoidal fuzzy number, which completes the proof for Theorem 2.1.

Theorem 2.2 [15,23]. Let $\tilde{\alpha}_1 = ([a_1, b_1, c_1, d_1]; \mu_{\tilde{\alpha}_1}, \nu_{\tilde{\alpha}_1})$, $\tilde{\alpha}_2 = ([a_2, b_2, c_2, d_2]; \mu_{\tilde{\alpha}_2}, \nu_{\tilde{\alpha}_2})$ be two intuitionistic trapezoidal fuzzy numbers, and $\lambda, \lambda_1, \lambda_2 > 0$. Then, we have:

$$(1) \tilde{\alpha}_1 \oplus \tilde{\alpha}_2 = \tilde{\alpha}_2 \oplus \tilde{\alpha}_1;$$

$$(2) \tilde{\alpha}_1 \otimes \tilde{\alpha}_2 = \tilde{\alpha}_2 \otimes \tilde{\alpha}_1;$$

$$(3) \lambda(\tilde{\alpha}_1 \oplus \tilde{\alpha}_2) = \lambda \tilde{\alpha}_1 \oplus \lambda \tilde{\alpha}_2;$$

$$(4) \lambda_1 \tilde{\alpha}_1 \oplus \lambda_2 \tilde{\alpha}_1 = (\lambda_1 + \lambda_2) \tilde{\alpha}_1;$$

$$(5) \tilde{\alpha}_1^{\lambda_1} \otimes \tilde{\alpha}_1^{\lambda_2} = \tilde{\alpha}_1^{\lambda_1 + \lambda_2};$$

$$(6) \tilde{\alpha}_1^{\lambda_1} \otimes \tilde{\alpha}_2^{\lambda_1} = (\tilde{\alpha}_1 \otimes \tilde{\alpha}_2)^{\lambda_1}.$$

Definition 2.4 [12]. For the intuitionistic trapezoidal fuzzy number $\tilde{\alpha} = ([a, b, c, d]; \mu_{\tilde{\alpha}}, \nu_{\tilde{\alpha}})$, its

$$\text{expected value is } I(\tilde{\alpha}) = \frac{1}{8} \times [(a + b + c + d) \times (1 + \mu_{\tilde{\alpha}} - \nu_{\tilde{\alpha}})]. \quad (4)$$

The score function and accuracy function of intuitionistic trapezoidal fuzzy numbers are introduced below.

Definition 2.5 [12]. Let $\tilde{\alpha} = ([a, b, c, d]; \mu_{\tilde{\alpha}}, \nu_{\tilde{\alpha}})$ be an intuitionistic trapezoidal fuzzy number; then, $S(\tilde{\alpha}) = I(\tilde{\alpha}) \times (\mu_{\tilde{\alpha}} - \nu_{\tilde{\alpha}})$ is called the score function of $\tilde{\alpha}$, where $I(\tilde{\alpha})$ is the expected value of intuitionistic trapezoidal fuzzy number $\tilde{\alpha}$.

Definition 2.6 [12]. Let $\tilde{\alpha} = ([a, b, c, d]; \mu_{\tilde{\alpha}}, \nu_{\tilde{\alpha}})$ be an intuitionistic trapezoidal fuzzy number; then, $H(\tilde{\alpha}) = I(\tilde{\alpha}) \times (\mu_{\tilde{\alpha}} + \nu_{\tilde{\alpha}})$ is called the accuracy function of $\tilde{\alpha}$, where $I(\tilde{\alpha})$ is the expected value of intuitionistic trapezoidal fuzzy number $\tilde{\alpha}$.

Definition 2.7 [12]. If $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$ are two random intuitionistic trapezoidal fuzzy numbers, then,

$$(1) \text{ If } S(\tilde{\alpha}_1) > S(\tilde{\alpha}_2), \text{ then, } \tilde{\alpha}_1 > \tilde{\alpha}_2;$$

$$(2) \text{ If } S(\tilde{\alpha}_1) = S(\tilde{\alpha}_2), \text{ and}$$

$$\text{if } H(\tilde{\alpha}_1) = H(\tilde{\alpha}_2), \text{ then, } \tilde{\alpha}_1 = \tilde{\alpha}_2;$$

$$\text{if } H(\tilde{\alpha}_1) > H(\tilde{\alpha}_2), \text{ then, } \tilde{\alpha}_1 > \tilde{\alpha}_2.$$

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2.2. Prioritized average (PA) operators

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The prioritized average (PA) operator was originally introduced by Yager [17], which was defined as follows:

Definition 2.8 [17]. Let $C = \{C_1, C_2, \dots, C_n\}$ be a collection of criteria and that there is a prioritization between the criteria expressed by the linear ordering $C_1 \succ C_2 \succ C_3 \dots \succ C_n$, indicate criteria C_j has a higher priority than C_k if $j < k$. The value $C_j(x)$ is the performance of any alternative x under criteria C_j , and satisfies $C_j(x) \in [0, 1]$. If

$$PA(C_i(x)) = \sum_{j=1}^n w_j C_j(x), \quad (5)$$

where $w_j = \frac{T_j}{\sum_{j=1}^n T_j}$, $T_j = \prod_{k=1}^{j-1} C_k(x)$ ($j = 2, \dots, n$), $T_1 = 1$. Then PA is called the prioritized average (PA) operator.

3. INTUITIONISTIC TRAPEZOIDAL FUZZY PRIORITIZED OPERATORS

The prioritized average operator [17] has only been used in situations in which the input arguments are the exact values. In this section, we shall investigate the PA operator under intuitionistic trapezoidal fuzzy environments. We shall propose two intuitionistic trapezoidal fuzzy prioritized operators, including the ITFPWA and ITFPWA operators, which can accommodate the situations where the input arguments are intuitionistic trapezoidal fuzzy information.

3.1. Intuitionistic trapezoidal fuzzy prioritized weighted average (ITFPWA) operators

Definition 3.1. Let $\tilde{\alpha}_i = ([a_i, b_i, c_i, d_i]; \mu_{\tilde{\alpha}_i}, \nu_{\tilde{\alpha}_i})$ ($i = 1, 2, \dots, n$) be a collection of intuitionistic trapezoidal fuzzy numbers, if

$$ITFPWA(\tilde{\alpha}_1, \tilde{\alpha}_2, \dots, \tilde{\alpha}_n) = \frac{T_1}{\sum_{j=1}^n T_j} \tilde{\alpha}_1 \oplus \frac{T_2}{\sum_{j=1}^n T_j} \tilde{\alpha}_2 \oplus \dots \oplus \frac{T_n}{\sum_{j=1}^n T_j} \tilde{\alpha}_n, \quad (6)$$

where $T_1 = 1$ and $T_j = \prod_{k=1}^{j-1} S(\tilde{\alpha}_k)$ ($j = 2, \dots, n$), then ITFPWA is called an intuitionistic trapezoidal fuzzy prioritized weighted average (ITFPWA) operator.

Based on operations of intuitionistic trapezoidal fuzzy numbers described in Definition 2.3, we can drive the Theorem 3.1.

Theorem 3.1. Let $\tilde{\alpha}_i = ([a_i, b_i, c_i, d_i]; \mu_{\tilde{\alpha}_i}, \nu_{\tilde{\alpha}_i})$ ($i = 1, 2, \dots, n$) be a collection of intuitionistic trapezoidal fuzzy numbers, then their aggregated value by using the ITFPWA operator is also an intuitionistic trapezoidal fuzzy number, and

$$ITFPWA(\tilde{\alpha}_1, \tilde{\alpha}_2, \dots, \tilde{\alpha}_n) = \left(\left[\sum_{j=1}^n \left(\frac{T_j a_j}{\sum_{j=1}^n T_j} \right), \sum_{j=1}^n \left(\frac{T_j b_j}{\sum_{j=1}^n T_j} \right), \sum_{j=1}^n \left(\frac{T_j c_j}{\sum_{j=1}^n T_j} \right), \sum_{j=1}^n \left(\frac{T_j d_j}{\sum_{j=1}^n T_j} \right) \right]; 1 - \prod_{j=1}^n (1 - \mu_{\tilde{\alpha}_j})^{\left(\frac{T_j}{\sum_{j=1}^n T_j} \right)}, \prod_{j=1}^n (\nu_{\tilde{\alpha}_j})^{\left(\frac{T_j}{\sum_{j=1}^n T_j} \right)} \right). \quad (7)$$

251 **Proof.** The first result follows quickly from Definition 3.1 and Theorem 2.1. In the following,
 252 we prove the second result by using mathematical induction on n . We first prove that Eq. (7)
 253 holds for $n = 2$. Since

$$254 \quad \frac{T_1}{\sum_{j=1}^n T_j} \tilde{\alpha}_1 = \left(\left[\frac{T_1 a_1}{\sum_{j=1}^n T_j}, \frac{T_1 b_1}{\sum_{j=1}^n T_j}, \frac{T_1 c_1}{\sum_{j=1}^n T_j}, \frac{T_1 d_1}{\sum_{j=1}^n T_j} \right]; 1 - (1 - \mu_{\tilde{\alpha}_1})^{T_1 / \sum_{j=1}^n T_j}, v_{\tilde{\alpha}_1}^{T_1 / \sum_{j=1}^n T_j} \right)$$

$$255 \quad \frac{T_2}{\sum_{j=1}^n T_j} \tilde{\alpha}_2 = \left(\left[\frac{T_2 a_2}{\sum_{j=1}^n T_j}, \frac{T_2 b_2}{\sum_{j=1}^n T_j}, \frac{T_2 c_2}{\sum_{j=1}^n T_j}, \frac{T_2 d_2}{\sum_{j=1}^n T_j} \right]; 1 - (1 - \mu_{\tilde{\alpha}_2})^{T_2 / \sum_{j=1}^n T_j}, v_{\tilde{\alpha}_2}^{T_2 / \sum_{j=1}^n T_j} \right),$$

256 then

$$\begin{aligned} & \frac{T_1}{\sum_{j=1}^n T_j} \tilde{\alpha}_1 \oplus \frac{T_2}{\sum_{j=1}^n T_j} \tilde{\alpha}_2 \\ &= \left(\left[\frac{T_1 a_1}{\sum_{j=1}^n T_j}, \frac{T_1 b_1}{\sum_{j=1}^n T_j}, \frac{T_1 c_1}{\sum_{j=1}^n T_j}, \frac{T_1 d_1}{\sum_{j=1}^n T_j} \right]; 1 - (1 - \mu_{\tilde{\alpha}_1})^{T_1 / \sum_{j=1}^n T_j}, v_{\tilde{\alpha}_1}^{T_1 / \sum_{j=1}^n T_j} \right) \oplus \left(\left[\frac{T_2 a_2}{\sum_{j=1}^n T_j}, \frac{T_2 b_2}{\sum_{j=1}^n T_j}, \frac{T_2 c_2}{\sum_{j=1}^n T_j}, \frac{T_2 d_2}{\sum_{j=1}^n T_j} \right]; 1 - (1 - \mu_{\tilde{\alpha}_2})^{T_2 / \sum_{j=1}^n T_j}, v_{\tilde{\alpha}_2}^{T_2 / \sum_{j=1}^n T_j} \right) \\ 257 \quad &= \left(\left[\frac{T_1 a_1}{\sum_{j=1}^n T_j} + \frac{T_2 a_2}{\sum_{j=1}^n T_j}, \frac{T_1 b_1}{\sum_{j=1}^n T_j} + \frac{T_2 b_2}{\sum_{j=1}^n T_j}, \frac{T_1 c_1}{\sum_{j=1}^n T_j} + \frac{T_2 c_2}{\sum_{j=1}^n T_j}, \frac{T_1 d_1}{\sum_{j=1}^n T_j} + \frac{T_2 d_2}{\sum_{j=1}^n T_j} \right]; \right. \\ & \quad \left. 1 - (1 - \mu_{\tilde{\alpha}_1})^{T_1 / \sum_{j=1}^n T_j} + 1 - (1 - \mu_{\tilde{\alpha}_2})^{T_2 / \sum_{j=1}^n T_j} - \left(1 - (1 - \mu_{\tilde{\alpha}_1})^{T_1 / \sum_{j=1}^n T_j} \right) \cdot \left(1 - (1 - \mu_{\tilde{\alpha}_2})^{T_2 / \sum_{j=1}^n T_j} \right), v_{\tilde{\alpha}_1}^{T_1 / \sum_{j=1}^n T_j} \cdot v_{\tilde{\alpha}_2}^{T_2 / \sum_{j=1}^n T_j} \right) \\ &= \left(\left[\frac{T_1 a_1}{\sum_{j=1}^n T_j} + \frac{T_2 a_2}{\sum_{j=1}^n T_j}, \frac{T_1 b_1}{\sum_{j=1}^n T_j} + \frac{T_2 b_2}{\sum_{j=1}^n T_j}, \frac{T_1 c_1}{\sum_{j=1}^n T_j} + \frac{T_2 c_2}{\sum_{j=1}^n T_j}, \frac{T_1 d_1}{\sum_{j=1}^n T_j} + \frac{T_2 d_2}{\sum_{j=1}^n T_j} \right]; 1 - (1 - \mu_{\tilde{\alpha}_1})^{T_1 / \sum_{j=1}^n T_j} \cdot (1 - \mu_{\tilde{\alpha}_2})^{T_2 / \sum_{j=1}^n T_j}, v_{\tilde{\alpha}_1}^{T_1 / \sum_{j=1}^n T_j} \cdot v_{\tilde{\alpha}_2}^{T_2 / \sum_{j=1}^n T_j} \right). \end{aligned}$$

258

259 If Eq. (7) holds for $n = k$, that is

$$260 \quad \text{ITFPWA}(\tilde{\alpha}_1, \tilde{\alpha}_2, \dots, \tilde{\alpha}_k) = \left(\left[\sum_{j=1}^k \left(\frac{T_j a_j}{\sum_{j=1}^k T_j} \right), \sum_{j=1}^k \left(\frac{T_j b_j}{\sum_{j=1}^k T_j} \right), \sum_{j=1}^k \left(\frac{T_j c_j}{\sum_{j=1}^k T_j} \right), \sum_{j=1}^k \left(\frac{T_j d_j}{\sum_{j=1}^k T_j} \right) \right]; 1 - \prod_{j=1}^k (1 - \mu_{\tilde{\alpha}_j})^{T_j / \sum_{j=1}^k T_j}, \prod_{j=1}^k (v_{\tilde{\alpha}_j})^{T_j / \sum_{j=1}^k T_j} \right)$$

261

262 then, when $n = k + 1$, by the operational laws in Definition 2.3, we have

$$\begin{aligned}
& \text{ITFPWA}(\tilde{\alpha}_1, \tilde{\alpha}_2, \dots, \tilde{\alpha}_{k+1}) \\
&= \frac{T_1}{\sum_{j=1}^n T_j} \tilde{\alpha}_1 \oplus \frac{T_2}{\sum_{j=1}^n T_j} \tilde{\alpha}_2 \oplus \dots \oplus \frac{T_k}{\sum_{j=1}^n T_j} \tilde{\alpha}_k \oplus \frac{T_{k+1}}{\sum_{j=1}^n T_j} \tilde{\alpha}_{k+1} \\
&= \left(\left[\sum_{j=1}^k \left(\frac{T_j a_j}{\sum_{j=1}^n T_j} \right), \sum_{j=1}^k \left(\frac{T_j b_j}{\sum_{j=1}^n T_j} \right), \sum_{j=1}^k \left(\frac{T_j c_j}{\sum_{j=1}^n T_j} \right), \sum_{j=1}^k \left(\frac{T_j d_j}{\sum_{j=1}^n T_j} \right) \right]; 1 - \prod_{j=1}^k (1 - \mu_{\tilde{\alpha}_j})^{\left(\frac{T_j}{\sum_{j=1}^n T_j} \right)}, \prod_{j=1}^k (v_{\tilde{\alpha}_j})^{\left(\frac{T_j}{\sum_{j=1}^n T_j} \right)} \right) \oplus \\
&\quad \left(\left[\frac{T_{k+1} a_{k+1}}{\sum_{j=1}^n T_j}, \frac{T_{k+1} b_{k+1}}{\sum_{j=1}^n T_j}, \frac{T_{k+1} c_{k+1}}{\sum_{j=1}^n T_j}, \frac{T_{k+1} d_{k+1}}{\sum_{j=1}^n T_j} \right]; 1 - (1 - \mu_{\tilde{\alpha}_{k+1}})^{T_{k+1}/\sum_{j=1}^n T_j}, v_{\tilde{\alpha}_{k+1}}^{T_{k+1}/\sum_{j=1}^n T_j} \right) \\
&= \left(\left[\sum_{j=1}^{k+1} \left(\frac{T_j a_j}{\sum_{j=1}^n T_j} \right), \sum_{j=1}^{k+1} \left(\frac{T_j b_j}{\sum_{j=1}^n T_j} \right), \sum_{j=1}^{k+1} \left(\frac{T_j c_j}{\sum_{j=1}^n T_j} \right), \sum_{j=1}^{k+1} \left(\frac{T_j d_j}{\sum_{j=1}^n T_j} \right) \right]; \right. \\
&\quad \left. 1 - \prod_{j=1}^k (1 - \mu_{\tilde{\alpha}_j})^{\left(\frac{T_j}{\sum_{j=1}^n T_j} \right)} + 1 - (1 - \mu_{\tilde{\alpha}_{k+1}})^{T_{k+1}/\sum_{j=1}^n T_j} - \left(1 - \prod_{j=1}^k (1 - \mu_{\tilde{\alpha}_j})^{\left(\frac{T_j}{\sum_{j=1}^n T_j} \right)} \right) \cdot \left(1 - (1 - \mu_{\tilde{\alpha}_{k+1}})^{T_{k+1}/\sum_{j=1}^n T_j} \right), \right. \\
&\quad \left. \left(\prod_{j=1}^k (v_{\tilde{\alpha}_j})^{\left(\frac{T_j}{\sum_{j=1}^n T_j} \right)} \right) \cdot \left(v_{\tilde{\alpha}_{k+1}}^{T_{k+1}/\sum_{j=1}^n T_j} \right) \right) \\
&= \left(\left[\sum_{j=1}^{k+1} \left(\frac{T_j a_j}{\sum_{j=1}^n T_j} \right), \sum_{j=1}^{k+1} \left(\frac{T_j b_j}{\sum_{j=1}^n T_j} \right), \sum_{j=1}^{k+1} \left(\frac{T_j c_j}{\sum_{j=1}^n T_j} \right), \sum_{j=1}^{k+1} \left(\frac{T_j d_j}{\sum_{j=1}^n T_j} \right) \right]; 1 - \prod_{j=1}^{k+1} (1 - \mu_{\tilde{\alpha}_j})^{\left(\frac{T_j}{\sum_{j=1}^n T_j} \right)}, \prod_{j=1}^{k+1} (v_{\tilde{\alpha}_j})^{\left(\frac{T_j}{\sum_{j=1}^n T_j} \right)} \right)
\end{aligned}$$

263

264 i.e., Eq. (7) holds for $n = k + 1$.265 Therefore, Eq. (7) holds for all n , which completes the proof of Theorem 3.1. $\square$

266 **Theorem 3.2.** Let $\tilde{\alpha}_i = ([a_i, b_i, c_i, d_i]; \mu_{\tilde{\alpha}_i}, v_{\tilde{\alpha}_i})$ ($i = 1, 2, \dots, n$) be a collection of intuitionistic
 267 trapezoidal fuzzy numbers. Then, the following properties hold.

268 (1) Idempotency: If $\tilde{\alpha}_i = \tilde{\alpha} = ([a, b, c, d]; \mu_{\tilde{\alpha}}, v_{\tilde{\alpha}})$ for all i , then

$$269 \quad \text{ITFPWA}(\tilde{\alpha}_1, \tilde{\alpha}_2, \dots, \tilde{\alpha}_n) = \tilde{\alpha}. \quad (8)$$

270 (2) Boundedness:

$$271 \quad \tilde{\alpha}_{\min} \leq \text{ITFPWA}(\tilde{\alpha}_1, \tilde{\alpha}_2, \dots, \tilde{\alpha}_n) \leq \tilde{\alpha}_{\max}, \quad (9)$$

272 where $\tilde{\alpha}_{\min} = \left(\left[\min_{1 \leq i \leq n} \{a_i\}, \min_{1 \leq i \leq n} \{b_i\}, \min_{1 \leq i \leq n} \{c_i\}, \min_{1 \leq i \leq n} \{d_i\} \right]; \min_{1 \leq i \leq n} \{\mu_{\tilde{\alpha}_i}\}, \max_{1 \leq i \leq n} \{v_{\tilde{\alpha}_i}\} \right)$ and

$$273 \quad \tilde{\alpha}_{\max} = \left(\left[\max_{1 \leq i \leq n} \{a_i\}, \max_{1 \leq i \leq n} \{b_i\}, \max_{1 \leq i \leq n} \{c_i\}, \max_{1 \leq i \leq n} \{d_i\} \right]; \max_{1 \leq i \leq n} \{\mu_{\tilde{\alpha}_i}\}, \min_{1 \leq i \leq n} \{v_{\tilde{\alpha}_i}\} \right).$$

274 **Proof.** (1) If $\tilde{\alpha}_i = \tilde{\alpha} = ([a, b, c, d]; \mu_{\tilde{\alpha}}, v_{\tilde{\alpha}})$ for all i , then

$$\begin{aligned}
& \text{ITFPWA}(\tilde{\alpha}_1, \tilde{\alpha}_2, \dots, \tilde{\alpha}_n) \\
275 \quad &= \left[\sum_{j=1}^n \left(\frac{T_j a_j}{\sum_{j=1}^n T_j} \right), \sum_{j=1}^n \left(\frac{T_j b_j}{\sum_{j=1}^n T_j} \right), \sum_{j=1}^n \left(\frac{T_j c_j}{\sum_{j=1}^n T_j} \right), \sum_{j=1}^n \left(\frac{T_j d_j}{\sum_{j=1}^n T_j} \right) \right]; 1 - \prod_{j=1}^n (1 - \mu_{\tilde{\alpha}_j})^{\left(\frac{T_j}{\sum_{j=1}^n T_j} \right)}, \prod_{j=1}^n (v_{\tilde{\alpha}_j})^{\left(\frac{T_j}{\sum_{j=1}^n T_j} \right)} \\
&= ([a, b, c, d]; \mu_{\tilde{\alpha}}, v_{\tilde{\alpha}}) \\
&= \tilde{\alpha}.
\end{aligned}$$

276 (2) First, by Definitions 2.5 and 2.6, and Theorem 3.1, we have

$$\begin{aligned}
& \text{ITFPWA}(\tilde{\alpha}_1, \tilde{\alpha}_2, \dots, \tilde{\alpha}_n) \\
277 \quad &= \left[\sum_{j=1}^n \left(\frac{T_j a_j}{\sum_{j=1}^n T_j} \right), \sum_{j=1}^n \left(\frac{T_j b_j}{\sum_{j=1}^n T_j} \right), \sum_{j=1}^n \left(\frac{T_j c_j}{\sum_{j=1}^n T_j} \right), \sum_{j=1}^n \left(\frac{T_j d_j}{\sum_{j=1}^n T_j} \right) \right]; 1 - \prod_{j=1}^n (1 - \mu_{\tilde{\alpha}_j})^{\left(\frac{T_j}{\sum_{j=1}^n T_j} \right)}, \prod_{j=1}^n (v_{\tilde{\alpha}_j})^{\left(\frac{T_j}{\sum_{j=1}^n T_j} \right)} \\
& S(\text{ITFPWA}(\tilde{\alpha}_1, \tilde{\alpha}_2, \dots, \tilde{\alpha}_n)) = \frac{1}{8} \times \left[\sum_{j=1}^n \left(\frac{T_j a_j}{\sum_{j=1}^n T_j} \right) + \sum_{j=1}^n \left(\frac{T_j b_j}{\sum_{j=1}^n T_j} \right) + \sum_{j=1}^n \left(\frac{T_j c_j}{\sum_{j=1}^n T_j} \right) + \sum_{j=1}^n \left(\frac{T_j d_j}{\sum_{j=1}^n T_j} \right) \right] \times \\
278 \quad & \left(1 + 1 - \prod_{j=1}^n (1 - \mu_{\tilde{\alpha}_j})^{\left(\frac{T_j}{\sum_{j=1}^n T_j} \right)} - \prod_{j=1}^n (v_{\tilde{\alpha}_j})^{\left(\frac{T_j}{\sum_{j=1}^n T_j} \right)} \right) \times \\
& \left(1 - \prod_{j=1}^n (1 - \mu_{\tilde{\alpha}_j})^{\left(\frac{T_j}{\sum_{j=1}^n T_j} \right)} - \prod_{j=1}^n (v_{\tilde{\alpha}_j})^{\left(\frac{T_j}{\sum_{j=1}^n T_j} \right)} \right) \\
& H(\text{ITFPWA}(\tilde{\alpha}_1, \tilde{\alpha}_2, \dots, \tilde{\alpha}_n)) = \frac{1}{8} \times \left[\sum_{j=1}^n \left(\frac{T_j a_j}{\sum_{j=1}^n T_j} \right) + \sum_{j=1}^n \left(\frac{T_j b_j}{\sum_{j=1}^n T_j} \right) + \sum_{j=1}^n \left(\frac{T_j c_j}{\sum_{j=1}^n T_j} \right) + \sum_{j=1}^n \left(\frac{T_j d_j}{\sum_{j=1}^n T_j} \right) \right] \times \\
279 \quad & \left(1 + 1 - \prod_{j=1}^n (1 - \mu_{\tilde{\alpha}_j})^{\left(\frac{T_j}{\sum_{j=1}^n T_j} \right)} - \prod_{j=1}^n (v_{\tilde{\alpha}_j})^{\left(\frac{T_j}{\sum_{j=1}^n T_j} \right)} \right) \times \\
& \left(1 - \prod_{j=1}^n (1 - \mu_{\tilde{\alpha}_j})^{\left(\frac{T_j}{\sum_{j=1}^n T_j} \right)} + \prod_{j=1}^n (v_{\tilde{\alpha}_j})^{\left(\frac{T_j}{\sum_{j=1}^n T_j} \right)} \right)
\end{aligned}$$

$$280 \quad S(\tilde{\alpha}_{\min}) = \frac{1}{8} \times \left(\min_{1 \leq i \leq n} \{a_i\} + \min_{1 \leq i \leq n} \{b_i\} + \min_{1 \leq i \leq n} \{c_i\} + \min_{1 \leq i \leq n} \{d_i\} \right) \times \left(1 + \min_{1 \leq i \leq n} \{\mu_{\tilde{\alpha}_i}\} - \max_{1 \leq i \leq n} \{v_{\tilde{\alpha}_i}\} \right) \times \left(\min_{1 \leq i \leq n} \{\mu_{\tilde{\alpha}_i}\} - \max_{1 \leq i \leq n} \{v_{\tilde{\alpha}_i}\} \right)$$

$$281 \quad H(\tilde{\alpha}_{\min}) = \frac{1}{8} \times \left(\min_{1 \leq i \leq n} \{a_i\} + \min_{1 \leq i \leq n} \{b_i\} + \min_{1 \leq i \leq n} \{c_i\} + \min_{1 \leq i \leq n} \{d_i\} \right) \times \left(1 + \min_{1 \leq i \leq n} \{\mu_{\tilde{\alpha}_i}\} - \max_{1 \leq i \leq n} \{v_{\tilde{\alpha}_i}\} \right) \times \left(\min_{1 \leq i \leq n} \{\mu_{\tilde{\alpha}_i}\} + \max_{1 \leq i \leq n} \{v_{\tilde{\alpha}_i}\} \right)$$

$$282 \quad S(\tilde{\alpha}_{\max}) = \frac{1}{8} \times \left(\max_{1 \leq i \leq n} \{a_i\} + \max_{1 \leq i \leq n} \{b_i\} + \max_{1 \leq i \leq n} \{c_i\} + \max_{1 \leq i \leq n} \{d_i\} \right) \times \left(1 + \max_{1 \leq i \leq n} \{\mu_{\tilde{\alpha}_i}\} - \min_{1 \leq i \leq n} \{v_{\tilde{\alpha}_i}\} \right) \times \left(\max_{1 \leq i \leq n} \{\mu_{\tilde{\alpha}_i}\} - \min_{1 \leq i \leq n} \{v_{\tilde{\alpha}_i}\} \right)$$

$$283 \quad H(\tilde{\alpha}_{\max}) = \frac{1}{8} \times \left(\max_{1 \leq i \leq n} \{a_i\} + \max_{1 \leq i \leq n} \{b_i\} + \max_{1 \leq i \leq n} \{c_i\} + \max_{1 \leq i \leq n} \{d_i\} \right) \times \left(1 + \max_{1 \leq i \leq n} \{\mu_{\tilde{\alpha}_i}\} - \min_{1 \leq i \leq n} \{v_{\tilde{\alpha}_i}\} \right) \times \left(\max_{1 \leq i \leq n} \{\mu_{\tilde{\alpha}_i}\} + \min_{1 \leq i \leq n} \{v_{\tilde{\alpha}_i}\} \right)$$

284

Obviously, we have $\min_{1 \leq i \leq n} \{a_i\} \leq \sum_{j=1}^n \left(\frac{T_j a_j}{\sum_{j=1}^n T_j} \right) \leq \max_{1 \leq i \leq n} \{a_i\}$, $\min_{1 \leq i \leq n} \{b_i\} \leq \sum_{j=1}^n \left(\frac{T_j b_j}{\sum_{j=1}^n T_j} \right) \leq \max_{1 \leq i \leq n} \{b_i\}$,

$\min_{1 \leq i \leq n} \{c_i\} \leq \sum_{j=1}^n \left(\frac{T_j c_j}{\sum_{j=1}^n T_j} \right) \leq \max_{1 \leq i \leq n} \{c_i\}$, and $\min_{1 \leq i \leq n} \{d_i\} \leq \sum_{j=1}^n \left(\frac{T_j d_j}{\sum_{j=1}^n T_j} \right) \leq \max_{1 \leq i \leq n} \{d_i\}$.

Because $\min_{1 \leq i \leq n} \{\mu_{\tilde{a}_i}\} \leq \mu_{\tilde{a}_i} \leq \max_{1 \leq i \leq n} \{\mu_{\tilde{a}_i}\}$ and $\min_{1 \leq i \leq n} \{\nu_{\tilde{a}_i}\} \leq \nu_{\tilde{a}_i} \leq \max_{1 \leq i \leq n} \{\nu_{\tilde{a}_i}\}$,

$$1 - \prod_{j=1}^n \left(1 - \mu_{\tilde{a}_j} \right)^{\left(T_j / \sum_{j=1}^n T_j \right)} \geq 1 - \prod_{j=1}^n \left(1 - \min_{1 \leq j \leq n} \{\mu_{\tilde{a}_j}\} \right)^{\left(T_j / \sum_{j=1}^n T_j \right)} = \min_{1 \leq j \leq n} \{\mu_{\tilde{a}_j}\} \quad (10)$$

$$1 - \prod_{j=1}^n \left(1 - \mu_{\tilde{a}_j} \right)^{\left(T_j / \sum_{j=1}^n T_j \right)} \leq 1 - \prod_{j=1}^n \left(1 - \max_{1 \leq j \leq n} \{\mu_{\tilde{a}_j}\} \right)^{\left(T_j / \sum_{j=1}^n T_j \right)} = \max_{1 \leq j \leq n} \{\mu_{\tilde{a}_j}\} \quad (11)$$

$$\prod_{j=1}^n \left(\nu_{\tilde{a}_j} \right)^{\left(T_j / \sum_{j=1}^n T_j \right)} \leq \prod_{j=1}^n \left(\max_{1 \leq j \leq n} \{\nu_{\tilde{a}_j}\} \right)^{\left(T_j / \sum_{j=1}^n T_j \right)} = \max_{1 \leq j \leq n} \{\nu_{\tilde{a}_j}\} \quad (12)$$

$$\prod_{j=1}^n \left(\nu_{\tilde{a}_j} \right)^{\left(T_j / \sum_{j=1}^n T_j \right)} \geq \prod_{j=1}^n \left(\min_{1 \leq j \leq n} \{\nu_{\tilde{a}_j}\} \right)^{\left(T_j / \sum_{j=1}^n T_j \right)} = \min_{1 \leq j \leq n} \{\nu_{\tilde{a}_j}\} \quad (13)$$

According to Eqs. (10)-(13), we obtain

$$1 - \prod_{j=1}^n \left(1 - \mu_{\tilde{a}_j} \right)^{\left(T_j / \sum_{j=1}^n T_j \right)} - \prod_{j=1}^n \left(\nu_{\tilde{a}_j} \right)^{\left(T_j / \sum_{j=1}^n T_j \right)} \geq \min_{1 \leq j \leq n} \{\mu_{\tilde{a}_j}\} - \max_{1 \leq j \leq n} \{\nu_{\tilde{a}_j}\}$$

$$1 - \prod_{j=1}^n \left(1 - \mu_{\tilde{a}_j} \right)^{\left(T_j / \sum_{j=1}^n T_j \right)} - \prod_{j=1}^n \left(\nu_{\tilde{a}_j} \right)^{\left(T_j / \sum_{j=1}^n T_j \right)} \leq \max_{1 \leq j \leq n} \{\mu_{\tilde{a}_j}\} - \min_{1 \leq j \leq n} \{\nu_{\tilde{a}_j}\}$$

$$S(\text{ITFPWA}(\tilde{\alpha}_1, \tilde{\alpha}_2, \dots, \tilde{\alpha}_n)) \leq S(\tilde{\alpha}_{\max})$$

$$S(\text{ITFPWA}(\tilde{\alpha}_1, \tilde{\alpha}_2, \dots, \tilde{\alpha}_n)) \geq S(\tilde{\alpha}_{\min})$$

If $S(\text{ITFPWA}(\tilde{\alpha}_1, \tilde{\alpha}_2, \dots, \tilde{\alpha}_n)) < S(\tilde{\alpha}_{\max})$ and $S(\text{ITFPWA}(\tilde{\alpha}_1, \tilde{\alpha}_2, \dots, \tilde{\alpha}_n)) > S(\tilde{\alpha}_{\min})$, then by

Definition 2.7,

$$\tilde{\alpha}_{\min} \leq \text{ITFPWA}(\tilde{\alpha}_1, \tilde{\alpha}_2, \dots, \tilde{\alpha}_n) \leq \tilde{\alpha}_{\max} .$$

If $S(\text{ITFPWA}(\tilde{\alpha}_1, \tilde{\alpha}_2, \dots, \tilde{\alpha}_n)) = S(\tilde{\alpha}_{\max})$, i.e.,

$$\begin{aligned} & \left(\max_{1 \leq i \leq n} \{a_i\} + \max_{1 \leq i \leq n} \{b_i\} + \max_{1 \leq i \leq n} \{c_i\} + \max_{1 \leq i \leq n} \{d_i\} \right) \times \left(1 + \max_{1 \leq i \leq n} \{\mu_{\tilde{a}_i}\} - \min_{1 \leq i \leq n} \{\nu_{\tilde{a}_i}\} \right) \times \left(\max_{1 \leq i \leq n} \{\mu_{\tilde{a}_i}\} - \min_{1 \leq i \leq n} \{\nu_{\tilde{a}_i}\} \right) \\ &= \left(\sum_{j=1}^n \left(\frac{T_j a_j}{\sum_{j=1}^n T_j} \right) + \sum_{j=1}^n \left(\frac{T_j b_j}{\sum_{j=1}^n T_j} \right) + \sum_{j=1}^n \left(\frac{T_j c_j}{\sum_{j=1}^n T_j} \right) + \sum_{j=1}^n \left(\frac{T_j d_j}{\sum_{j=1}^n T_j} \right) \right) \times \left(1 + 1 - \prod_{j=1}^n \left(1 - \mu_{\tilde{a}_j} \right)^{\left(T_j / \sum_{j=1}^n T_j \right)} - \prod_{j=1}^n \left(\nu_{\tilde{a}_j} \right)^{\left(T_j / \sum_{j=1}^n T_j \right)} \right) \times \\ & \left(1 - \prod_{j=1}^n \left(1 - \mu_{\tilde{a}_j} \right)^{\left(T_j / \sum_{j=1}^n T_j \right)} - \prod_{j=1}^n \left(\nu_{\tilde{a}_j} \right)^{\left(T_j / \sum_{j=1}^n T_j \right)} \right) \end{aligned}$$

302

303 then by Eqs. (10)-(13), it follows that

$$304 \quad \max_{1 \leq i \leq n} \{a_i\} + \max_{1 \leq i \leq n} \{b_i\} + \max_{1 \leq i \leq n} \{c_i\} + \max_{1 \leq i \leq n} \{d_i\} = \sum_{j=1}^n \left(\frac{T_j a_j}{\sum_{j=1}^n T_j} \right) + \sum_{j=1}^n \left(\frac{T_j b_j}{\sum_{j=1}^n T_j} \right) + \sum_{j=1}^n \left(\frac{T_j c_j}{\sum_{j=1}^n T_j} \right) + \sum_{j=1}^n \left(\frac{T_j d_j}{\sum_{j=1}^n T_j} \right)$$

$$305 \quad 1 - \prod_{j=1}^n (1 - \mu_{\tilde{a}_j})^{\left(\frac{T_j}{\sum_{j=1}^n T_j} \right)} = \max_{1 \leq i \leq n} \{ \mu_{\tilde{a}_i} \}, \quad \prod_{j=1}^n (v_{\tilde{a}_j})^{\left(\frac{T_j}{\sum_{j=1}^n T_j} \right)} = \min_{1 \leq i \leq n} \{ v_{\tilde{a}_i} \};$$

306 thus,

$$307 \quad H(\text{ITFPWA}(\tilde{\alpha}_1, \tilde{\alpha}_2, \dots, \tilde{\alpha}_n)) = H(\tilde{\alpha}_{\max}),$$

308 which implies that

$$309 \quad \text{ITFPWA}(\tilde{\alpha}_1, \tilde{\alpha}_2, \dots, \tilde{\alpha}_n) = \tilde{\alpha}_{\max}.$$

310 If $S(\text{ITFPWA}(\tilde{\alpha}_1, \tilde{\alpha}_2, \dots, \tilde{\alpha}_n)) = S(\tilde{\alpha}_{\min})$, i.e.,

$$311 \quad \begin{aligned} & \left(\min_{1 \leq i \leq n} \{a_i\} + \min_{1 \leq i \leq n} \{b_i\} + \min_{1 \leq i \leq n} \{c_i\} + \min_{1 \leq i \leq n} \{d_i\} \right) \times \left(1 + \min_{1 \leq i \leq n} \{ \mu_{\tilde{a}_i} \} - \max_{1 \leq i \leq n} \{ v_{\tilde{a}_i} \} \right) \times \left(\min_{1 \leq i \leq n} \{ \mu_{\tilde{a}_i} \} - \max_{1 \leq i \leq n} \{ v_{\tilde{a}_i} \} \right) \\ &= \left(\sum_{j=1}^n \left(\frac{T_j a_j}{\sum_{j=1}^n T_j} \right) + \sum_{j=1}^n \left(\frac{T_j b_j}{\sum_{j=1}^n T_j} \right) + \sum_{j=1}^n \left(\frac{T_j c_j}{\sum_{j=1}^n T_j} \right) + \sum_{j=1}^n \left(\frac{T_j d_j}{\sum_{j=1}^n T_j} \right) \right) \times \left(1 + 1 - \prod_{j=1}^n (1 - \mu_{\tilde{a}_j})^{\left(\frac{T_j}{\sum_{j=1}^n T_j} \right)} - \prod_{j=1}^n (v_{\tilde{a}_j})^{\left(\frac{T_j}{\sum_{j=1}^n T_j} \right)} \right) \times \\ & \quad \left(1 - \prod_{j=1}^n (1 - \mu_{\tilde{a}_j})^{\left(\frac{T_j}{\sum_{j=1}^n T_j} \right)} - \prod_{j=1}^n (v_{\tilde{a}_j})^{\left(\frac{T_j}{\sum_{j=1}^n T_j} \right)} \right) \end{aligned}$$

312 then by Eqs. (10)-(13),

$$313 \quad \min_{1 \leq i \leq n} \{a_i\} + \min_{1 \leq i \leq n} \{b_i\} + \min_{1 \leq i \leq n} \{c_i\} + \min_{1 \leq i \leq n} \{d_i\} = \sum_{j=1}^n \left(\frac{T_j a_j}{\sum_{j=1}^n T_j} \right) + \sum_{j=1}^n \left(\frac{T_j b_j}{\sum_{j=1}^n T_j} \right) + \sum_{j=1}^n \left(\frac{T_j c_j}{\sum_{j=1}^n T_j} \right) + \sum_{j=1}^n \left(\frac{T_j d_j}{\sum_{j=1}^n T_j} \right),$$

$$314 \quad 1 - \prod_{j=1}^n (1 - \mu_{\tilde{a}_j})^{\left(\frac{T_j}{\sum_{j=1}^n T_j} \right)} = \min_{1 \leq i \leq n} \{ \mu_{\tilde{a}_i} \}, \quad \prod_{j=1}^n (v_{\tilde{a}_j})^{\left(\frac{T_j}{\sum_{j=1}^n T_j} \right)} = \max_{1 \leq i \leq n} \{ v_{\tilde{a}_i} \}$$

315 thus,

$$316 \quad H(\text{ITFPWA}(\tilde{\alpha}_1, \tilde{\alpha}_2, \dots, \tilde{\alpha}_n)) = H(\tilde{\alpha}_{\min}),$$

317 which implies that

$$318 \quad \text{ITFPWA}(\tilde{\alpha}_1, \tilde{\alpha}_2, \dots, \tilde{\alpha}_n) = \tilde{\alpha}_{\min}.$$

319 From the above analysis, we can conclude that Eq. (9) always holds. $\square$

320

321 3.2. Intuitionistic trapezoidal fuzzy prioritized weighted geometric (ITFPWG) operators

322

323 In the following, we define an intuitionistic trapezoidal fuzzy prioritized weighted geometric (ITFPWG) operator based on the ITFPWA operator and the geometric mean.

324 **Definition 3.2.** Let $\tilde{\alpha}_i = ([a_i, b_i, c_i, d_i]; \mu_{\tilde{\alpha}_i}, v_{\tilde{\alpha}_i})$ ($i = 1, 2, \dots, n$) be a collection of intuitionistic trapezoidal fuzzy numbers, if

$$327 \quad \text{ITFPWG}(\tilde{\alpha}_1, \tilde{\alpha}_2, \dots, \tilde{\alpha}_n) = \tilde{\alpha}_1^{\frac{T_1}{\sum_{j=1}^n T_j}} \otimes \tilde{\alpha}_2^{\frac{T_2}{\sum_{j=1}^n T_j}} \otimes \dots \otimes \tilde{\alpha}_n^{\frac{T_n}{\sum_{j=1}^n T_j}}, \quad (14)$$

where $T_1 = 1$ and $T_j = \prod_{k=1}^{j-1} S(\tilde{\alpha}_k)$ ($j = 2, \dots, n$), then ITFPWG is called an intuitionistic trapezoidal fuzzy prioritized weighted geometric (ITFPWG) operator.

Theorem 3.3. Let $\tilde{\alpha}_i = ([a_i, b_i, c_i, d_i]; \mu_{\tilde{\alpha}_i}, \nu_{\tilde{\alpha}_i})$ ($i = 1, 2, \dots, n$) be a collection of intuitionistic trapezoidal fuzzy numbers, then their aggregated value by using the ITFPWG operator is also an ITFN, and

$$\text{ITFPWG}(\tilde{\alpha}_1, \tilde{\alpha}_2, \dots, \tilde{\alpha}_n) = \left[\left[\prod_{j=1}^n a_j^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)}, \prod_{j=1}^n b_j^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)}, \prod_{j=1}^n c_j^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)}, \prod_{j=1}^n d_j^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)} \right]; \left[\prod_{j=1}^n \mu_{\tilde{\alpha}_j}^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)}, 1 - \prod_{j=1}^n (1 - \nu_{\tilde{\alpha}_j})^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)} \right] \right]. \quad (15)$$

Proof. The first result follows quickly from Definition 3.2 and Theorem 2.1. In the following, we prove the second result by using mathematical induction on n . We first prove that Eq. (15) holds for $n = 2$. Since

$$\tilde{\alpha}_1^{\frac{T_1}{\sum_{j=1}^n T_j}} = \left[\left[a_1^{\frac{T_1}{\sum_{j=1}^n T_j}}, b_1^{\frac{T_1}{\sum_{j=1}^n T_j}}, c_1^{\frac{T_1}{\sum_{j=1}^n T_j}}, d_1^{\frac{T_1}{\sum_{j=1}^n T_j}} \right]; \mu_{\tilde{\alpha}_1}^{\frac{T_1}{\sum_{j=1}^n T_j}}, 1 - (1 - \nu_{\tilde{\alpha}_1})^{\frac{T_1}{\sum_{j=1}^n T_j}} \right]$$

$$\tilde{\alpha}_2^{\frac{T_2}{\sum_{j=1}^n T_j}} = \left[\left[a_2^{\frac{T_2}{\sum_{j=1}^n T_j}}, b_2^{\frac{T_2}{\sum_{j=1}^n T_j}}, c_2^{\frac{T_2}{\sum_{j=1}^n T_j}}, d_2^{\frac{T_2}{\sum_{j=1}^n T_j}} \right]; \mu_{\tilde{\alpha}_2}^{\frac{T_2}{\sum_{j=1}^n T_j}}, 1 - (1 - \nu_{\tilde{\alpha}_2})^{\frac{T_2}{\sum_{j=1}^n T_j}} \right],$$

then

$$\begin{aligned} & \tilde{\alpha}_1^{\frac{T_1}{\sum_{j=1}^n T_j}} \otimes \tilde{\alpha}_2^{\frac{T_2}{\sum_{j=1}^n T_j}} \\ &= \left[\left[a_1^{\frac{T_1}{\sum_{j=1}^n T_j}}, b_1^{\frac{T_1}{\sum_{j=1}^n T_j}}, c_1^{\frac{T_1}{\sum_{j=1}^n T_j}}, d_1^{\frac{T_1}{\sum_{j=1}^n T_j}} \right]; \mu_{\tilde{\alpha}_1}^{\frac{T_1}{\sum_{j=1}^n T_j}}, 1 - (1 - \nu_{\tilde{\alpha}_1})^{\frac{T_1}{\sum_{j=1}^n T_j}} \right] \otimes \left[\left[a_2^{\frac{T_2}{\sum_{j=1}^n T_j}}, b_2^{\frac{T_2}{\sum_{j=1}^n T_j}}, c_2^{\frac{T_2}{\sum_{j=1}^n T_j}}, d_2^{\frac{T_2}{\sum_{j=1}^n T_j}} \right]; \mu_{\tilde{\alpha}_2}^{\frac{T_2}{\sum_{j=1}^n T_j}}, 1 - (1 - \nu_{\tilde{\alpha}_2})^{\frac{T_2}{\sum_{j=1}^n T_j}} \right] \\ &= \left[\left[a_1^{\frac{T_1}{\sum_{j=1}^n T_j}} a_2^{\frac{T_2}{\sum_{j=1}^n T_j}}, b_1^{\frac{T_1}{\sum_{j=1}^n T_j}} b_2^{\frac{T_2}{\sum_{j=1}^n T_j}}, c_1^{\frac{T_1}{\sum_{j=1}^n T_j}} c_2^{\frac{T_2}{\sum_{j=1}^n T_j}}, d_1^{\frac{T_1}{\sum_{j=1}^n T_j}} d_2^{\frac{T_2}{\sum_{j=1}^n T_j}} \right]; \right. \\ & \quad \left. \mu_{\tilde{\alpha}_1}^{\frac{T_1}{\sum_{j=1}^n T_j}} \mu_{\tilde{\alpha}_2}^{\frac{T_2}{\sum_{j=1}^n T_j}}, 1 - (1 - \nu_{\tilde{\alpha}_1})^{\frac{T_1}{\sum_{j=1}^n T_j}} + 1 - (1 - \nu_{\tilde{\alpha}_2})^{\frac{T_2}{\sum_{j=1}^n T_j}} - \left(1 - (1 - \nu_{\tilde{\alpha}_1})^{\frac{T_1}{\sum_{j=1}^n T_j}} \right) \cdot \left(1 - (1 - \nu_{\tilde{\alpha}_2})^{\frac{T_2}{\sum_{j=1}^n T_j}} \right) \right] \\ &= \left[\left[a_1^{\frac{T_1}{\sum_{j=1}^n T_j}} a_2^{\frac{T_2}{\sum_{j=1}^n T_j}}, b_1^{\frac{T_1}{\sum_{j=1}^n T_j}} b_2^{\frac{T_2}{\sum_{j=1}^n T_j}}, c_1^{\frac{T_1}{\sum_{j=1}^n T_j}} c_2^{\frac{T_2}{\sum_{j=1}^n T_j}}, d_1^{\frac{T_1}{\sum_{j=1}^n T_j}} d_2^{\frac{T_2}{\sum_{j=1}^n T_j}} \right]; \mu_{\tilde{\alpha}_1}^{\frac{T_1}{\sum_{j=1}^n T_j}} \mu_{\tilde{\alpha}_2}^{\frac{T_2}{\sum_{j=1}^n T_j}}, 1 - (1 - \nu_{\tilde{\alpha}_1})^{\frac{T_1}{\sum_{j=1}^n T_j}} \cdot (1 - \nu_{\tilde{\alpha}_2})^{\frac{T_2}{\sum_{j=1}^n T_j}} \right]. \end{aligned}$$

If Eq. (15) holds for $n = k$, that is

$$\text{ITFPWG}(\tilde{\alpha}_1, \tilde{\alpha}_2, \dots, \tilde{\alpha}_k) = \left[\left[\prod_{j=1}^k a_j^{\left(\frac{T_j}{\sum_{j=1}^k T_j}\right)}, \prod_{j=1}^k b_j^{\left(\frac{T_j}{\sum_{j=1}^k T_j}\right)}, \prod_{j=1}^k c_j^{\left(\frac{T_j}{\sum_{j=1}^k T_j}\right)}, \prod_{j=1}^k d_j^{\left(\frac{T_j}{\sum_{j=1}^k T_j}\right)} \right]; \prod_{j=1}^k \mu_{\tilde{\alpha}_j}^{\left(\frac{T_j}{\sum_{j=1}^k T_j}\right)}, 1 - \prod_{j=1}^k (1 - \nu_{\tilde{\alpha}_j})^{\left(\frac{T_j}{\sum_{j=1}^k T_j}\right)} \right]$$

then, when $n = k + 1$, by the operational laws in Definition 2.3, we have

$$\begin{aligned}
& \text{ITFPWG}(\tilde{\alpha}_1, \tilde{\alpha}_2, \dots, \tilde{\alpha}_{k+1}) \\
&= \tilde{\alpha}_1^{\frac{T_1}{\sum_{j=1}^n T_j}} \otimes \tilde{\alpha}_2^{\frac{T_2}{\sum_{j=1}^n T_j}} \otimes \dots \otimes \tilde{\alpha}_k^{\frac{T_k}{\sum_{j=1}^n T_j}} \otimes \tilde{\alpha}_{k+1}^{\frac{T_{k+1}}{\sum_{j=1}^n T_j}} \\
&= \left[\prod_{j=1}^k a_j^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)}, \prod_{j=1}^k b_j^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)}, \prod_{j=1}^k c_j^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)}, \prod_{j=1}^k d_j^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)} \right]; \prod_{j=1}^k \mu_{\tilde{\alpha}_j}^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)}, 1 - \prod_{j=1}^k (1 - \nu_{\tilde{\alpha}_j})^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)} \otimes \\
&\quad \left[\left[a_{k+1}^{\frac{T_{k+1}}{\sum_{j=1}^n T_j}}, b_{k+1}^{\frac{T_{k+1}}{\sum_{j=1}^n T_j}}, c_{k+1}^{\frac{T_{k+1}}{\sum_{j=1}^n T_j}}, d_{k+1}^{\frac{T_{k+1}}{\sum_{j=1}^n T_j}} \right]; \mu_{\tilde{\alpha}_{k+1}}^{\frac{T_{k+1}}{\sum_{j=1}^n T_j}}, 1 - (1 - \nu_{\tilde{\alpha}_{k+1}})^{\frac{T_{k+1}}{\sum_{j=1}^n T_j}} \right] \\
&= \left[\left[\prod_{j=1}^{k+1} a_j^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)}, \prod_{j=1}^{k+1} b_j^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)}, \prod_{j=1}^{k+1} c_j^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)}, \prod_{j=1}^{k+1} d_j^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)} \right]; \prod_{j=1}^{k+1} \mu_{\tilde{\alpha}_j}^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)}, \right. \\
&\quad \left. 1 - \prod_{j=1}^k (1 - \nu_{\tilde{\alpha}_j})^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)} + 1 - (1 - \nu_{\tilde{\alpha}_{k+1}})^{\frac{T_{k+1}}{\sum_{j=1}^n T_j}} - \left(1 - \prod_{j=1}^k (1 - \nu_{\tilde{\alpha}_j})^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)} \right) \cdot \left(1 - (1 - \nu_{\tilde{\alpha}_{k+1}})^{\frac{T_{k+1}}{\sum_{j=1}^n T_j}} \right) \right] \\
&= \left[\left[\prod_{j=1}^{k+1} a_j^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)}, \prod_{j=1}^{k+1} b_j^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)}, \prod_{j=1}^{k+1} c_j^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)}, \prod_{j=1}^{k+1} d_j^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)} \right]; \prod_{j=1}^{k+1} \mu_{\tilde{\alpha}_j}^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)}, 1 - \prod_{j=1}^{k+1} (1 - \nu_{\tilde{\alpha}_j})^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)} \right]
\end{aligned}$$

i.e., Eq. (15) holds for $n = k + 1$. Therefore, Eq. (15) holds for all n , which completes the proof of Theorem 3.3.

Theorem 3.4. Let $\tilde{\alpha}_i = ([a_i, b_i, c_i, d_i]; \mu_{\tilde{\alpha}_i}, \nu_{\tilde{\alpha}_i})$ ($i = 1, 2, \dots, n$) be a collection of intuitionistic trapezoidal fuzzy numbers. Then, the following properties hold.

(1) Idempotency: If $\tilde{\alpha}_i = \tilde{\alpha} = ([a, b, c, d]; \mu_{\tilde{\alpha}}, \nu_{\tilde{\alpha}})$ for all i , then

$$\text{ITFPWG}(\tilde{\alpha}_1, \tilde{\alpha}_2, \dots, \tilde{\alpha}_n) = \tilde{\alpha}. \quad (16)$$

(2) Boundedness:

$$\tilde{\alpha}_{\min} \leq \text{ITFPWG}(\tilde{\alpha}_1, \tilde{\alpha}_2, \dots, \tilde{\alpha}_n) \leq \tilde{\alpha}_{\max}, \quad (17)$$

where $\tilde{\alpha}_{\min} = \left(\left[\min_{1 \leq i \leq n} \{a_i\}, \min_{1 \leq i \leq n} \{b_i\}, \min_{1 \leq i \leq n} \{c_i\}, \min_{1 \leq i \leq n} \{d_i\} \right]; \min_{1 \leq i \leq n} \{\mu_{\tilde{\alpha}_i}\}, \max_{1 \leq i \leq n} \{\nu_{\tilde{\alpha}_i}\} \right)$ and

$$\tilde{\alpha}_{\max} = \left(\left[\max_{1 \leq i \leq n} \{a_i\}, \max_{1 \leq i \leq n} \{b_i\}, \max_{1 \leq i \leq n} \{c_i\}, \max_{1 \leq i \leq n} \{d_i\} \right]; \max_{1 \leq i \leq n} \{\mu_{\tilde{\alpha}_i}\}, \min_{1 \leq i \leq n} \{\nu_{\tilde{\alpha}_i}\} \right).$$

Proof. (1) If $\tilde{\alpha}_i = \tilde{\alpha} = ([a, b, c, d]; \mu_{\tilde{\alpha}}, \nu_{\tilde{\alpha}})$ for all i , then

$$\begin{aligned}
& \text{ITFPWG}(\tilde{\alpha}_1, \tilde{\alpha}_2, \dots, \tilde{\alpha}_n) \\
&= \left[\left[\prod_{j=1}^n a_j^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)}, \prod_{j=1}^n b_j^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)}, \prod_{j=1}^n c_j^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)}, \prod_{j=1}^n d_j^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)} \right]; \prod_{j=1}^n \mu_{\tilde{\alpha}_j}^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)}, 1 - \prod_{j=1}^n (1 - \nu_{\tilde{\alpha}_j})^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)} \right] \\
&= ([a, b, c, d]; \mu_{\tilde{\alpha}}, \nu_{\tilde{\alpha}}) \\
&= \tilde{\alpha}.
\end{aligned}$$

(2) First, by Definitions 2.5 and 2.6, and Theorem 3.3, we have

$$\begin{aligned}
& \text{ITFPWG}(\tilde{\alpha}_1, \tilde{\alpha}_2, \dots, \tilde{\alpha}_n) \\
358 \quad &= \left(\left[\prod_{j=1}^n a_j^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)}, \prod_{j=1}^n b_j^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)}, \prod_{j=1}^n c_j^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)}, \prod_{j=1}^n d_j^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)} \right]; \prod_{j=1}^n \mu_{\tilde{\alpha}_j}^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)}, 1 - \prod_{j=1}^n (1 - \nu_{\tilde{\alpha}_j})^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)} \right) \\
& S(\text{ITFPWG}(\tilde{\alpha}_1, \tilde{\alpha}_2, \dots, \tilde{\alpha}_n)) = \frac{1}{8} \times \left(\prod_{j=1}^n a_j^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)} + \prod_{j=1}^n b_j^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)} + \prod_{j=1}^n c_j^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)} + \prod_{j=1}^n d_j^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)} \right) \times \\
359 \quad & \left(1 + \prod_{j=1}^n \mu_{\tilde{\alpha}_j}^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)} - 1 + \prod_{j=1}^n (1 - \nu_{\tilde{\alpha}_j})^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)} \right) \times \\
& \left(\prod_{j=1}^n \mu_{\tilde{\alpha}_j}^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)} - 1 + \prod_{j=1}^n (1 - \nu_{\tilde{\alpha}_j})^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)} \right) \\
& H(\text{ITFPWG}(\tilde{\alpha}_1, \tilde{\alpha}_2, \dots, \tilde{\alpha}_n)) = \frac{1}{8} \times \left(\prod_{j=1}^n a_j^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)} + \prod_{j=1}^n b_j^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)} + \prod_{j=1}^n c_j^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)} + \prod_{j=1}^n d_j^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)} \right) \times \\
360 \quad & \left(1 + \prod_{j=1}^n \mu_{\tilde{\alpha}_j}^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)} - 1 + \prod_{j=1}^n (1 - \nu_{\tilde{\alpha}_j})^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)} \right) \times \\
& \left(\prod_{j=1}^n \mu_{\tilde{\alpha}_j}^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)} + 1 - \prod_{j=1}^n (1 - \nu_{\tilde{\alpha}_j})^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)} \right) \\
361 \quad & S(\tilde{\alpha}_{\min}) = \frac{1}{8} \times \left(\min_{1 \leq i \leq n} \{a_i\} + \min_{1 \leq i \leq n} \{b_i\} + \min_{1 \leq i \leq n} \{c_i\} + \min_{1 \leq i \leq n} \{d_i\} \right) \times \left(1 + \min_{1 \leq i \leq n} \{\mu_{\tilde{\alpha}_i}\} - \max_{1 \leq i \leq n} \{\nu_{\tilde{\alpha}_i}\} \right) \times \left(\min_{1 \leq i \leq n} \{\mu_{\tilde{\alpha}_i}\} - \max_{1 \leq i \leq n} \{\nu_{\tilde{\alpha}_i}\} \right) \\
362 \quad & H(\tilde{\alpha}_{\min}) = \frac{1}{8} \times \left(\min_{1 \leq i \leq n} \{a_i\} + \min_{1 \leq i \leq n} \{b_i\} + \min_{1 \leq i \leq n} \{c_i\} + \min_{1 \leq i \leq n} \{d_i\} \right) \times \left(1 + \min_{1 \leq i \leq n} \{\mu_{\tilde{\alpha}_i}\} - \max_{1 \leq i \leq n} \{\nu_{\tilde{\alpha}_i}\} \right) \times \left(\min_{1 \leq i \leq n} \{\mu_{\tilde{\alpha}_i}\} + \max_{1 \leq i \leq n} \{\nu_{\tilde{\alpha}_i}\} \right) \\
363 \quad & S(\tilde{\alpha}_{\max}) = \frac{1}{8} \times \left(\max_{1 \leq i \leq n} \{a_i\} + \max_{1 \leq i \leq n} \{b_i\} + \max_{1 \leq i \leq n} \{c_i\} + \max_{1 \leq i \leq n} \{d_i\} \right) \times \left(1 + \max_{1 \leq i \leq n} \{\mu_{\tilde{\alpha}_i}\} - \min_{1 \leq i \leq n} \{\nu_{\tilde{\alpha}_i}\} \right) \times \left(\max_{1 \leq i \leq n} \{\mu_{\tilde{\alpha}_i}\} - \min_{1 \leq i \leq n} \{\nu_{\tilde{\alpha}_i}\} \right) \\
364 \quad & H(\tilde{\alpha}_{\max}) = \frac{1}{8} \times \left(\max_{1 \leq i \leq n} \{a_i\} + \max_{1 \leq i \leq n} \{b_i\} + \max_{1 \leq i \leq n} \{c_i\} + \max_{1 \leq i \leq n} \{d_i\} \right) \times \left(1 + \max_{1 \leq i \leq n} \{\mu_{\tilde{\alpha}_i}\} - \min_{1 \leq i \leq n} \{\nu_{\tilde{\alpha}_i}\} \right) \times \left(\max_{1 \leq i \leq n} \{\mu_{\tilde{\alpha}_i}\} + \min_{1 \leq i \leq n} \{\nu_{\tilde{\alpha}_i}\} \right) \\
365 \quad & \text{Obviously, we have } \min_{1 \leq i \leq n} \{a_i\} \leq \prod_{j=1}^n a_j^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)} \leq \max_{1 \leq i \leq n} \{a_i\}, \quad \min_{1 \leq i \leq n} \{b_i\} \leq \prod_{j=1}^n b_j^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)} \leq \max_{1 \leq i \leq n} \{b_i\}, \\
366 \quad & \min_{1 \leq i \leq n} \{c_i\} \leq \prod_{j=1}^n c_j^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)} \leq \max_{1 \leq i \leq n} \{c_i\}, \text{ and } \min_{1 \leq i \leq n} \{d_i\} \leq \prod_{j=1}^n d_j^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)} \leq \max_{1 \leq i \leq n} \{d_i\}. \\
367 \quad & \text{Because } \min_{1 \leq i \leq n} \{\mu_{\tilde{\alpha}_i}\} \leq \mu_{\tilde{\alpha}_i} \leq \max_{1 \leq i \leq n} \{\mu_{\tilde{\alpha}_i}\} \text{ and } \min_{1 \leq i \leq n} \{\nu_{\tilde{\alpha}_i}\} \leq \nu_{\tilde{\alpha}_i} \leq \max_{1 \leq i \leq n} \{\nu_{\tilde{\alpha}_i}\}, \\
368 \quad & 1 - \prod_{j=1}^n (1 - \nu_{\tilde{\alpha}_j})^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)} \geq 1 - \prod_{j=1}^n (1 - \min_{1 \leq j \leq n} \{\nu_{\tilde{\alpha}_j}\})^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)} = \min_{1 \leq j \leq n} \{\nu_{\tilde{\alpha}_j}\} \quad (18)
\end{aligned}$$

$$1 - \prod_{j=1}^n (1 - v_{\tilde{\alpha}_j})^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)} \leq 1 - \prod_{j=1}^n \left(1 - \max_{1 \leq j \leq n} \{v_{\tilde{\alpha}_j}\}\right)^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)} = \max_{1 \leq j \leq n} \{v_{\tilde{\alpha}_j}\} \quad (19)$$

$$\prod_{j=1}^n \mu_{\tilde{\alpha}_j}^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)} \leq \prod_{j=1}^n \left(\max_{1 \leq j \leq n} \{\mu_{\tilde{\alpha}_j}\}\right)^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)} = \max_{1 \leq j \leq n} \{\mu_{\tilde{\alpha}_j}\} \quad (20)$$

$$\prod_{j=1}^n \mu_{\tilde{\alpha}_j}^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)} \geq \prod_{j=1}^n \left(\min_{1 \leq j \leq n} \{\mu_{\tilde{\alpha}_j}\}\right)^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)} = \min_{1 \leq j \leq n} \{\mu_{\tilde{\alpha}_j}\} \quad (21)$$

According to Eqs. (18)-(21), we obtain

$$\prod_{j=1}^n \mu_{\tilde{\alpha}_j}^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)} - 1 + \prod_{j=1}^n (1 - v_{\tilde{\alpha}_j})^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)} \geq \min_{1 \leq j \leq n} \{\mu_{\tilde{\alpha}_j}\} - \max_{1 \leq j \leq n} \{v_{\tilde{\alpha}_j}\}$$

$$\prod_{j=1}^n \mu_{\tilde{\alpha}_j}^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)} - 1 + \prod_{j=1}^n (1 - v_{\tilde{\alpha}_j})^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)} \leq \max_{1 \leq j \leq n} \{\mu_{\tilde{\alpha}_j}\} - \min_{1 \leq j \leq n} \{v_{\tilde{\alpha}_j}\}$$

$$S(\text{ITFPWG}(\tilde{\alpha}_1, \tilde{\alpha}_2, \dots, \tilde{\alpha}_n)) \leq S(\tilde{\alpha}_{\max})$$

$$S(\text{ITFPWG}(\tilde{\alpha}_1, \tilde{\alpha}_2, \dots, \tilde{\alpha}_n)) \geq S(\tilde{\alpha}_{\min})$$

If $S(\text{ITFPWG}(\tilde{\alpha}_1, \tilde{\alpha}_2, \dots, \tilde{\alpha}_n)) < S(\tilde{\alpha}_{\max})$ and $S(\text{ITFPWG}(\tilde{\alpha}_1, \tilde{\alpha}_2, \dots, \tilde{\alpha}_n)) > S(\tilde{\alpha}_{\min})$, then by Definition 2.7,

$$\tilde{\alpha}_{\min} \leq \text{ITFPWG}(\tilde{\alpha}_1, \tilde{\alpha}_2, \dots, \tilde{\alpha}_n) \leq \tilde{\alpha}_{\max}.$$

If $S(\text{ITFPWG}(\tilde{\alpha}_1, \tilde{\alpha}_2, \dots, \tilde{\alpha}_n)) = S(\tilde{\alpha}_{\max})$, i.e.,

$$\begin{aligned} & \left(\max_{1 \leq i \leq n} \{a_i\} + \max_{1 \leq i \leq n} \{b_i\} + \max_{1 \leq i \leq n} \{c_i\} + \max_{1 \leq i \leq n} \{d_i\} \right) \times \left(1 + \max_{1 \leq i \leq n} \{\mu_{\tilde{\alpha}_i}\} - \min_{1 \leq i \leq n} \{v_{\tilde{\alpha}_i}\} \right) \times \left(\max_{1 \leq i \leq n} \{\mu_{\tilde{\alpha}_i}\} - \min_{1 \leq i \leq n} \{v_{\tilde{\alpha}_i}\} \right) \\ &= \left(\prod_{j=1}^n a_j^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)} + \prod_{j=1}^n b_j^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)} + \prod_{j=1}^n c_j^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)} + \prod_{j=1}^n d_j^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)} \right) \times \left(1 + \prod_{j=1}^n \mu_{\tilde{\alpha}_j}^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)} - 1 + \prod_{j=1}^n (1 - v_{\tilde{\alpha}_j})^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)} \right) \times \\ & \quad \left(\prod_{j=1}^n \mu_{\tilde{\alpha}_j}^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)} - 1 + \prod_{j=1}^n (1 - v_{\tilde{\alpha}_j})^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)} \right) \end{aligned}$$

then by Eqs. (18)-(21), it follows that

$$\max_{1 \leq i \leq n} \{a_i\} + \max_{1 \leq i \leq n} \{b_i\} + \max_{1 \leq i \leq n} \{c_i\} + \max_{1 \leq i \leq n} \{d_i\} = \prod_{j=1}^n a_j^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)} + \prod_{j=1}^n b_j^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)} + \prod_{j=1}^n c_j^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)} + \prod_{j=1}^n d_j^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)}$$

$$\prod_{j=1}^n \mu_{\tilde{\alpha}_j}^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)} = \max_{1 \leq i \leq n} \{\mu_{\tilde{\alpha}_i}\}, \quad 1 - \prod_{j=1}^n (1 - v_{\tilde{\alpha}_j})^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)} = \min_{1 \leq i \leq n} \{v_{\tilde{\alpha}_i}\};$$

thus,

$$H(\text{ITFPWG}(\tilde{\alpha}_1, \tilde{\alpha}_2, \dots, \tilde{\alpha}_n)) = H(\tilde{\alpha}_{\max}),$$

which implies that

$$\text{ITFPWG}(\tilde{\alpha}_1, \tilde{\alpha}_2, \dots, \tilde{\alpha}_n) = \tilde{\alpha}_{\max}.$$

If $S(\text{ITFPWG}(\tilde{\alpha}_1, \tilde{\alpha}_2, \dots, \tilde{\alpha}_n)) = S(\tilde{\alpha}_{\min})$, i.e.,

$$\begin{aligned}
& \left(\min_{1 \leq i \leq n} \{a_i\} + \min_{1 \leq i \leq n} \{b_i\} + \min_{1 \leq i \leq n} \{c_i\} + \min_{1 \leq i \leq n} \{d_i\} \right) \times \left(1 + \min_{1 \leq i \leq n} \{\mu_{\tilde{a}_i}\} - \max_{1 \leq i \leq n} \{\nu_{\tilde{a}_i}\} \right) \times \left(\min_{1 \leq i \leq n} \{\mu_{\tilde{a}_i}\} - \max_{1 \leq i \leq n} \{\nu_{\tilde{a}_i}\} \right) \\
390 \quad & = \left(\prod_{j=1}^n a_j^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)} + \prod_{j=1}^n b_j^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)} + \prod_{j=1}^n c_j^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)} + \prod_{j=1}^n d_j^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)} \right) \times \left(1 + \prod_{j=1}^n \mu_{\tilde{a}_j}^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)} - 1 + \prod_{j=1}^n (1 - \nu_{\tilde{a}_j})^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)} \right) \times \\
& \quad \left(\prod_{j=1}^n \mu_{\tilde{a}_j}^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)} - 1 + \prod_{j=1}^n (1 - \nu_{\tilde{a}_j})^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)} \right)
\end{aligned}$$

391 then by Eqs. (18)-(21),

$$\begin{aligned}
392 \quad & \min_{1 \leq i \leq n} \{a_i\} + \min_{1 \leq i \leq n} \{b_i\} + \min_{1 \leq i \leq n} \{c_i\} + \min_{1 \leq i \leq n} \{d_i\} = \prod_{j=1}^n a_j^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)} + \prod_{j=1}^n b_j^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)} + \prod_{j=1}^n c_j^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)} + \prod_{j=1}^n d_j^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)}, \\
393 \quad & \prod_{j=1}^n \mu_{\tilde{a}_j}^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)} = \min_{1 \leq i \leq n} \{\mu_{\tilde{a}_i}\}, \quad 1 - \prod_{j=1}^n (1 - \nu_{\tilde{a}_j})^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)} = \max_{1 \leq i \leq n} \{\nu_{\tilde{a}_i}\}
\end{aligned}$$

394 thus,

$$395 \quad H(\text{ITFPWG}(\tilde{\alpha}_1, \tilde{\alpha}_2, \dots, \tilde{\alpha}_n)) = H(\tilde{\alpha}_{\min}),$$

396 which implies that

$$397 \quad \text{ITFPWG}(\tilde{\alpha}_1, \tilde{\alpha}_2, \dots, \tilde{\alpha}_n) = \tilde{\alpha}_{\min}.$$

398 From the above analysis, we can conclude that Eq. (17) always holds.

399 **Lemma 3.1 [5,16].** Let $x_i > 0$, $\lambda_i > 0$, $i = 1, 2, \dots, n$, and $\sum_{i=1}^n \lambda_i = 1$, then

$$400 \quad \prod_{i=1}^n (x_i)^{\lambda_i} \leq \sum_{i=1}^n \lambda_i x_i$$

401 with equality if and only if $x_1 = x_2 = \dots = x_n$.

402 **Theorem 3.5.** Let $\tilde{\alpha}_i = ([a_i, b_i, c_i, d_i]; \mu_{\tilde{\alpha}_i}, \nu_{\tilde{\alpha}_i}) \in \Omega$ ($i = 1, 2, \dots, n$) be a collection of intuitionistic
403 trapezoidal fuzzy numbers and $\lambda > 0$. Then,

$$404 \quad \text{ITFPWG}(\tilde{\alpha}_1, \tilde{\alpha}_2, \dots, \tilde{\alpha}_n) \leq \text{ITFPWA}(\tilde{\alpha}_1, \tilde{\alpha}_2, \dots, \tilde{\alpha}_n). \quad (22)$$

405 **Proof.** Because $\sum_{i=1}^n w_i = 1$, then by Lemma 3.1, we have

$$406 \quad \prod_{j=1}^n a_j^{\frac{T_j}{\sum_{j=1}^n T_j}} \leq \sum_{j=1}^n \left(\frac{T_j a_j}{\sum_{j=1}^n T_j} \right), \quad \prod_{j=1}^n b_j^{\frac{T_j}{\sum_{j=1}^n T_j}} \leq \sum_{j=1}^n \left(\frac{T_j b_j}{\sum_{j=1}^n T_j} \right), \quad \prod_{j=1}^n c_j^{\frac{T_j}{\sum_{j=1}^n T_j}} \leq \sum_{j=1}^n \left(\frac{T_j c_j}{\sum_{j=1}^n T_j} \right),$$

$$407 \quad \prod_{j=1}^n d_j^{\frac{T_j}{\sum_{j=1}^n T_j}} \leq \sum_{j=1}^n \left(\frac{T_j d_j}{\sum_{j=1}^n T_j} \right),$$

$$408 \quad \prod_{j=1}^n \mu_{\tilde{\alpha}_j}^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)} \leq \sum_{j=1}^n \left(\frac{T_j \mu_{\tilde{\alpha}_j}}{\sum_{j=1}^n T_j} \right) = 1 - \sum_{j=1}^n \left(\frac{T_j (1 - \mu_{\tilde{\alpha}_j})}{\sum_{j=1}^n T_j} \right) \leq 1 - \prod_{j=1}^n (1 - \mu_{\tilde{\alpha}_j})^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)}$$

$$1 - \prod_{j=1}^n (1 - v_{\tilde{\alpha}_j})^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)} \geq 1 - \sum_{j=1}^n \left(\frac{T_j (1 - v_{\tilde{\alpha}_j})}{\sum_{j=1}^n T_j} \right) = \sum_{j=1}^n \left(\frac{T_j v_{\tilde{\alpha}_j}}{\sum_{j=1}^n T_j} \right) \geq \prod_{j=1}^n (v_{\tilde{\alpha}_j})^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)}.$$

Thus, we have $S(\text{ITFPWG}(\tilde{\alpha}_1, \tilde{\alpha}_2, \dots, \tilde{\alpha}_n)) \leq S(\text{ITFPWA}(\tilde{\alpha}_1, \tilde{\alpha}_2, \dots, \tilde{\alpha}_n))$.

If $S(\text{ITFPWG}(\tilde{\alpha}_1, \tilde{\alpha}_2, \dots, \tilde{\alpha}_n)) < S(\text{ITFPWA}(\tilde{\alpha}_1, \tilde{\alpha}_2, \dots, \tilde{\alpha}_n))$, then by Definition 2.7,

$$\text{ITFPWG}(\tilde{\alpha}_1, \tilde{\alpha}_2, \dots, \tilde{\alpha}_n) < \text{ITFPWA}(\tilde{\alpha}_1, \tilde{\alpha}_2, \dots, \tilde{\alpha}_n).$$

If $S(\text{ITFPWG}(\tilde{\alpha}_1, \tilde{\alpha}_2, \dots, \tilde{\alpha}_n)) = S(\text{ITFPWA}(\tilde{\alpha}_1, \tilde{\alpha}_2, \dots, \tilde{\alpha}_n))$, i.e.,

$$\begin{aligned} & \left(\prod_{j=1}^n a_j^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)} + \prod_{j=1}^n b_j^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)} + \prod_{j=1}^n c_j^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)} + \prod_{j=1}^n d_j^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)} \right) \times \left(1 + \prod_{j=1}^n \mu_{\tilde{\alpha}_j}^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)} - 1 + \prod_{j=1}^n (1 - v_{\tilde{\alpha}_j})^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)} \right) \times \\ & \left(\prod_{j=1}^n \mu_{\tilde{\alpha}_j}^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)} - 1 + \prod_{j=1}^n (1 - v_{\tilde{\alpha}_j})^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)} \right) \\ & = \left(\sum_{j=1}^n \left(\frac{T_j a_j}{\sum_{j=1}^n T_j} \right) + \sum_{j=1}^n \left(\frac{T_j b_j}{\sum_{j=1}^n T_j} \right) + \sum_{j=1}^n \left(\frac{T_j c_j}{\sum_{j=1}^n T_j} \right) + \sum_{j=1}^n \left(\frac{T_j d_j}{\sum_{j=1}^n T_j} \right) \right) \times \left(1 + 1 - \prod_{j=1}^n (1 - \mu_{\tilde{\alpha}_j})^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)} - \prod_{j=1}^n (v_{\tilde{\alpha}_j})^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)} \right) \times \\ & \left(1 - \prod_{j=1}^n (1 - \mu_{\tilde{\alpha}_j})^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)} - \prod_{j=1}^n (v_{\tilde{\alpha}_j})^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)} \right) \end{aligned}$$

then it follows that

$$\begin{aligned} & \prod_{j=1}^n a_j^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)} + \prod_{j=1}^n b_j^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)} + \prod_{j=1}^n c_j^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)} + \prod_{j=1}^n d_j^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)} = \sum_{j=1}^n \left(\frac{T_j a_j}{\sum_{j=1}^n T_j} \right) + \sum_{j=1}^n \left(\frac{T_j b_j}{\sum_{j=1}^n T_j} \right) + \sum_{j=1}^n \left(\frac{T_j c_j}{\sum_{j=1}^n T_j} \right) + \sum_{j=1}^n \left(\frac{T_j d_j}{\sum_{j=1}^n T_j} \right) \\ & \prod_{j=1}^n \mu_{\tilde{\alpha}_j}^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)} = 1 - \prod_{j=1}^n (1 - \mu_{\tilde{\alpha}_j})^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)}, \quad 1 - \prod_{j=1}^n (1 - v_{\tilde{\alpha}_j})^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)} = \prod_{j=1}^n (v_{\tilde{\alpha}_j})^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)}; \end{aligned}$$

thus,

$$H(\text{ITFPWG}(\tilde{\alpha}_1, \tilde{\alpha}_2, \dots, \tilde{\alpha}_n)) = H(\text{ITFPWA}(\tilde{\alpha}_1, \tilde{\alpha}_2, \dots, \tilde{\alpha}_n)),$$

which implies that

$$\text{ITFPWG}(\tilde{\alpha}_1, \tilde{\alpha}_2, \dots, \tilde{\alpha}_n) = \text{ITFPWA}(\tilde{\alpha}_1, \tilde{\alpha}_2, \dots, \tilde{\alpha}_n).$$

From the above analysis, we can conclude that Eq. (22) always holds.

According to Theorem 3.5, we can show that the values obtained by the ITFPWG operator are not greater than the values obtained by the ITFPWA operator for any $\lambda > 0$.

3.3. An approach to multiple attribute group decision making with intuitionistic trapezoidal fuzzy information

In this subsection, we utilize the proposed aggregation operators to handle a multiple attribute group decision making with intuitionistic trapezoidal fuzzy information.

For a group decision making problem with intuitionistic trapezoidal fuzzy information, let $X = \{x_1, x_2, \dots, x_m\}$ is the set of alternatives. Let $C = \{c_1, c_2, \dots, c_n\}$ be a collection of attributes and that there is a prioritization between the attributes expressed by the linear ordering $c_1 \succ c_2 \succ c_3 \succ \dots \succ c_n$, indicate attribute c_j has a higher priority than c_k if $j < k$. Let $D = \{d_1, d_2, \dots, d_l\}$ is the set of decision makers and that there is a prioritization between the decision makers expressed by the linear ordering $d_1 \succ d_2 \succ d_3 \succ \dots \succ d_l$, indicate decision maker d_p has a higher priority than d_q if $p < q$. Suppose that each decision maker provides his/her own intuitionistic trapezoidal fuzzy decision matrix $\tilde{A}^{(k)} = (\tilde{a}_{ij}^{(k)})_{m \times n}$ ($k = 1, 2, \dots, l$), where $\tilde{a}_{ij}^{(k)} = ([a_{ij}^{(k)}, b_{ij}^{(k)}, c_{ij}^{(k)}, d_{ij}^{(k)}]; \mu_{ij}^{(k)}, \nu_{ij}^{(k)})$ is an intuitionistic trapezoidal fuzzy number provided by the decision maker $d_k \in D$, $\mu_{ij}^{(k)}$ denotes the extent to which alternative x_i belongs to trapezoidal fuzzy number $[a_{ij}^{(k)}, b_{ij}^{(k)}, c_{ij}^{(k)}, d_{ij}^{(k)}]$ on the criteria c_j , $\nu_{ij}^{(k)}$ denotes the extent to which alternative x_i does not belong to trapezoidal fuzzy number $[a_{ij}^{(k)}, b_{ij}^{(k)}, c_{ij}^{(k)}, d_{ij}^{(k)}]$ on the criteria c_j , $\mu_{ij}^{(k)} \in [0, 1]$, $\nu_{ij}^{(k)} \in [0, 1]$, $\mu_{ij}^{(k)} + \nu_{ij}^{(k)} \leq 1$, $i = 1, 2, \dots, m$, $j = 1, 2, \dots, n$.

To eliminate the effect from different physical dimensions to decision results, the decision making matrix should be standardized firstly. Consider that there are benefit attributes (i.e., the bigger the attribute values the better) and cost attributes (i.e., the smaller the attribute values the better) in multiple attribute group decision making problems. We may transform the intuitionistic trapezoidal fuzzy decision matrix $\tilde{A}^{(k)} = (\tilde{a}_{ij}^{(k)})_{m \times n} = ([a_{ij}^{(k)}, b_{ij}^{(k)}, c_{ij}^{(k)}, d_{ij}^{(k)}]; \mu_{ij}^{(k)}, \nu_{ij}^{(k)})$ into the standardized intuitionistic trapezoidal fuzzy decision matrix $\tilde{R}^{(k)} = (\tilde{r}_{ij}^{(k)})_{m \times n} = ([e_{ij}^{(k)}, f_{ij}^{(k)}, g_{ij}^{(k)}, h_{ij}^{(k)}]; \mu_{ij}^{(k)}, \nu_{ij}^{(k)})$ by the following method given in [11,12,23]:

For cost type of criteria:

$$\begin{aligned} e_{ij}^{(k)} &= \frac{\max_{1 \leq j \leq n} \{d_{ij}^{(k)}\} - d_{ij}^{(k)}}{\max_{1 \leq j \leq n} \{d_{ij}^{(k)}\} - \min_{1 \leq j \leq n} \{d_{ij}^{(k)}\}}, & f_{ij}^{(k)} &= \frac{\max_{1 \leq j \leq n} \{d_{ij}^{(k)}\} - b_{ij}^{(k)}}{\max_{1 \leq j \leq n} \{d_{ij}^{(k)}\} - \min_{1 \leq j \leq n} \{d_{ij}^{(k)}\}}, \\ g_{ij}^{(k)} &= \frac{\max_{1 \leq j \leq n} \{d_{ij}^{(k)}\} - c_{ij}^{(k)}}{\max_{1 \leq j \leq n} \{d_{ij}^{(k)}\} - \min_{1 \leq j \leq n} \{d_{ij}^{(k)}\}}, & h_{ij}^{(k)} &= \frac{\max_{1 \leq j \leq n} \{d_{ij}^{(k)}\} - d_{ij}^{(k)}}{\max_{1 \leq j \leq n} \{d_{ij}^{(k)}\} - \min_{1 \leq j \leq n} \{d_{ij}^{(k)}\}}. \end{aligned} \quad (23)$$

For benefit type of criteria:

$$\begin{aligned} e_{ij}^{(k)} &= \frac{a_{ij}^{(k)} - \min_{1 \leq j \leq n} \{a_{ij}^{(k)}\}}{\max_{1 \leq j \leq n} \{d_{ij}^{(k)}\} - \min_{1 \leq j \leq n} \{a_{ij}^{(k)}\}}, & f_{ij}^{(k)} &= \frac{b_{ij}^{(k)} - \min_{1 \leq j \leq n} \{a_{ij}^{(k)}\}}{\max_{1 \leq j \leq n} \{d_{ij}^{(k)}\} - \min_{1 \leq j \leq n} \{a_{ij}^{(k)}\}}, \\ g_{ij}^{(k)} &= \frac{c_{ij}^{(k)} - \min_{1 \leq j \leq n} \{a_{ij}^{(k)}\}}{\max_{1 \leq j \leq n} \{d_{ij}^{(k)}\} - \min_{1 \leq j \leq n} \{a_{ij}^{(k)}\}}, & h_{ij}^{(k)} &= \frac{d_{ij}^{(k)} - \min_{1 \leq j \leq n} \{a_{ij}^{(k)}\}}{\max_{1 \leq j \leq n} \{d_{ij}^{(k)}\} - \min_{1 \leq j \leq n} \{a_{ij}^{(k)}\}}. \end{aligned} \quad (24)$$

In the following, we utilize the ITPFWA (or ITPFWG) operator to develop an approach to multi attribute group decision making under intuitionistic trapezoidal fuzzy environment. The main steps can be summarized as follows:

Step 1. Transform the intuitionistic trapezoidal fuzzy decision matrices

$\tilde{A}^{(k)} = (\tilde{a}_{ij}^{(k)})_{m \times n} = ([a_{ij}^{(k)}, b_{ij}^{(k)}, c_{ij}^{(k)}, d_{ij}^{(k)}]; \mu_{ij}^{(k)}, \nu_{ij}^{(k)})$ ($k = 1, 2, \dots, l$) into the standardized

intuitionistic trapezoidal fuzzy decision matrices

$\tilde{R}^{(k)} = (\tilde{r}_{ij}^{(k)})_{m \times n} = \left(\left([e_{ij}^{(k)}, f_{ij}^{(k)}, g_{ij}^{(k)}, h_{ij}^{(k)}]; \mu_{ij}^{(k)}, \nu_{ij}^{(k)} \right) \right)_{m \times n} \quad (k = 1, 2, \dots, l)$ using Eqs. (23) and (24).

Step 2: Calculate the matrices $T^{(p)} = (T_{ij}^{(p)})_{m \times n}$, $(p = 1, 2, \dots, l)$ based on the following equations:

$$T_{ij}^{(p)} = \prod_{k=1}^{p-1} S(\tilde{r}_{ij}^{(k)}), \quad p = 2, \dots, l, \quad i = 1, 2, \dots, m, \quad j = 1, 2, \dots, n, \quad (25)$$

$$T_{ij}^{(1)} = 1, \quad i = 1, 2, \dots, m, \quad j = 1, 2, \dots, n. \quad (26)$$

Step 3: Utilize the ITFPWA operator:

$$\begin{aligned} \tilde{r}_{ij} &= \text{ITFPWA}(\tilde{r}_{ij}^{(1)}, \tilde{r}_{ij}^{(2)}, \dots, \tilde{r}_{ij}^{(l)}) \\ &= \frac{T_{ij}^{(1)}}{\sum_{p=1}^l T_{ij}^{(p)}} \tilde{r}_{ij}^{(1)} \oplus \frac{T_{ij}^{(2)}}{\sum_{p=1}^l T_{ij}^{(p)}} \tilde{r}_{ij}^{(2)} \oplus \dots \oplus \frac{T_{ij}^{(l)}}{\sum_{p=1}^l T_{ij}^{(p)}} \tilde{r}_{ij}^{(l)} \\ &= \left(\left[\frac{\sum_{k=1}^l \left(\frac{T_{ij}^{(k)} e_{ij}^{(k)}}{\sum_{p=1}^l T_{ij}^{(p)}} \right)}{\sum_{p=1}^l T_{ij}^{(p)}}, \frac{\sum_{k=1}^l \left(\frac{T_{ij}^{(k)} f_{ij}^{(k)}}{\sum_{p=1}^l T_{ij}^{(p)}} \right)}{\sum_{p=1}^l T_{ij}^{(p)}}, \frac{\sum_{k=1}^l \left(\frac{T_{ij}^{(k)} g_{ij}^{(k)}}{\sum_{p=1}^l T_{ij}^{(p)}} \right)}{\sum_{p=1}^l T_{ij}^{(p)}}, \frac{\sum_{k=1}^l \left(\frac{T_{ij}^{(k)} h_{ij}^{(k)}}{\sum_{p=1}^l T_{ij}^{(p)}} \right)}{\sum_{p=1}^l T_{ij}^{(p)}} \right]; 1 - \prod_{k=1}^l (1 - \mu_{ij}^{(k)})^{\left(\frac{T_{ij}^{(k)}}{\sum_{p=1}^l T_{ij}^{(p)}} \right)}, \prod_{k=1}^l (\nu_{ij}^{(k)})^{\left(\frac{T_{ij}^{(k)}}{\sum_{p=1}^l T_{ij}^{(p)}} \right)} \right) \end{aligned} \quad (27)$$

or the ITFPWG operator:

$$\begin{aligned} \tilde{r}_{ij} &= \text{ITFPWG}(\tilde{r}_{ij}^{(1)}, \tilde{r}_{ij}^{(2)}, \dots, \tilde{r}_{ij}^{(l)}) \\ &= (\tilde{r}_{ij}^{(1)})^{\left(\frac{T_{ij}^{(1)}}{\sum_{p=1}^l T_{ij}^{(p)}} \right)} \otimes (\tilde{r}_{ij}^{(2)})^{\left(\frac{T_{ij}^{(2)}}{\sum_{p=1}^l T_{ij}^{(p)}} \right)} \otimes \dots \otimes (\tilde{r}_{ij}^{(l)})^{\left(\frac{T_{ij}^{(l)}}{\sum_{p=1}^l T_{ij}^{(p)}} \right)} \\ &= \left(\left[\prod_{k=1}^l (e_{ij}^{(k)})^{\left(\frac{T_{ij}^{(k)}}{\sum_{p=1}^l T_{ij}^{(p)}} \right)}, \prod_{k=1}^l (f_{ij}^{(k)})^{\left(\frac{T_{ij}^{(k)}}{\sum_{p=1}^l T_{ij}^{(p)}} \right)}, \prod_{k=1}^l (g_{ij}^{(k)})^{\left(\frac{T_{ij}^{(k)}}{\sum_{p=1}^l T_{ij}^{(p)}} \right)}, \prod_{k=1}^l (h_{ij}^{(k)})^{\left(\frac{T_{ij}^{(k)}}{\sum_{p=1}^l T_{ij}^{(p)}} \right)} \right]; \prod_{k=1}^l (\mu_{ij}^{(k)})^{\left(\frac{T_{ij}^{(k)}}{\sum_{p=1}^l T_{ij}^{(p)}} \right)}, 1 - \prod_{k=1}^l (1 - \nu_{ij}^{(k)})^{\left(\frac{T_{ij}^{(k)}}{\sum_{p=1}^l T_{ij}^{(p)}} \right)} \right) \end{aligned} \quad (28)$$

to aggregate all the individual intuitionistic trapezoidal fuzzy decision matrices $\tilde{R}^{(k)} = (\tilde{r}_{ij}^{(k)})_{m \times n}$

$(k = 1, 2, \dots, l)$ into the collective intuitionistic trapezoidal fuzzy decision matrix

$$\tilde{R} = (\tilde{r}_{ij})_{m \times n} = \left(\left([e_{ij}, f_{ij}, g_{ij}, h_{ij}]; \mu_{ij}, \nu_{ij} \right) \right)_{m \times n}.$$

Step 4: Calculate the matrix $T = (T_{ij})_{m \times n}$ based on following equations:

$$T_{ij} = \prod_{k=1}^{j-1} S(\tilde{r}_{ij}), \quad (i = 1, 2, \dots, m, \quad j = 2, \dots, n), \quad (29)$$

$$T_{i1} = 1, \quad (i = 1, 2, \dots, m). \quad (30)$$

Step 5: Utilize the ITFPWA operator:

$$\begin{aligned}
\tilde{r}_i &= \text{ITFPWA}(\tilde{r}_{i1}, \tilde{r}_{i2}, \dots, \tilde{r}_{in}) \\
&= \frac{T_{i1}}{\sum_{j=1}^n T_{ij}} \tilde{r}_{i1} \oplus \frac{T_{i2}}{\sum_{j=1}^n T_{ij}} \tilde{r}_{i2} \oplus \dots \oplus \frac{T_{in}}{\sum_{j=1}^n T_{ij}} \tilde{r}_{in} \\
&= \left[\left[\sum_{j=1}^n \left(\frac{T_{ij} e_{ij}}{\sum_{j=1}^n T_{ij}} \right), \sum_{j=1}^n \left(\frac{T_{ij} f_{ij}}{\sum_{j=1}^n T_{ij}} \right), \sum_{j=1}^n \left(\frac{T_{ij} g_{ij}}{\sum_{j=1}^n T_{ij}} \right), \sum_{j=1}^n \left(\frac{T_{ij} h_{ij}}{\sum_{j=1}^n T_{ij}} \right) \right]; 1 - \prod_{j=1}^n (1 - \mu_{ij})^{\left(\frac{T_{ij}}{\sum_{j=1}^n T_{ij}} \right)}, \prod_{j=1}^n (\nu_{ij})^{\left(\frac{T_{ij}}{\sum_{j=1}^n T_{ij}} \right)} \right]
\end{aligned} \tag{31}$$

or the ITFPWG operator:

$$\begin{aligned}
\tilde{r}_i &= \text{ITFPWG}(\tilde{r}_{i1}, \tilde{r}_{i2}, \dots, \tilde{r}_{in}) \\
&= (\tilde{r}_{i1})^{\left(\frac{T_{i1}}{\sum_{j=1}^n T_{ij}} \right)} \otimes (\tilde{r}_{i2})^{\left(\frac{T_{i2}}{\sum_{j=1}^n T_{ij}} \right)} \otimes \dots \otimes (\tilde{r}_{in})^{\left(\frac{T_{in}}{\sum_{j=1}^n T_{ij}} \right)} \\
&= \left[\left[\prod_{j=1}^n (e_{ij})^{\left(\frac{T_{ij}}{\sum_{j=1}^n T_{ij}} \right)}, \prod_{j=1}^n (f_{ij})^{\left(\frac{T_{ij}}{\sum_{j=1}^n T_{ij}} \right)}, \prod_{j=1}^n (g_{ij})^{\left(\frac{T_{ij}}{\sum_{j=1}^n T_{ij}} \right)}, \prod_{j=1}^n (h_{ij})^{\left(\frac{T_{ij}}{\sum_{j=1}^n T_{ij}} \right)} \right]; \right. \\
&\quad \left. \prod_{j=1}^n (\mu_{ij})^{\left(\frac{T_{ij}}{\sum_{j=1}^n T_{ij}} \right)}, 1 - \prod_{j=1}^n (1 - \nu_{ij})^{\left(\frac{T_{ij}}{\sum_{j=1}^n T_{ij}} \right)} \right]
\end{aligned} \tag{32}$$

to derive the collective overall preference intuitionistic trapezoidal fuzzy values $\tilde{r}_i = ([e_i, f_i, g_i, h_i]; \mu_i, \nu_i)$ ($i = 1, 2, \dots, m$) of the alternatives x_i ($i = 1, 2, \dots, m$).

Step 6: Calculate the score functions and the accuracy functions of $\tilde{r}_i$ ($i = 1, 2, \dots, m$) as follows:

$$S(\tilde{r}_i) = \frac{1}{8} \times (e_i + f_i + g_i + h_i) \times (1 + \mu_i - \nu_i) \times (\mu_i - \nu_i) \tag{33}$$

$$H(\tilde{r}_i) = \frac{1}{8} \times (e_i + f_i + g_i + h_i) \times (1 + \mu_i - \nu_i) \times (\mu_i + \nu_i). \tag{34}$$

Step 7: Rank all the alternatives x_i ($i = 1, 2, \dots, m$) and then select the best alternative(s) by Definition 2.7.

Step 8: End.

3.4. An illustrative example

In this subsection, we use a practical example (adapted from [20]) to illustrate the application of our method.

Example 3.1 [20]. In order to strengthen academic education, promote the building of teaching body, the school of management in a Chinese university wants to introduce overseas outstanding teachers. This introduction has been raised great attention from the school, university president d_1 , dean of management school d_2 , and human resource officer d_3 sets up the panel of decision makers which will take the whole responsibility for this introduction. They made strict evaluation for 5 candidates x_i ($i = 1, 2, 3, 4, 5$) from the following four aspects: (1) morality c_1 ; (2) research capability c_2 ; (3) teaching skill c_3 ; and (4) education background c_4 . University president has the absolute priority for decision making, dean of the management school comes next. That is, there is a prioritization between three decision makers expressed by the linear ordering $d_1 \succ d_2 \succ d_3$. In addition, this introduction will be in strict accordance with the principle of combine ability with political integrity. In three

508 decision makers' opinion, there exists the prioritization relationship among these attributes,
 509 for example, the morality of the candidate is the most important, but the education
 510 background of the candidate is not so important comparing with other attributes. Therefore,
 511 the prioritization relationship can be denoted by: $c_1 \succ c_2 \succ c_3 \succ c_4$. Suppose that five
 512 candidates x_i ($i=1,2,3,4,5$) are to be evaluated by three decision makers d_k ($k=1,2,3$)
 513 under the above four attributes c_j ($j=1,2,3,4$), and three intuitionistic trapezoidal fuzzy
 514 decision matrices $\tilde{A}^{(k)} = (\tilde{\alpha}_{ij}^{(k)})_{m \times n}$ ($k=1,2,3$) are listed in Tables 1-3, respectively.

515 **Table 1.** The intuitionistic trapezoidal fuzzy decision matrix $\tilde{A}^{(1)}$ provided by d_1 .

1	c_1	c_2	c_3	c_4
x_1	([2, 3, 4, 6]; 0.6, 0.3)	([4, 6, 8, 9]; 0.6, 0.3)	([2, 3, 4, 6]; 0.6, 0.4)	([3, 5, 6, 9]; 0.7, 0.1)
x_2	([1, 2, 3, 6]; 0.6, 0.4)	([6, 7, 8, 9]; 0.8, 0.2)	([3, 5, 6, 8]; 0.8, 0.1)	([4, 6, 8, 9]; 0.7, 0.3)
x_3	([2, 3, 4, 5]; 0.5, 0.3)	([5, 6, 7, 9]; 0.7, 0.3)	([2, 3, 5, 6]; 0.6, 0.4)	([3, 5, 6, 8]; 0.7, 0.1)
x_4	([1, 2, 3, 5]; 0.6, 0.4)	([6, 7, 8, 9]; 0.8, 0.1)	([3, 4, 6, 7]; 0.8, 0.1)	([4, 6, 7, 9]; 0.6, 0.3)
x_5	([2, 3, 4, 5]; 0.6, 0.3)	([5, 6, 7, 8]; 0.6, 0.3)	([2, 3, 5, 9]; 0.6, 0.3)	([3, 4, 6, 8]; 0.7, 0.2)

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517 **Table 2.** The intuitionistic trapezoidal fuzzy decision matrix $\tilde{A}^{(2)}$ provided by d_2 .

2	c_1	c_2	c_3	c_4
x_1	([5, 6, 7, 9]; 0.8, 0.1)	([2, 3, 4, 7]; 0.7, 0.3)	([3, 4, 6, 8]; 0.5, 0.4)	([5, 6, 7, 9]; 0.7, 0.3)
x_2	([1, 4, 5, 7]; 0.5, 0.4)	([2, 4, 5, 6]; 0.5, 0.4)	([2, 3, 4, 6]; 0.7, 0.3)	([2, 6, 7, 8]; 0.8, 0.1)
x_3	([5, 6, 8, 9]; 0.8, 0.1)	([2, 3, 4, 5]; 0.6, 0.3)	([3, 4, 5, 8]; 0.5, 0.4)	([5, 6, 7, 8]; 0.7, 0.2)
x_4	([1, 4, 5, 8]; 0.5, 0.4)	([2, 4, 5, 7]; 0.5, 0.4)	([2, 3, 4, 5]; 0.6, 0.3)	([2, 5, 7, 8]; 0.8, 0.1)
x_5	([5, 6, 7, 8]; 0.8, 0.2)	([2, 3, 4, 6]; 0.6, 0.3)	([3, 4, 6, 9]; 0.5, 0.3)	([5, 6, 7, 9]; 0.7, 0.1)

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Table 3. The intuitionistic trapezoidal fuzzy decision matrix $\tilde{A}^{(3)}$ provided by d_3 .

3	c_1	c_2	c_3	c_4
x_1	([2, 3, 5, 7]; 0.5, 0.4)	([4, 5, 7, 8]; 0.5, 0.4)	([2, 3, 6, 8]; 0.6, 0.3)	([4, 5, 6, 8]; 0.8, 0.1)
x_2	([4, 6, 7, 9]; 0.6, 0.3)	([2, 3, 5, 6]; 0.7, 0.2)	([2, 3, 5, 6]; 0.7, 0.3)	([6, 7, 8, 9]; 0.4, 0.6)
x_3	([2, 3, 5, 6]; 0.5, 0.4)	([4, 6, 7, 8]; 0.5, 0.4)	([2, 3, 6, 8]; 0.6, 0.4)	([4, 5, 6, 7]; 0.8, 0.1)
x_4	([4, 6, 7, 8]; 0.6, 0.4)	([2, 3, 4, 5]; 0.8, 0.2)	([2, 3, 4, 6]; 0.6, 0.3)	([6, 7, 8, 9]; 0.3, 0.5)
x_5	([2, 4, 5, 6]; 0.5, 0.3)	([4, 6, 7, 8]; 0.6, 0.4)	([2, 5, 6, 8]; 0.6, 0.4)	([4, 5, 6, 8]; 0.8, 0.2)

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534 **Step 1.** Considering that all the criteria c_j ($j=1,2,3,4$) are the benefit type criteria, we

535 transform the intuitionistic trapezoidal fuzzy decision matrices $\tilde{A}^{(k)} = (\tilde{a}_{ij}^{(k)})_{5 \times 4}$ ($k=1,2,3$) into

536 the standardized intuitionistic trapezoidal fuzzy matrix $\tilde{R}^{(k)} = (\tilde{r}_{ij}^{(k)})_{5 \times 4}$ ($k=1,2,3$) (see Tables

537 4-6) by using Eq. (24):

538 **Table 4.** The intuitionistic trapezoidal fuzzy decision matrix $\tilde{R}^{(1)}$ provided by d_1 .

4	c_1	c_2	c_3	c_4
x_1	([0, 0.1429, 0.2857, 0.5714]; 0.6, 0.3)	([0.2857, 0.5714, 0.8571, 1.0000]; 0.6, 0.3)	([0, 0.1429, 0.2857, 0.5714]; 0.6, 0.4)	([0.1429, 0.4286, 0.5714, 1.0000]; 0.7, 0.1)
x_2	([0, 0.1250, 0.2500, 0.6250]; 0.6, 0.4)	([0.6250, 0.7500, 0.8750, 1.0000]; 0.8, 0.2)	([0.2500, 0.5000, 0.6250, 0.8750]; 0.8, 0.1)	([0.3750, 0.6250, 0.8750, 1.0000]; 0.7, 0.3)
x_3	([0, 0.1429, 0.2857, 0.4286]; 0.5, 0.3)	([0.4286, 0.5714, 0.7143, 1.0000]; 0.7, 0.3)	([0, 0.1429, 0.4286, 0.5714]; 0.6, 0.4)	([0.1429, 0.4286, 0.5714, 0.8571]; 0.7, 0.1)
x_4	([0, 0.1250, 0.2500, 0.5000]; 0.6, 0.4)	([0.6250, 0.7500, 0.8750, 1.0000]; 0.8, 0.1)	([0.2500, 0.3750, 0.6250, 0.7500]; 0.8, 0.1)	([0.3750, 0.6250, 0.7500, 1.0000]; 0.6, 0.3)
x_5	([0, 0.1429, 0.2857, 0.4286]; 0.6, 0.3)	([0.4286, 0.5714, 0.7143, 0.8571]; 0.6, 0.3)	([0, 0.1429, 0.4286, 1.0000]; 0.6, 0.3)	([0.1429, 0.2857, 0.5714, 0.8571]; 0.7, 0.2)

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Table 5. The intuitionistic trapezoidal fuzzy decision matrix $\tilde{R}^{(2)}$ provided by d_2 .

5	c_1	c_2	c_3	c_4
x_1	([0.4286, 0.5714, 0.7143, 1.0000]; 0.8, 0.1)	([0, 0.1429, 0.2857, 0.7143]; 0.7, 0.3)	([0.1429, 0.2857, 0.5714, 0.8571]; 0.5, 0.4)	([0.4286, 0.5714, 0.7143, 1.0000]; 0.7, 0.3)
x_2	([0, 0.4286, 0.5714, 0.8571]; 0.5, 0.4)	([0.1429, 0.4286, 0.5714, 0.7143]; 0.5, 0.4)	([0.1429, 0.2857, 0.4286, 0.7143]; 0.7, 0.3)	([0.1429, 0.7143, 0.8571, 1.0000]; 0.8, 0.1)
x_3	([0.4286, 0.5714, 0.8571, 1.0000]; 0.8, 0.1)	([0, 0.1429, 0.2857, 0.4286]; 0.6, 0.3)	([0.1429, 0.2857, 0.4286, 0.8571]; 0.5, 0.4)	([0.4286, 0.5714, 0.7143, 0.8571]; 0.7, 0.2)
x_4	([0, 0.4286, 0.5714, 1.0000]; 0.5, 0.4)	([0.1429, 0.4286, 0.5714, 0.8571]; 0.5, 0.4)	([0.1429, 0.2857, 0.4286, 0.5714]; 0.6, 0.3)	([0.1429, 0.5714, 0.8571, 1.0000]; 0.8, 0.1)
x_5	([0.4286, 0.5714, 0.7143, 0.8571]; 0.8, 0.2)	([0, 0.1429, 0.2857, 0.5714]; 0.6, 0.3)	([0.1429, 0.2857, 0.5714, 1.0000]; 0.5, 0.3)	([0.4286, 0.5714, 0.7143, 1.0000]; 0.7, 0.1)

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Table 6. The intuitionistic trapezoidal fuzzy decision matrix $\tilde{R}^{(3)}$ provided by d_3 .

6	c_1	c_2	c_3	c_4
x_1	([0, 0.1667, 0.5000, 0.8333]; 0.5, 0.4)	([0.3333, 0.5000, 0.8333, 1.0000]; 0.5, 0.4)	([0, 0.1667, 0.6667, 1.0000]; 0.6, 0.3)	([0.3333, 0.5000, 0.6667, 1.0000]; 0.8, 0.1)
x_2	([0.2857, 0.5714, 0.7143, 1.0000]; 0.6, 0.3)	([0, 0.1429, 0.4286, 0.5714]; 0.7, 0.2)	([0, 0.1429, 0.4286, 0.5714]; 0.7, 0.3)	([0.5714, 0.7143, 0.8571, 1.0000]; 0.4, 0.6)
x_3	([0, 0.1667, 0.5000, 0.6667]; 0.5, 0.4)	([0.3333, 0.6667, 0.8333, 1.0000]; 0.5, 0.4)	([0, 0.1667, 0.6667, 1.0000]; 0.6, 0.4)	([0.3333, 0.5000, 0.6667, 0.8333]; 0.8, 0.1)
x_4	([0.2857, 0.5714, 0.7143, 0.8571]; 0.6, 0.4)	([0, 0.1429, 0.2857, 0.4286]; 0.8, 0.2)	([0, 0.1429, 0.2857, 0.5714]; 0.6, 0.3)	([0.5714, 0.7143, 0.8571, 1.0000]; 0.3, 0.5)
x_5	([0, 0.3333, 0.5000, 0.6667]; 0.5, 0.3)	([0.3333, 0.6667, 0.8333, 1.0000]; 0.6, 0.4)	([0, 0.5000, 0.6667, 1.0000]; 0.6, 0.4)	([0.3333, 0.5000, 0.6667, 1.0000]; 0.8, 0.2)

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Step 2: We first utilize Eqs. (25) and (26) to calculate the matrices $T^{(1)}$, $T^{(2)}$, and $T^{(3)}$ as follows:

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$$\begin{aligned}
575 \quad T^{(1)} &= \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \quad T^{(2)} = \begin{pmatrix} 0.0488 & 0.1323 & 0.0300 & 0.2571 \\ 0.0300 & 0.3900 & 0.3347 & 0.2012 \\ 0.0257 & 0.1900 & 0.0343 & 0.2400 \\ 0.0263 & 0.4834 & 0.2975 & 0.1341 \\ 0.0418 & 0.1254 & 0.0766 & 0.1741 \end{pmatrix}, \\
576 \quad T^{(3)} &= \begin{pmatrix} 0.0197 & 0.0106 & 0.0008 & 0.0489 \\ 0.0008 & 0.0100 & 0.0368 & 0.0813 \\ 0.0109 & 0.0079 & 0.0008 & 0.0579 \\ 0.0007 & 0.0133 & 0.0207 & 0.0513 \\ 0.0129 & 0.0061 & 0.0046 & 0.0567 \end{pmatrix}.
\end{aligned}$$

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578 **Step 3:** Utilize the ITFPWA operator (Eq. (27)) to aggregate all the individual intuitionistic
579 trapezoidal fuzzy decision matrices $\tilde{R}^{(k)} = (\tilde{r}_{ij}^{(k)})_{5 \times 4}$ ($k = 1, 2, 3$) into the collective intuitionistic

580 trapezoidal fuzzy decision matrix $\tilde{R} = (\tilde{r}_{ij})_{5 \times 4}$ (see Table 7).

581 **Table 7.** The collective intuitionistic trapezoidal fuzzy decision matrix $\tilde{R}$.

7	c_1	c_2	c_3	c_4
x_1	([0.0196, 0.1629, 0.3092, 0.5958]; 0.6109, 0.2868)	([0.2531, 0.5211, 0.7908, 0.9669]; 0.6123, 0.3008)	([0.0042, 0.1470, 0.2943, 0.5801]; 0.5974, 0.3999)	([0.2062, 0.4594, 0.6031, 1.0000]; 0.7045, 0.1241)
x_2	([0.0002, 0.1342, 0.2597, 0.6320]; 0.5974, 0.3999)	([0.4862, 0.6561, 0.7873, 0.9174]; 0.7411, 0.2426)	([0.2171, 0.4381, 0.5718, 0.8276]; 0.7768, 0.1347)	([0.3510, 0.6447, 0.8711, 1.0000]; 0.7059, 0.2638)
x_3	([0.0106, 0.1537, 0.3021, 0.4453]; 0.5112, 0.2928)	([0.3600, 0.5041, 0.6471, 0.9094]; 0.6849, 0.3006)	([0.0047, 0.1476, 0.4288, 0.5812]; 0.5970, 0.4000)	([0.2042, 0.4582, 0.6021, 0.8561]; 0.7054, 0.1137)
x_4	([0.0002, 0.1331, 0.2585, 0.5130]; 0.5977, 0.4000)	([0.4637, 0.6408, 0.7717, 0.9488]; 0.7311, 0.1574)	([0.2219, 0.3512, 0.5753, 0.7069]; 0.7636, 0.1304)	([0.3572, 0.6228, 0.7668, 1.0000]; 0.6211, 0.2709)
x_5	([0.0170, 0.1622, 0.3053, 0.4485]; 0.6098, 0.2952)	([0.3806, 0.5245, 0.6674, 0.8263]; 0.6000, 0.3005)	([0.0101, 0.1545, 0.4397, 1.0000]; 0.5936, 0.3004)	([0.1920, 0.3360, 0.5960, 0.8839]; 0.7056, 0.1813)

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583 **Step 4:** Calculate the matrix $T = (T_{ij})_{5 \times 4}$ based on Eqs. (29) and (30):

$$584 \quad T = \begin{pmatrix} 1 & 0.0583 & 0.0075 & 0.0002 \\ 1 & 0.0303 & 0.0081 & 0.0022 \\ 1 & 0.0303 & 0.0049 & 0.0002 \\ 1 & 0.0268 & 0.0085 & 0.0020 \\ 1 & 0.0482 & 0.0056 & 0.0004 \end{pmatrix}.$$

Step 5: Utilize the ITPFWA operator (Eq. (31)) to aggregate all the preference values $\tilde{r}_{ij}$ ($j=1,2,3,4$) in the i th line of $\tilde{R}$, and derive the collective overall preference value $\tilde{r}_i$ ($i=1,2,3,4,5$) of the alternative x_i ($i=1,2,3,4,5$).

$$\tilde{r}_1 = ([0.0323, 0.1824, 0.3355, 0.6161]; 0.6109, 0.2882),$$

$$\tilde{r}_2 = ([0.0168, 0.1528, 0.2788, 0.6426]; 0.6046, 0.3905),$$

$$\tilde{r}_3 = ([0.0209, 0.1640, 0.3129, 0.4596]; 0.5180, 0.2934),$$

$$\tilde{r}_4 = ([0.0147, 0.1489, 0.2754, 0.5268]; 0.6037, 0.3866),$$

$$\tilde{r}_5 = ([0.0336, 0.1788, 0.3227, 0.4689]; 0.6093, 0.2954).$$

Step 6: Calculate the score functions and the accuracy functions of $\tilde{r}_i$ ($i=1,2,3,4,5$) as follows:

$$S(\tilde{r}_1) = 0.0622, S(\tilde{r}_2) = 0.0355, S(\tilde{r}_3) = 0.0329, S(\tilde{r}_4) = 0.0319, S(\tilde{r}_5) = 0.0518.$$

Step 7: Because $S(\tilde{r}_1) > S(\tilde{r}_5) > S(\tilde{r}_2) > S(\tilde{r}_3) > S(\tilde{r}_4)$, we have $x_1 \succ x_5 \succ x_2 \succ x_3 \succ x_4$. Therefore, the best candidate is x_1 .

If we deal with Example 3.1 using the ITPFWG operator instead of the ITPFWA operator, then the main steps are shown as follows:

Step 1': See Step 1.

Step 2': See Step 2.

Step 3': Utilize the ITPFWG operator (Eq. (28)) to aggregate all the individual intuitionistic trapezoidal fuzzy decision matrices $\tilde{R}^{(k)} = (\tilde{r}_{ij}^{(k)})_{5 \times 4}$ ($k=1,2,3$) into the collective intuitionistic

trapezoidal fuzzy decision matrix $\tilde{R}' = (\tilde{r}'_{ij})_{5 \times 4}$ (see Table 8).

Table 8. The collective intuitionistic trapezoidal fuzzy decision matrix $\tilde{R}'$.

8	c_1	c_2	c_3	c_4
x_1	([0, 0.1526, 0.3010, 0.5903]; 0.6059, 0.2939)	([0, 0.4861, 0.7546, 0.9618]; 0.6098, 0.3010)	([0, 0.1458, 0.2917, 0.5785]; 0.5968, 0.3999)	([0.1831, 0.4562, 0.6005, 1.0000]; 0.7035, 0.1435)
x_2	([0, 0.1297, 0.2563, 0.6310]; 0.5968, 0.3999)	([0, 0.6342, 0.7731, 0.9069]; 0.7012, 0.2616)	([0, 0.4218, 0.5643, 0.8233]; 0.7716, 0.1592)	([0.3310, 0.6437, 0.8710, 1.0000]; 0.6899, 0.2972)
x_3	([0, 0.1481, 0.2953, 0.4397]; 0.5059, 0.2968)	([0, 0.4591, 0.6183, 0.8743]; 0.6816, 0.3007)	([0, 0.1462, 0.4287, 0.5794]; 0.5964, 0.4000)	([0.1818, 0.4551, 0.5996, 0.8561]; 0.7042, 0.1194)
x_4	([0, 0.1291, 0.2555, 0.5091]; 0.5972, 0.4000)	([0, 0.6168, 0.7550, 0.9443]; 0.6873, 0.2113)	([0, 0.3474, 0.5670, 0.7023]; 0.7463, 0.1530)	([0.3424, 0.6223, 0.7658, 1.0000]; 0.6015, 0.2902)
x_5	([0, 0.1525, 0.2983, 0.4429]; 0.6055, 0.2963)	([0, 0.4905, 0.6459, 0.8202]; 0.6000, 0.3006)	([0, 0.1508, 0.4382, 1.0000]; 0.5923, 0.3005)	([0.1735, 0.3234, 0.5940, 0.8823]; 0.7043, 0.1866)

607

608 **Step 4':** Calculate the matrix $T' = (T'_{ij})_{5 \times 4}$ based on Eqs. (29) and (30):

$$T' = \begin{pmatrix} 1 & 0.0534 & 0.0059 & 0.0002 \\ 1 & 0.0300 & 0.0055 & 0.0012 \\ 1 & 0.0279 & 0.0036 & 0.0001 \\ 1 & 0.0264 & 0.0054 & 0.0010 \\ 1 & 0.0452 & 0.0043 & 0.0003 \end{pmatrix}$$

Step 5': Utilize the ITPFWG operator (Eq. (32)) to aggregate all the preference values $\tilde{r}'_{ij}$ ($j=1,2,3,4$) in the i th line of $\tilde{R}'$, and derive the collective overall preference value $\tilde{r}'_i$ ($i=1,2,3,4,5$) of the alternative x_i ($i=1,2,3,4,5$).

$$\tilde{r}'_1 = ([0, 0.1618, 0.3153, 0.6050]; 0.6060, 0.2949),$$

$$\tilde{r}'_2 = ([0, 0.1369, 0.2661, 0.6389]; 0.6005, 0.3951),$$

$$\tilde{r}'_3 = ([0, 0.1527, 0.3017, 0.4484]; 0.5103, 0.2972),$$

$$\tilde{r}'_4 = ([0, 0.1353, 0.2641, 0.5184]; 0.6001, 0.3946),$$

$$\tilde{r}'_5 = ([0, 0.1604, 0.3090, 0.4564]; 0.6053, 0.2965).$$

Step 6': Calculate the score functions and the accuracy functions of $\tilde{r}'_i$ ($i=1,2,3,4,5$) as follows:

$$S(\tilde{r}'_1) = 0.0552, S(\tilde{r}'_2) = 0.0322, S(\tilde{r}'_3) = 0.0292, S(\tilde{r}'_4) = 0.0284, S(\tilde{r}'_5) = 0.0468.$$

Step 7': Because $S(\tilde{r}'_1) > S(\tilde{r}'_5) > S(\tilde{r}'_2) > S(\tilde{r}'_3) > S(\tilde{r}'_4)$, we have $x_1 \succ x_5 \succ x_2 \succ x_3 \succ x_4$. Therefore, the best candidate is x_1 .

In Steps 6 and 6', observe that the score functions obtained with the ITPFWA operator always exceed those obtained with the ITPFWG operator for the same aggregation values, as illustrated in Theorem 3.5.

4. INTERVAL-VALUED INTUITIONISTIC TRAPEZOIDAL FUZZY PRIORITIZED OPERATORS

Wang [9] proposed the concept of interval-valued intuitionistic trapezoidal fuzzy numbers in which membership and non-membership functions take the form of interval numbers. The interval-valued intuitionistic trapezoidal fuzzy number therefore provides a more accurate description of the fuzzy nature of the objective world than the intuitionistic trapezoidal fuzzy number. To use the interval-valued intuitionistic trapezoidal fuzzy number in the area of decision making has more theoretical value and practical significance.

Let I be a closed unit interval, i.e., $I = [0, 1]$. Let $[I] = \{[a, b] : a \leq b, a, b \in I\}$.

4.1. Interval-valued intuitionistic trapezoidal fuzzy numbers

Definition 4.1 [9]. Let $\hat{\alpha}$ be an intuitionistic fuzzy set in R , whose membership function and non-membership function are defined as follows:

$$\mu_{\hat{a}}(x) = \begin{cases} \frac{x-a}{b-a} \mu_{\hat{a}}, & a \leq x < b, \\ \mu_{\hat{a}}, & b \leq x \leq c, \\ \frac{d-x}{d-c} \mu_{\hat{a}}, & c < x \leq d, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\nu_{\hat{a}}(x) = \begin{cases} \frac{(b-x) + \nu_{\hat{a}}(x-a_1)}{b-a_1}, & a_1 \leq x < b, \\ \nu_{\hat{a}}, & b \leq x \leq c, \\ \frac{(x-c) + \nu_{\hat{a}}(d_1-x)}{d_1-c}, & c < x \leq d_1, \\ 0, & \text{otherwise,} \end{cases}$$

respectively, where $\mu_{\hat{a}} = [\mu_{\hat{a}}^-, \mu_{\hat{a}}^+] \in [I]$, $\nu_{\hat{a}} = [\nu_{\hat{a}}^-, \nu_{\hat{a}}^+] \in [I]$, $0 \leq \mu_{\hat{a}}^+ + \nu_{\hat{a}}^+ \leq 1$, and $a, a_1, b, c, d, d_1 \in R$. Then, $\hat{a}$ is called an interval-valued intuitionistic trapezoidal fuzzy number. For convenience, $\hat{a}$ is denoted by $\hat{a} = \langle ([a, b, c, d]; [\mu_{\hat{a}}^-, \mu_{\hat{a}}^+]), ([a_1, b, c, d_1]; [\nu_{\hat{a}}^-, \nu_{\hat{a}}^+]) \rangle$.

The values $\mu_{\hat{a}} = [\mu_{\hat{a}}^-, \mu_{\hat{a}}^+]$ and $\nu_{\hat{a}} = [\nu_{\hat{a}}^-, \nu_{\hat{a}}^+]$ represent the maximum degree of membership and the minimum degree of non-membership, respectively.

If $[a, b, c, d] = [a_1, b, c, d_1]$, then the interval-valued intuitionistic trapezoidal fuzzy number $\hat{a}$ can be simply denoted by $\hat{a} = ([a, b, c, d]; [\mu_{\hat{a}}^-, \mu_{\hat{a}}^+], [\nu_{\hat{a}}^-, \nu_{\hat{a}}^+])$. If $b = c$, then an interval-valued intuitionistic trapezoidal fuzzy number reduces to an interval-valued intuitionistic triangle fuzzy number. If $\mu_{\hat{a}}^- = \mu_{\hat{a}}^+$ and $\nu_{\hat{a}}^- = \nu_{\hat{a}}^+$, then an interval-valued intuitionistic trapezoidal fuzzy number reduces to an intuitionistic trapezoidal fuzzy number.

Let $\hat{a} = ([a, b, c, d]; [\mu_{\hat{a}}^-, \mu_{\hat{a}}^+], [\nu_{\hat{a}}^-, \nu_{\hat{a}}^+])$ be an interval-valued intuitionistic trapezoidal fuzzy number. If $a, b, c, d \in [0, 1]$, then $\hat{a}$ is called a standardized interval-valued intuitionistic trapezoidal fuzzy number.

Definition 4.2 [6]. Let $\hat{a}_1 = ([a_1, b_1, c_1, d_1]; [\mu_{\hat{a}_1}^-, \mu_{\hat{a}_1}^+], [\nu_{\hat{a}_1}^-, \nu_{\hat{a}_1}^+])$, $\hat{a}_2 = ([a_2, b_2, c_2, d_2]; [\mu_{\hat{a}_2}^-, \mu_{\hat{a}_2}^+], [\nu_{\hat{a}_2}^-, \nu_{\hat{a}_2}^+])$ be two interval-valued intuitionistic trapezoidal fuzzy numbers, and $\lambda \geq 0$. Then, the operation laws for interval-valued intuitionistic trapezoidal fuzzy numbers are defined as follows:

- (1) $\hat{a}_1 \oplus \hat{a}_2 = ([a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2]; [\mu_{\hat{a}_1}^- + \mu_{\hat{a}_2}^- - \mu_{\hat{a}_1}^- \mu_{\hat{a}_2}^-, \mu_{\hat{a}_1}^+ + \mu_{\hat{a}_2}^+ - \mu_{\hat{a}_1}^+ \mu_{\hat{a}_2}^+], [\nu_{\hat{a}_1}^- \nu_{\hat{a}_2}^-, \nu_{\hat{a}_1}^+ \nu_{\hat{a}_2}^+])$;
- (2) $\hat{a}_1 \otimes \hat{a}_2 = ([a_1 a_2, b_1 b_2, c_1 c_2, d_1 d_2]; [\mu_{\hat{a}_1}^- \mu_{\hat{a}_2}^-, \mu_{\hat{a}_1}^+ \mu_{\hat{a}_2}^+], [\nu_{\hat{a}_1}^- + \nu_{\hat{a}_2}^- - \nu_{\hat{a}_1}^- \nu_{\hat{a}_2}^-, \nu_{\hat{a}_1}^+ + \nu_{\hat{a}_2}^+ - \nu_{\hat{a}_1}^+ \nu_{\hat{a}_2}^+])$, where $a_1, a_2 \geq 0$;
- (3) $\lambda \hat{a}_1 = ([\lambda a_1, \lambda b_1, \lambda c_1, \lambda d_1]; [1 - (1 - \mu_{\hat{a}_1}^-)^\lambda, 1 - (1 - \mu_{\hat{a}_1}^+)^\lambda], [(v_{\hat{a}_1}^-)^\lambda, (v_{\hat{a}_1}^+)^\lambda])$, where $a_1 \geq 0$;
- (4) $\hat{a}_1^\lambda = ([a_1^\lambda, b_1^\lambda, c_1^\lambda, d_1^\lambda]; [(\mu_{\hat{a}_1}^-)^\lambda, (\mu_{\hat{a}_1}^+)^\lambda], [1 - (1 - \nu_{\hat{a}_1}^-)^\lambda, 1 - (1 - \nu_{\hat{a}_1}^+)^\lambda])$, where $a_1 \geq 0$.

Theorem 4.1. Let $\hat{\alpha}_1 = ([a_1, b_1, c_1, d_1]; [\mu_{\hat{\alpha}_1}^-, \mu_{\hat{\alpha}_1}^+], [v_{\hat{\alpha}_1}^-, v_{\hat{\alpha}_1}^+])$,
 $\hat{\alpha}_2 = ([a_2, b_2, c_2, d_2]; [\mu_{\hat{\alpha}_2}^-, \mu_{\hat{\alpha}_2}^+], [v_{\hat{\alpha}_2}^-, v_{\hat{\alpha}_2}^+])$ be two interval-valued intuitionistic trapezoidal fuzzy
 numbers, and $\lambda \geq 0$. Then, $\hat{\alpha}_1 \oplus \hat{\alpha}_2$, $\hat{\alpha}_1 \otimes \hat{\alpha}_2$, $\lambda \hat{\alpha}_1$, and $\hat{\alpha}_1^\lambda$ are also interval-valued
 intuitionistic trapezoidal fuzzy numbers.

Theorem 4.2. Let $\hat{\alpha}_1 = ([a_1, b_1, c_1, d_1]; [\mu_{\hat{\alpha}_1}^-, \mu_{\hat{\alpha}_1}^+], [v_{\hat{\alpha}_1}^-, v_{\hat{\alpha}_1}^+])$,
 $\hat{\alpha}_2 = ([a_2, b_2, c_2, d_2]; [\mu_{\hat{\alpha}_2}^-, \mu_{\hat{\alpha}_2}^+], [v_{\hat{\alpha}_2}^-, v_{\hat{\alpha}_2}^+])$ be two interval-valued intuitionistic trapezoidal fuzzy
 numbers, and $\lambda, \lambda_1, \lambda_2 > 0$. Then, we have:

- (1) $\hat{\alpha}_1 \oplus \hat{\alpha}_2 = \hat{\alpha}_2 \oplus \hat{\alpha}_1$;
- (2) $\hat{\alpha}_1 \otimes \hat{\alpha}_2 = \hat{\alpha}_2 \otimes \hat{\alpha}_1$;
- (3) $\lambda(\hat{\alpha}_1 \oplus \hat{\alpha}_2) = \lambda \hat{\alpha}_1 \oplus \lambda \hat{\alpha}_2$;
- (4) $\lambda_1 \hat{\alpha}_1 \oplus \lambda_2 \hat{\alpha}_1 = (\lambda_1 + \lambda_2) \hat{\alpha}_1$;
- (5) $\hat{\alpha}_1^{\lambda_1} \otimes \hat{\alpha}_1^{\lambda_2} = \hat{\alpha}_1^{\lambda_1 + \lambda_2}$;
- (6) $\hat{\alpha}_1^{\lambda_1} \otimes \hat{\alpha}_2^{\lambda_2} = (\hat{\alpha}_1 \otimes \hat{\alpha}_2)^{\lambda_1}$.

Motivated by Definitions 2.4, 2.5, and 2.6, we introduce the expected value, the score
 function, the accuracy function of the interval-valued intuitionistic trapezoidal fuzzy number
 as follows:

Definition 4.3. For the interval-valued intuitionistic trapezoidal fuzzy number
 $\hat{\alpha} = ([a, b, c, d]; [\mu_{\hat{\alpha}}^-, \mu_{\hat{\alpha}}^+], [v_{\hat{\alpha}}^-, v_{\hat{\alpha}}^+])$, its expected value is

$$I(\hat{\alpha}) = \frac{1}{8} \times \left[(a+b+c+d) \times \left(1 + \frac{\mu_{\hat{\alpha}}^- - v_{\hat{\alpha}}^- + \mu_{\hat{\alpha}}^+ - v_{\hat{\alpha}}^+}{2} \right) \right].$$

Definition 4.4. Let $\hat{\alpha} = ([a, b, c, d]; [\mu_{\hat{\alpha}}^-, \mu_{\hat{\alpha}}^+], [v_{\hat{\alpha}}^-, v_{\hat{\alpha}}^+])$ be an interval-valued intuitionistic
 trapezoidal fuzzy number; then, $S(\hat{\alpha}) = I(\hat{\alpha}) \times \left(\frac{\mu_{\hat{\alpha}}^- - v_{\hat{\alpha}}^- + \mu_{\hat{\alpha}}^+ - v_{\hat{\alpha}}^+}{2} \right)$ is called the score function
 of $\hat{\alpha}$, where $I(\hat{\alpha})$ is the expected value of interval-valued intuitionistic trapezoidal fuzzy
 number $\hat{\alpha}$.

Definition 4.5. Let $\hat{\alpha} = ([a, b, c, d]; [\mu_{\hat{\alpha}}^-, \mu_{\hat{\alpha}}^+], [v_{\hat{\alpha}}^-, v_{\hat{\alpha}}^+])$ be an interval-valued intuitionistic
 trapezoidal fuzzy number; then, $H(\hat{\alpha}) = I(\hat{\alpha}) \times \left(\frac{\mu_{\hat{\alpha}}^- + v_{\hat{\alpha}}^- + \mu_{\hat{\alpha}}^+ + v_{\hat{\alpha}}^+}{2} \right)$ is called the accuracy
 function of $\hat{\alpha}$, where $I(\hat{\alpha})$ is the expected value of interval-valued intuitionistic trapezoidal
 fuzzy number $\hat{\alpha}$.

Based on the score function and the accuracy function, we give an order relation between
 any two interval-valued intuitionistic trapezoidal fuzzy numbers:

Definition 4.6. If $\hat{\alpha}_1$ and $\hat{\alpha}_2$ are two random interval-valued intuitionistic trapezoidal fuzzy
 numbers, then,

- (1) If $S(\hat{\alpha}_1) > S(\hat{\alpha}_2)$, then, $\hat{\alpha}_1 > \hat{\alpha}_2$;
- (2) If $S(\hat{\alpha}_1) = S(\hat{\alpha}_2)$, and
 if $H(\hat{\alpha}_1) = H(\hat{\alpha}_2)$, then, $\hat{\alpha}_1 = \hat{\alpha}_2$;
 if $H(\hat{\alpha}_1) > H(\hat{\alpha}_2)$, then, $\hat{\alpha}_1 > \hat{\alpha}_2$.

4.2. Interval-valued intuitionistic trapezoidal fuzzy prioritized weighted average (IVITFPWA) operators

In this subsection, we extend the ITPFWA operator to interval-valued intuitionistic trapezoidal fuzzy environment and develop the interval-valued intuitionistic trapezoidal fuzzy prioritized weighted average (IVITFPWA) operator.

Definition 4.7. Let $\hat{\alpha}_i = ([a_i, b_i, c_i, d_i]; [\mu_{\hat{\alpha}_i}^-, \mu_{\hat{\alpha}_i}^+], [v_{\hat{\alpha}_i}^-, v_{\hat{\alpha}_i}^+])$ ($i = 1, 2, \dots, n$) be a collection of interval-valued intuitionistic trapezoidal fuzzy numbers, if

$$\text{IVITFPWA}(\hat{\alpha}_1, \hat{\alpha}_2, \dots, \hat{\alpha}_n) = \frac{T_1}{\sum_{j=1}^n T_j} \hat{\alpha}_1 \oplus \frac{T_2}{\sum_{j=1}^n T_j} \hat{\alpha}_2 \oplus \dots \oplus \frac{T_n}{\sum_{j=1}^n T_j} \hat{\alpha}_n, \quad (35)$$

where $T_1 = 1$ and $T_j = \prod_{k=1}^{j-1} S(\hat{\alpha}_k)$ ($j = 2, \dots, n$), then IVITFPWA is called an interval-valued intuitionistic trapezoidal fuzzy prioritized weighted average (IVITFPWA) operator.

Based on operations of the interval-valued intuitionistic trapezoidal fuzzy numbers described in Definition 4.2, we can drive the following result:

Theorem 4.3. Let $\hat{\alpha}_i = ([a_i, b_i, c_i, d_i]; [\mu_{\hat{\alpha}_i}^-, \mu_{\hat{\alpha}_i}^+], [v_{\hat{\alpha}_i}^-, v_{\hat{\alpha}_i}^+])$ ($i = 1, 2, \dots, n$) be a collection of interval-valued intuitionistic trapezoidal fuzzy numbers, then their aggregated value by using the IVITFPWA operator is also an interval-valued intuitionistic trapezoidal fuzzy number, and

$$\text{IVITFPWA}(\hat{\alpha}_1, \hat{\alpha}_2, \dots, \hat{\alpha}_n) = \left[\left[\sum_{j=1}^n \left(\frac{T_j a_j}{\sum_{j=1}^n T_j} \right), \sum_{j=1}^n \left(\frac{T_j b_j}{\sum_{j=1}^n T_j} \right), \sum_{j=1}^n \left(\frac{T_j c_j}{\sum_{j=1}^n T_j} \right), \sum_{j=1}^n \left(\frac{T_j d_j}{\sum_{j=1}^n T_j} \right) \right]; \left[1 - \prod_{j=1}^n (1 - \mu_{\hat{\alpha}_j}^-)^{\left(\frac{T_j}{\sum_{j=1}^n T_j} \right)}, 1 - \prod_{j=1}^n (1 - \mu_{\hat{\alpha}_j}^+)^{\left(\frac{T_j}{\sum_{j=1}^n T_j} \right)}, \left[\prod_{j=1}^n (v_{\hat{\alpha}_j}^-)^{\left(\frac{T_j}{\sum_{j=1}^n T_j} \right)}, \prod_{j=1}^n (v_{\hat{\alpha}_j}^+)^{\left(\frac{T_j}{\sum_{j=1}^n T_j} \right)} \right] \right] \right]. \quad (36)$$

Theorem 4.4. Let $\hat{\alpha}_i = ([a_i, b_i, c_i, d_i]; [\mu_{\hat{\alpha}_i}^-, \mu_{\hat{\alpha}_i}^+], [v_{\hat{\alpha}_i}^-, v_{\hat{\alpha}_i}^+])$ ($i = 1, 2, \dots, n$) be a collection of interval-valued intuitionistic trapezoidal fuzzy numbers. Then, the following properties hold.

(1) Idempotency: If $\hat{\alpha}_i = \hat{\alpha} = ([a, b, c, d]; [\mu_{\hat{\alpha}}^-, \mu_{\hat{\alpha}}^+], [v_{\hat{\alpha}}^-, v_{\hat{\alpha}}^+])$ for all i , then

$$\text{IVITFPWA}(\hat{\alpha}_1, \hat{\alpha}_2, \dots, \hat{\alpha}_n) = \hat{\alpha}. \quad (37)$$

(2) Boundedness:

$$\hat{\alpha}_{\min} \leq \text{IVITFPWA}(\hat{\alpha}_1, \hat{\alpha}_2, \dots, \hat{\alpha}_n) \leq \hat{\alpha}_{\max}, \quad (38)$$

where

$$\hat{\alpha}_{\min} = \left(\left[\min_{1 \leq i \leq n} \{a_i\}, \min_{1 \leq i \leq n} \{b_i\}, \min_{1 \leq i \leq n} \{c_i\}, \min_{1 \leq i \leq n} \{d_i\} \right]; \left[\min_{1 \leq i \leq n} \{\mu_{\hat{\alpha}_i}^-\}, \min_{1 \leq i \leq n} \{\mu_{\hat{\alpha}_i}^+\} \right], \left[\max_{1 \leq i \leq n} \{v_{\hat{\alpha}_i}^-\}, \max_{1 \leq i \leq n} \{v_{\hat{\alpha}_i}^+\} \right] \right) \text{ and}$$

$$\hat{\alpha}_{\max} = \left(\left[\max_{1 \leq i \leq n} \{a_i\}, \max_{1 \leq i \leq n} \{b_i\}, \max_{1 \leq i \leq n} \{c_i\}, \max_{1 \leq i \leq n} \{d_i\} \right]; \left[\max_{1 \leq i \leq n} \{\mu_{\hat{\alpha}_i}^-\}, \max_{1 \leq i \leq n} \{\mu_{\hat{\alpha}_i}^+\} \right], \left[\min_{1 \leq i \leq n} \{v_{\hat{\alpha}_i}^-\}, \min_{1 \leq i \leq n} \{v_{\hat{\alpha}_i}^+\} \right] \right).$$

4.3. Interval-valued intuitionistic trapezoidal fuzzy prioritized weighted geometric (IVITFPWG) operators

In the following, we define an interval-valued intuitionistic trapezoidal fuzzy prioritized weighted geometric (IVITFPWG) operator based on the IVITFPWA operator and the geometric mean.

Definition 4.8. Let $\hat{\alpha}_i = ([a_i, b_i, c_i, d_i]; [\mu_{\hat{\alpha}_i}^-, \mu_{\hat{\alpha}_i}^+], [v_{\hat{\alpha}_i}^-, v_{\hat{\alpha}_i}^+])$ ($i = 1, 2, \dots, n$) be a collection of interval-valued intuitionistic trapezoidal fuzzy numbers, if

$$\text{IVITFPWG}(\hat{\alpha}_1, \hat{\alpha}_2, \dots, \hat{\alpha}_n) = \hat{\alpha}_1^{\frac{T_1}{\sum_{j=1}^n T_j}} \otimes \hat{\alpha}_2^{\frac{T_2}{\sum_{j=1}^n T_j}} \otimes \dots \otimes \hat{\alpha}_n^{\frac{T_n}{\sum_{j=1}^n T_j}}, \quad (39)$$

where $T_1 = 1$ and $T_j = \prod_{k=1}^{j-1} S(\hat{\alpha}_k)$ ($j = 2, \dots, n$), then IVITFPWG is called an interval-valued intuitionistic trapezoidal fuzzy prioritized weighted geometric (IVITFPWG) operators.

Theorem 4.5. Let $\hat{\alpha}_i = ([a_i, b_i, c_i, d_i]; [\mu_{\hat{\alpha}_i}^-, \mu_{\hat{\alpha}_i}^+], [v_{\hat{\alpha}_i}^-, v_{\hat{\alpha}_i}^+])$ ($i = 1, 2, \dots, n$) be a collection of interval-valued intuitionistic trapezoidal fuzzy numbers, then their aggregated value by using the IVITFPWG operator is also an interval-valued intuitionistic trapezoidal fuzzy number, and

$$\text{IVITFPWG}(\hat{\alpha}_1, \hat{\alpha}_2, \dots, \hat{\alpha}_n) = \left[\left[\prod_{j=1}^n a_j^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)}, \prod_{j=1}^n b_j^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)}, \prod_{j=1}^n c_j^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)}, \prod_{j=1}^n d_j^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)} \right]; \left[\prod_{j=1}^n (\mu_{\hat{\alpha}_j}^-)^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)}, \prod_{j=1}^n (\mu_{\hat{\alpha}_j}^+)^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)} \right], \left[1 - \prod_{j=1}^n (1 - v_{\hat{\alpha}_j}^-)^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)}, 1 - \prod_{j=1}^n (1 - v_{\hat{\alpha}_j}^+)^{\left(\frac{T_j}{\sum_{j=1}^n T_j}\right)} \right] \right]. \quad (40)$$

Theorem 4.6. Let $\hat{\alpha}_i = ([a_i, b_i, c_i, d_i]; [\mu_{\hat{\alpha}_i}^-, \mu_{\hat{\alpha}_i}^+], [v_{\hat{\alpha}_i}^-, v_{\hat{\alpha}_i}^+])$ ($i = 1, 2, \dots, n$) be a collection of interval-valued intuitionistic trapezoidal fuzzy numbers. Then, the following properties hold.

(1) Idempotency: If $\hat{\alpha}_i = \hat{\alpha} = ([a, b, c, d]; [\mu_{\hat{\alpha}}^-, \mu_{\hat{\alpha}}^+], [v_{\hat{\alpha}}^-, v_{\hat{\alpha}}^+])$ for all i , then

$$\text{IVITFPWG}(\hat{\alpha}_1, \hat{\alpha}_2, \dots, \hat{\alpha}_n) = \hat{\alpha}. \quad (41)$$

(2) Boundedness:

$$\hat{\alpha}_{\min} \leq \text{IVITFPWG}(\hat{\alpha}_1, \hat{\alpha}_2, \dots, \hat{\alpha}_n) \leq \hat{\alpha}_{\max}, \quad (42)$$

where

$$\hat{\alpha}_{\min} = \left(\left[\min_{1 \leq i \leq n} \{a_i\}, \min_{1 \leq i \leq n} \{b_i\}, \min_{1 \leq i \leq n} \{c_i\}, \min_{1 \leq i \leq n} \{d_i\} \right]; \left[\min_{1 \leq i \leq n} \{\mu_{\hat{\alpha}_i}^-\}, \min_{1 \leq i \leq n} \{\mu_{\hat{\alpha}_i}^+\} \right], \left[\max_{1 \leq i \leq n} \{v_{\hat{\alpha}_i}^-\}, \max_{1 \leq i \leq n} \{v_{\hat{\alpha}_i}^+\} \right] \right) \text{ and}$$

$$\hat{\alpha}_{\max} = \left(\left[\max_{1 \leq i \leq n} \{a_i\}, \max_{1 \leq i \leq n} \{b_i\}, \max_{1 \leq i \leq n} \{c_i\}, \max_{1 \leq i \leq n} \{d_i\} \right]; \left[\max_{1 \leq i \leq n} \{\mu_{\hat{\alpha}_i}^-\}, \max_{1 \leq i \leq n} \{\mu_{\hat{\alpha}_i}^+\} \right], \left[\min_{1 \leq i \leq n} \{v_{\hat{\alpha}_i}^-\}, \min_{1 \leq i \leq n} \{v_{\hat{\alpha}_i}^+\} \right] \right).$$

Theorem 4.7. Let $\hat{\alpha}_i = ([a_i, b_i, c_i, d_i]; [\mu_{\hat{\alpha}_i}^-, \mu_{\hat{\alpha}_i}^+], [v_{\hat{\alpha}_i}^-, v_{\hat{\alpha}_i}^+])$ ($i = 1, 2, \dots, n$) be a collection of interval-valued intuitionistic trapezoidal fuzzy numbers and $\lambda > 0$. Then,

$$\text{IVITFPWG}(\hat{\alpha}_1, \hat{\alpha}_2, \dots, \hat{\alpha}_n) \leq \text{IVITFPWA}(\hat{\alpha}_1, \hat{\alpha}_2, \dots, \hat{\alpha}_n). \quad (43)$$

According to Theorem 4.7, we can show that the values obtained by the IVITFPWG operator are not greater than the values obtained by the IVITFPWA operator for any $\lambda > 0$.

4.4. An approach to multiple attribute group decision making with interval-valued intuitionistic trapezoidal fuzzy information

In this subsection, we utilize the proposed aggregation operators to deal with a multiple attribute group decision making with interval-valued intuitionistic trapezoidal fuzzy information.

For a group decision making problem with interval-valued intuitionistic trapezoidal fuzzy information, let $X = \{x_1, x_2, \dots, x_m\}$ is the set of alternatives. Let $C = \{c_1, c_2, \dots, c_n\}$ be a collection of attributes and that there is a prioritization between the attributes expressed by the linear ordering $c_1 \succ c_2 \succ c_3 \succ \dots \succ c_n$, indicate attribute c_j has a higher priority than c_k if $j < k$. Let $D = \{d_1, d_2, \dots, d_l\}$ is the set of decision makers and that there is a prioritization

between the decision makers expressed by the linear ordering $d_1 \succ d_2 \succ d_3 \succ \dots \succ d_l$, indicate decision maker d_p has a higher priority than d_q if $p < q$. Suppose that each decision maker provides his/her own interval-valued intuitionistic trapezoidal fuzzy decision matrix $\hat{A}^{(k)} = (\hat{a}_{ij}^{(k)})_{m \times n}$ ($k=1,2,\dots,l$), where $\hat{a}_{ij}^{(k)} = \left([a_{ij}^{(k)}, b_{ij}^{(k)}, c_{ij}^{(k)}, d_{ij}^{(k)}]; [\mu_{ij}^{(k)-}, \mu_{ij}^{(k)+}], [\nu_{ij}^{(k)-}, \nu_{ij}^{(k)+}] \right)$ is an interval-valued intuitionistic trapezoidal fuzzy number provided by the decision maker $d_k \in D$, $[\mu_{ij}^{(k)-}, \mu_{ij}^{(k)+}]$ denotes the extent to which alternative x_i belongs to trapezoidal fuzzy number $[a_{ij}^{(k)}, b_{ij}^{(k)}, c_{ij}^{(k)}, d_{ij}^{(k)}]$ on the criteria c_j , $[\nu_{ij}^{(k)-}, \nu_{ij}^{(k)+}]$ denotes the extent to which alternative x_i does not belong to trapezoidal fuzzy number $[a_{ij}^{(k)}, b_{ij}^{(k)}, c_{ij}^{(k)}, d_{ij}^{(k)}]$ on the criteria c_j , $[\mu_{ij}^{(k)-}, \mu_{ij}^{(k)+}] \in [I]$, $[\nu_{ij}^{(k)-}, \nu_{ij}^{(k)+}] \in [I]$, $(\mu_{ij}^{(k)+} + \nu_{ij}^{(k)+}) \leq 1$, $i=1,2,\dots,m$, $j=1,2,\dots,n$.

To eliminate the effect from different physical dimensions to decision results, the decision making matrix should be standardized firstly. Consider that there are benefit attributes (i.e., the bigger the attribute values the better) and cost attributes (i.e., the smaller the attribute values the better) in multiple attribute group decision making problems. We may transform the interval-valued intuitionistic trapezoidal fuzzy decision matrix $\hat{A}^{(k)} = (\hat{a}_{ij}^{(k)})_{m \times n} = \left(\left([a_{ij}^{(k)}, b_{ij}^{(k)}, c_{ij}^{(k)}, d_{ij}^{(k)}]; [\mu_{ij}^{(k)-}, \mu_{ij}^{(k)+}], [\nu_{ij}^{(k)-}, \nu_{ij}^{(k)+}] \right) \right)_{m \times n}$ into the standardized interval-valued intuitionistic trapezoidal fuzzy decision matrix $\hat{R}^{(k)} = (\hat{r}_{ij}^{(k)})_{m \times n} = \left(\left([e_{ij}^{(k)}, f_{ij}^{(k)}, g_{ij}^{(k)}, h_{ij}^{(k)}]; [\mu_{ij}^{(k)-}, \mu_{ij}^{(k)+}], [\nu_{ij}^{(k)-}, \nu_{ij}^{(k)+}] \right) \right)_{m \times n}$ by the following method given in [11,12,23]:

For cost type of criteria:

$$\begin{aligned} e_{ij}^{(k)} &= \frac{\max_{1 \leq j \leq n} \{d_{ij}^{(k)}\} - a_{ij}^{(k)}}{\max_{1 \leq j \leq n} \{d_{ij}^{(k)}\} - \min_{1 \leq j \leq n} \{a_{ij}^{(k)}\}}, & f_{ij}^{(k)} &= \frac{\max_{1 \leq j \leq n} \{d_{ij}^{(k)}\} - b_{ij}^{(k)}}{\max_{1 \leq j \leq n} \{d_{ij}^{(k)}\} - \min_{1 \leq j \leq n} \{a_{ij}^{(k)}\}}, \\ g_{ij}^{(k)} &= \frac{\max_{1 \leq j \leq n} \{d_{ij}^{(k)}\} - c_{ij}^{(k)}}{\max_{1 \leq j \leq n} \{d_{ij}^{(k)}\} - \min_{1 \leq j \leq n} \{a_{ij}^{(k)}\}}, & h_{ij}^{(k)} &= \frac{\max_{1 \leq j \leq n} \{d_{ij}^{(k)}\} - d_{ij}^{(k)}}{\max_{1 \leq j \leq n} \{d_{ij}^{(k)}\} - \min_{1 \leq j \leq n} \{a_{ij}^{(k)}\}}. \end{aligned} \quad (44)$$

For benefit type of criteria:

$$\begin{aligned} e_{ij}^{(k)} &= \frac{a_{ij}^{(k)} - \min_{1 \leq j \leq n} \{a_{ij}^{(k)}\}}{\max_{1 \leq j \leq n} \{d_{ij}^{(k)}\} - \min_{1 \leq j \leq n} \{a_{ij}^{(k)}\}}, & f_{ij}^{(k)} &= \frac{b_{ij}^{(k)} - \min_{1 \leq j \leq n} \{a_{ij}^{(k)}\}}{\max_{1 \leq j \leq n} \{d_{ij}^{(k)}\} - \min_{1 \leq j \leq n} \{a_{ij}^{(k)}\}}, \\ g_{ij}^{(k)} &= \frac{c_{ij}^{(k)} - \min_{1 \leq j \leq n} \{a_{ij}^{(k)}\}}{\max_{1 \leq j \leq n} \{d_{ij}^{(k)}\} - \min_{1 \leq j \leq n} \{a_{ij}^{(k)}\}}, & h_{ij}^{(k)} &= \frac{d_{ij}^{(k)} - \min_{1 \leq j \leq n} \{a_{ij}^{(k)}\}}{\max_{1 \leq j \leq n} \{d_{ij}^{(k)}\} - \min_{1 \leq j \leq n} \{a_{ij}^{(k)}\}}. \end{aligned} \quad (45)$$

In the following, we utilize the IVITFPWA (or IVITFPWG) operator to develop an approach to multi attribute group decision making under interval-valued intuitionistic trapezoidal fuzzy environment. The main steps can be summarized as follows:

Step 1. Transform the interval-valued intuitionistic trapezoidal fuzzy decision matrices

$\hat{A}^{(k)} = (\hat{a}_{ij}^{(k)})_{m \times n} = \left(\left([a_{ij}^{(k)}, b_{ij}^{(k)}, c_{ij}^{(k)}, d_{ij}^{(k)}]; [\mu_{ij}^{(k)-}, \mu_{ij}^{(k)+}], [\nu_{ij}^{(k)-}, \nu_{ij}^{(k)+}] \right) \right)_{m \times n}$ ($k=1,2,\dots,l$) into the standardized interval-valued intuitionistic trapezoidal fuzzy decision matrices

803 $\hat{R}^{(k)} = (\hat{r}_{ij}^{(k)})_{m \times n} = \left(\left(\left[e_{ij}^{(k)}, f_{ij}^{(k)}, g_{ij}^{(k)}, h_{ij}^{(k)} \right]; \left[(\mu_{ij}^{(k)})^-, (\mu_{ij}^{(k)})^+ \right], \left[(\nu_{ij}^{(k)})^-, (\nu_{ij}^{(k)})^+ \right] \right) \right)_{m \times n} \quad (k=1,2,\dots,l)$ using
 804 Eqs. (44) and (45).

805 **Step 2:** Calculate the matrices $T^{(p)} = (T_{ij}^{(p)})_{m \times n}$, $(p=1,2,\dots,l)$ based on the following
 806 equations:

807
$$T_{ij}^{(p)} = \prod_{k=1}^{p-1} S(\hat{r}_{ij}^{(k)}), \quad p=2,\dots,l, \quad i=1,2,\dots,m, \quad j=1,2,\dots,n, \quad (46)$$

808
$$T_{ij}^{(1)} = 1, \quad i=1,2,\dots,m, \quad j=1,2,\dots,n. \quad (47)$$

809 **Step 3:** Utilize the IVITFPWA operator:

810
$$\begin{aligned} \hat{r}_{ij} &= \text{IVITFPWA}(\hat{r}_{ij}^{(1)}, \hat{r}_{ij}^{(2)}, \dots, \hat{r}_{ij}^{(l)}) \\ &= \frac{T_{ij}^{(1)}}{\sum_{p=1}^l T_{ij}^{(p)}} \hat{r}_{ij}^{(1)} \oplus \frac{T_{ij}^{(2)}}{\sum_{p=1}^l T_{ij}^{(p)}} \hat{r}_{ij}^{(2)} \oplus \dots \oplus \frac{T_{ij}^{(l)}}{\sum_{p=1}^l T_{ij}^{(p)}} \hat{r}_{ij}^{(l)} \\ &= \left[\begin{aligned} &\left[\sum_{k=1}^l \frac{T_{ij}^{(k)} e_{ij}^{(k)}}{\sum_{p=1}^l T_{ij}^{(p)}}, \sum_{k=1}^l \frac{T_{ij}^{(k)} f_{ij}^{(k)}}{\sum_{p=1}^l T_{ij}^{(p)}}, \sum_{k=1}^l \frac{T_{ij}^{(k)} g_{ij}^{(k)}}{\sum_{p=1}^l T_{ij}^{(p)}}, \sum_{k=1}^l \frac{T_{ij}^{(k)} h_{ij}^{(k)}}{\sum_{p=1}^l T_{ij}^{(p)}} \right]; \\ &\left[1 - \prod_{k=1}^l \left(1 - (\mu_{ij}^{(k)})^- \right)^{\left(\frac{T_{ij}^{(k)}}{\sum_{p=1}^l T_{ij}^{(p)}} \right)}, 1 - \prod_{k=1}^l \left(1 - (\mu_{ij}^{(k)})^+ \right)^{\left(\frac{T_{ij}^{(k)}}{\sum_{p=1}^l T_{ij}^{(p)}} \right)} \right], \left[\prod_{k=1}^l \left((\nu_{ij}^{(k)})^- \right)^{\left(\frac{T_{ij}^{(k)}}{\sum_{p=1}^l T_{ij}^{(p)}} \right)}, \prod_{k=1}^l \left((\nu_{ij}^{(k)})^+ \right)^{\left(\frac{T_{ij}^{(k)}}{\sum_{p=1}^l T_{ij}^{(p)}} \right)} \right] \end{aligned} \right] \end{aligned} \quad (48)$$

811 or the IVITFPWG operator:

812
$$\begin{aligned} \hat{r}_{ij} &= \text{IVITFPWG}(\hat{r}_{ij}^{(1)}, \hat{r}_{ij}^{(2)}, \dots, \hat{r}_{ij}^{(l)}) \\ &= \left(\hat{r}_{ij}^{(1)} \right)^{\left(\frac{T_{ij}^{(1)}}{\sum_{p=1}^l T_{ij}^{(p)}} \right)} \otimes \left(\hat{r}_{ij}^{(2)} \right)^{\left(\frac{T_{ij}^{(2)}}{\sum_{p=1}^l T_{ij}^{(p)}} \right)} \otimes \dots \otimes \left(\hat{r}_{ij}^{(l)} \right)^{\left(\frac{T_{ij}^{(l)}}{\sum_{p=1}^l T_{ij}^{(p)}} \right)} \\ &= \left[\begin{aligned} &\left[\prod_{k=1}^l \left(e_{ij}^{(k)} \right)^{\left(\frac{T_{ij}^{(k)}}{\sum_{p=1}^l T_{ij}^{(p)}} \right)}, \prod_{k=1}^l \left(f_{ij}^{(k)} \right)^{\left(\frac{T_{ij}^{(k)}}{\sum_{p=1}^l T_{ij}^{(p)}} \right)}, \prod_{k=1}^l \left(g_{ij}^{(k)} \right)^{\left(\frac{T_{ij}^{(k)}}{\sum_{p=1}^l T_{ij}^{(p)}} \right)}, \prod_{k=1}^l \left(h_{ij}^{(k)} \right)^{\left(\frac{T_{ij}^{(k)}}{\sum_{p=1}^l T_{ij}^{(p)}} \right)} \right]; \\ &\left[\prod_{k=1}^l \left((\mu_{ij}^{(k)})^- \right)^{\left(\frac{T_{ij}^{(k)}}{\sum_{p=1}^l T_{ij}^{(p)}} \right)}, \prod_{k=1}^l \left((\mu_{ij}^{(k)})^+ \right)^{\left(\frac{T_{ij}^{(k)}}{\sum_{p=1}^l T_{ij}^{(p)}} \right)} \right], \left[1 - \prod_{k=1}^l \left(1 - (\nu_{ij}^{(k)})^- \right)^{\left(\frac{T_{ij}^{(k)}}{\sum_{p=1}^l T_{ij}^{(p)}} \right)}, 1 - \prod_{k=1}^l \left(1 - (\nu_{ij}^{(k)})^+ \right)^{\left(\frac{T_{ij}^{(k)}}{\sum_{p=1}^l T_{ij}^{(p)}} \right)} \right] \end{aligned} \right] \end{aligned} \quad (49)$$

813 to aggregate all the individual interval-valued intuitionistic trapezoidal fuzzy decision matrices

814 $\hat{R}^{(k)} = (\hat{r}_{ij}^{(k)})_{m \times n} \quad (k=1,2,\dots,l)$ into the collective interval-valued intuitionistic trapezoidal fuzzy

815 decision matrix $\hat{R} = (\hat{r}_{ij})_{m \times n} = \left(\left(\left[e_{ij}, f_{ij}, g_{ij}, h_{ij} \right]; \left[(\mu_{ij})^-, (\mu_{ij})^+ \right], \left[(\nu_{ij})^-, (\nu_{ij})^+ \right] \right) \right)_{m \times n}$.

816 **Step 4:** Calculate the matrix $T = (T_{ij})_{m \times n}$ based on following equations:

817
$$T_{ij} = \prod_{k=1}^{j-1} S(\hat{r}_{ij}), \quad (i=1,2,\dots,m, \quad j=2,\dots,n), \quad (50)$$

818
$$T_{i1} = 1, \quad (i=1,2,\dots,m). \quad (51)$$

819 **Step 5:** Utilize the IVITFPWA operator:

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$$\hat{r}_i = \text{IVITFPWA}(\hat{r}_{i1}, \hat{r}_{i2}, \dots, \hat{r}_{in})$$

821

$$= \frac{T_{i1}}{\sum_{j=1}^n T_{ij}} \hat{r}_{i1} \oplus \frac{T_{i2}}{\sum_{j=1}^n T_{ij}} \hat{r}_{i2} \oplus \dots \oplus \frac{T_{in}}{\sum_{j=1}^n T_{ij}} \hat{r}_{in} \quad (52)$$

$$= \left[\left[\sum_{j=1}^n \left(\frac{T_{ij} e_{ij}}{\sum_{j=1}^n T_{ij}} \right), \sum_{j=1}^n \left(\frac{T_{ij} f_{ij}}{\sum_{j=1}^n T_{ij}} \right), \sum_{j=1}^n \left(\frac{T_{ij} g_{ij}}{\sum_{j=1}^n T_{ij}} \right), \sum_{j=1}^n \left(\frac{T_{ij} h_{ij}}{\sum_{j=1}^n T_{ij}} \right) \right]; \right. \\ \left. \left[1 - \prod_{j=1}^n (1 - \mu_{ij}^-)^{\left(\frac{T_{ij}}{\sum_{j=1}^n T_{ij}} \right)}, 1 - \prod_{j=1}^n (1 - \mu_{ij}^+)^{\left(\frac{T_{ij}}{\sum_{j=1}^n T_{ij}} \right)} \right], \left[\prod_{j=1}^n (v_{ij}^-)^{\left(\frac{T_{ij}}{\sum_{j=1}^n T_{ij}} \right)}, \prod_{j=1}^n (v_{ij}^+)^{\left(\frac{T_{ij}}{\sum_{j=1}^n T_{ij}} \right)} \right] \right]$$

822

or the IVITFPWG operator:

$$\hat{r}_i = \text{IVITFPWG}(\hat{r}_{i1}, \hat{r}_{i2}, \dots, \hat{r}_{in})$$

823

$$= (\hat{r}_{i1})^{\left(\frac{T_{i1}}{\sum_{j=1}^n T_{ij}} \right)} \otimes (\hat{r}_{i2})^{\left(\frac{T_{i2}}{\sum_{j=1}^n T_{ij}} \right)} \otimes \dots \otimes (\hat{r}_{in})^{\left(\frac{T_{in}}{\sum_{j=1}^n T_{ij}} \right)} \quad (53)$$

$$= \left[\left[\prod_{j=1}^n (e_{ij})^{\left(\frac{T_{ij}}{\sum_{j=1}^n T_{ij}} \right)}, \prod_{j=1}^n (f_{ij})^{\left(\frac{T_{ij}}{\sum_{j=1}^n T_{ij}} \right)}, \prod_{j=1}^n (g_{ij})^{\left(\frac{T_{ij}}{\sum_{j=1}^n T_{ij}} \right)}, \prod_{j=1}^n (h_{ij})^{\left(\frac{T_{ij}}{\sum_{j=1}^n T_{ij}} \right)} \right]; \right. \\ \left. \left[\prod_{j=1}^n (\mu_{ij}^-)^{\left(\frac{T_{ij}}{\sum_{j=1}^n T_{ij}} \right)}, \prod_{j=1}^n (\mu_{ij}^+)^{\left(\frac{T_{ij}}{\sum_{j=1}^n T_{ij}} \right)} \right], \left[1 - \prod_{j=1}^n (1 - v_{ij}^-)^{\left(\frac{T_{ij}}{\sum_{j=1}^n T_{ij}} \right)}, 1 - \prod_{j=1}^n (1 - v_{ij}^+)^{\left(\frac{T_{ij}}{\sum_{j=1}^n T_{ij}} \right)} \right] \right]$$

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to derive the collective overall preference interval-valued intuitionistic trapezoidal fuzzy

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values $\hat{r}_i = ([e_i, f_i, g_i, h_i], [\mu_i^-, \mu_i^+], [v_i^-, v_i^+])$ ($i = 1, 2, \dots, m$) of the alternatives x_i ($i = 1, 2, \dots, m$).

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Step 6: Calculate the score functions and the accuracy functions of $\hat{r}_i$ ($i = 1, 2, \dots, m$) as

827

follows:

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$$S(\hat{r}_i) = \frac{1}{8} \times (e_i + f_i + g_i + h_i) \times \left(1 + \frac{\mu_i^- - v_i^- + \mu_i^+ - v_i^+}{2} \right) \times \left(\frac{\mu_i^- - v_i^- + \mu_i^+ - v_i^+}{2} \right) \quad (54)$$

829

$$H(\hat{r}_i) = \frac{1}{8} \times (e_i + f_i + g_i + h_i) \times \left(1 + \frac{\mu_i^- - v_i^- + \mu_i^+ - v_i^+}{2} \right) \times \left(\frac{\mu_i^- + v_i^- + \mu_i^+ + v_i^+}{2} \right) .$$

830 (55)

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Step 7: Rank all the alternatives x_i ($i = 1, 2, \dots, m$) and then select the best alternative(s) by

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Definition 4.6.

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Step 8: End.

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4.5. An illustrative example

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In this subsection, we continue to investigate Example 3.1 as an illustration of our approach.

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Example 4.1 [20]. Suppose that five candidates x_i ($i = 1, 2, 3, 4, 5$) are to be evaluated by

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three decision makers d_k ($k = 1, 2, 3$) under the above four attributes c_j ($j = 1, 2, 3, 4$). After a

841

serious consideration, the decision makers construct, respectively, three interval-valued

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intuitionistic trapezoidal fuzzy decision matrices $\hat{A}^{(k)} = (\hat{\alpha}_{ij}^{(k)})_{m \times n}$ ($k = 1, 2, 3$) listed in Tables 9-

843

11.

Table 9. The interval-valued intuitionistic trapezoidal fuzzy decision matrix $\hat{A}^{(1)}$ provided by d_1 .

9	c_1	c_2	c_3	c_4
x_1	$([2, 3, 4, 6]; [0.6, 0.7], [0.3, 0.3])$	$([4, 6, 8, 9]; [0.6, 0.6], [0.3, 0.4])$	$([2, 3, 4, 6]; [0.6, 0.6], [0.4, 0.4])$	$([3, 5, 6, 9]; [0.7, 0.8], [0.1, 0.2])$
x_2	$([1, 2, 3, 6]; [0.6, 0.6], [0.4, 0.4])$	$([6, 7, 8, 9]; [0.7, 0.8], [0.1, 0.2])$	$([3, 5, 6, 8]; [0.7, 0.8], [0.1, 0.2])$	$([4, 6, 8, 9]; [0.5, 0.6], [0.1, 0.3])$
x_3	$([2, 3, 4, 5]; [0.5, 0.6], [0.3, 0.4])$	$([5, 6, 7, 9]; [0.5, 0.6], [0.2, 0.3])$	$([2, 3, 5, 6]; [0.4, 0.7], [0.1, 0.3])$	$([3, 5, 6, 8]; [0.7, 0.8], [0.1, 0.2])$
x_4	$([1, 2, 3, 5]; [0.6, 0.6], [0.3, 0.4])$	$([6, 7, 8, 9]; [0.8, 0.8], [0.1, 0.2])$	$([3, 4, 6, 7]; [0.8, 0.9], [0.1, 0.1])$	$([4, 6, 7, 9]; [0.6, 0.6], [0.3, 0.4])$
x_5	$([2, 3, 4, 5]; [0.6, 0.7], [0.3, 0.3])$	$([5, 6, 7, 8]; [0.6, 0.7], [0.2, 0.3])$	$([2, 3, 5, 9]; [0.6, 0.7], [0.3, 0.3])$	$([3, 4, 6, 8]; [0.5, 0.7], [0.2, 0.3])$

Table 10. The interval-valued intuitionistic trapezoidal fuzzy decision matrix $\hat{A}^{(2)}$ provided by d_2 .

10	c_1	c_2	c_3	c_4
x_1	$([5, 6, 7, 9]; [0.8, 0.8], [0.1, 0.2])$	$([2, 3, 4, 7]; [0.7, 0.7], [0.2, 0.3])$	$([3, 4, 6, 8]; [0.5, 0.5], [0.4, 0.5])$	$([5, 6, 7, 9]; [0.6, 0.7], [0.3, 0.3])$
x_2	$([1, 4, 5, 7]; [0.5, 0.6], [0.3, 0.4])$	$([2, 4, 5, 6]; [0.5, 0.6], [0.4, 0.4])$	$([2, 3, 4, 6]; [0.5, 0.7], [0.2, 0.3])$	$([2, 6, 7, 8]; [0.6, 0.8], [0.1, 0.2])$
x_3	$([5, 6, 8, 9]; [0.5, 0.7], [0.1, 0.2])$	$([2, 3, 4, 5]; [0.6, 0.6], [0.3, 0.3])$	$([3, 4, 5, 8]; [0.5, 0.6], [0.3, 0.4])$	$([5, 6, 7, 8]; [0.5, 0.7], [0.2, 0.3])$
x_4	$([1, 4, 5, 8]; [0.5, 0.6], [0.4, 0.4])$	$([2, 4, 5, 7]; [0.5, 0.5], [0.4, 0.5])$	$([2, 3, 4, 5]; [0.6, 0.7], [0.2, 0.3])$	$([2, 5, 7, 8]; [0.8, 0.9], [0.1, 0.1])$
x_5	$([5, 6, 7, 8]; [0.5, 0.7], [0.2, 0.3])$	$([2, 3, 4, 6]; [0.5, 0.6], [0.3, 0.4])$	$([3, 4, 6, 9]; [0.4, 0.6], [0.3, 0.4])$	$([5, 6, 7, 9]; [0.7, 0.8], [0.1, 0.2])$

Table 11. The interval-valued intuitionistic trapezoidal fuzzy decision matrix $\hat{A}^{(3)}$ provided by d_3 .

11	c_1	c_2	c_3	c_4
x_1	([2, 3, 5, 7]; [0.5, 0.6], [0.4, 0.4])	([4, 5, 7, 8]; [0.5, 0.5], [0.4, 0.5])	([2, 3, 6, 8]; [0.6, 0.7], [0.3, 0.3])	([4, 5, 6, 8]; [0.8, 0.8], [0.1, 0.2])
x_2	([4, 6, 7, 9]; [0.6, 0.7], [0.1, 0.3])	([2, 3, 5, 6]; [0.7, 0.8], [0.1, 0.2])	([2, 3, 5, 6]; [0.5, 0.7], [0.2, 0.3])	([6, 7, 8, 9]; [0.4, 0.5], [0.5, 0.5])
x_3	([2, 3, 5, 6]; [0.5, 0.6], [0.3, 0.4])	([4, 6, 7, 8]; [0.5, 0.6], [0.3, 0.4])	([2, 3, 6, 8]; [0.4, 0.6], [0.3, 0.4])	([4, 5, 6, 7]; [0.7, 0.8], [0.1, 0.2])
x_4	([4, 6, 7, 8]; [0.6, 0.6], [0.3, 0.4])	([2, 3, 4, 5]; [0.7, 0.8], [0.1, 0.2])	([2, 3, 4, 6]; [0.6, 0.7], [0.2, 0.3])	([6, 7, 8, 9]; [0.3, 0.4], [0.5, 0.6])
x_5	([2, 4, 5, 6]; [0.4, 0.5], [0.3, 0.5])	([4, 6, 7, 8]; [0.4, 0.6], [0.3, 0.4])	([2, 5, 6, 8]; [0.5, 0.6], [0.3, 0.4])	([4, 5, 6, 8]; [0.5, 0.8], [0.1, 0.2])

Step 1. Using Eqs. (44) and (45), we transform the interval-valued intuitionistic trapezoidal fuzzy decision matrices

$\hat{A}^{(k)} = (\hat{a}_{ij}^{(k)})_{5 \times 4}$ ($k=1,2,3$) into the standardized interval-valued intuitionistic trapezoidal fuzzy matrix $\hat{R}^{(k)} = (\hat{r}_{ij}^{(k)})_{5 \times 4}$ ($k=1,2,3$) (see Tables 12-14):

Table 12. The interval-valued intuitionistic trapezoidal fuzzy decision matrix $\hat{R}^{(1)}$ provided by d_1 .

12	c_1	c_2	c_3	c_4
x_1	([0, 0.1429, 0.2857, 0.5714]; [0.6, 0.7], [0.3, 0.3])	([0.2857, 0.5714, 0.8571, 1.0000]; [0.6, 0.6], [0.3, 0.4])	([0, 0.1429, 0.2857, 0.5714]; [0.6, 0.6], [0.4, 0.4])	([0.1429, 0.4286, 0.5714, 1.0000]; [0.7, 0.8], [0.1, 0.2])
x_2	([0, 0.1250, 0.2500, 0.6250]; [0.6, 0.6], [0.4, 0.4])	([0.6250, 0.7500, 0.8750, 1.0000]; [0.7, 0.8], [0.1, 0.2])	([0.2500, 0.5000, 0.6250, 0.8750]; [0.7, 0.8], [0.1, 0.2])	([0.3750, 0.6250, 0.8750, 1.0000]; [0.5, 0.6], [0.1, 0.3])
x_3	([0, 0.1429, 0.2857, 0.4286]; [0.5, 0.6], [0.3, 0.4])	([0.4286, 0.5714, 0.7143, 1.0000]; [0.5, 0.6], [0.2, 0.3])	([0, 0.1429, 0.4286, 0.5714]; [0.4, 0.7], [0.1, 0.3])	([0.1429, 0.4286, 0.5714, 0.8571]; [0.7, 0.8], [0.1, 0.2])
x_4	([0, 0.1250, 0.2500, 0.5000]; [0.6, 0.6], [0.3, 0.4])	([0.6250, 0.7500, 0.8750, 1.0000]; [0.8, 0.8], [0.1, 0.2])	([0.2500, 0.3750, 0.6250, 0.7500]; [0.8, 0.9], [0.1, 0.1])	([0.3750, 0.6250, 0.7500, 1.0000]; [0.6, 0.6], [0.3, 0.4])
x_5	([0, 0.1429, 0.2857, 0.4286]; [0.6, 0.7], [0.3, 0.3])	([0.4286, 0.5714, 0.7143, 0.8571]; [0.6, 0.7], [0.2, 0.3])	([0, 0.1429, 0.4286, 1.0000]; [0.6, 0.7], [0.3, 0.3])	([0.1429, 0.2857, 0.5714, 0.8571]; [0.5, 0.7], [0.2, 0.3])

Table 13. The interval-valued intuitionistic trapezoidal fuzzy decision matrix $\hat{R}^{(2)}$ provided by d_2 .

13	c_1	c_2	c_3	c_4
x_1	([0.4286, 0.5714, 0.7143, 1.0000]; [0.8, 0.8], [0.1, 0.2])	([0, 0.1429, 0.2857, 0.7143]; [0.7, 0.7], [0.2, 0.3])	([0.1429, 0.2857, 0.5714, 0.8571]; [0.5, 0.5], [0.4, 0.5])	([0.4286, 0.5714, 0.7143, 1.0000]; [0.6, 0.7], [0.3, 0.3])
x_2	([0, 0.4286, 0.5714, 0.8571]; [0.5, 0.6], [0.3, 0.4])	([0.1429, 0.4286, 0.5714, 0.7143]; [0.5, 0.6], [0.4, 0.4])	([0.1429, 0.2857, 0.4286, 0.7143]; [0.5, 0.7], [0.2, 0.3])	([0.1429, 0.7143, 0.8571, 1.0000]; [0.6, 0.8], [0.1, 0.2])
x_3	([0.4286, 0.5714, 0.8571, 1.0000]; [0.5, 0.7], [0.1, 0.2])	([0, 0.1429, 0.2857, 0.4286]; [0.6, 0.6], [0.3, 0.3])	([0.1429, 0.2857, 0.4286, 0.8571]; [0.5, 0.6], [0.3, 0.4])	([0.4286, 0.5714, 0.7143, 0.8571]; [0.5, 0.7], [0.2, 0.3])
x_4	([0, 0.4286, 0.5714, 1.0000]; [0.5, 0.6], [0.4, 0.4])	([0.1429, 0.4286, 0.5714, 0.8571]; [0.5, 0.5], [0.4, 0.5])	([0.1429, 0.2857, 0.4286, 0.5714]; [0.6, 0.7], [0.2, 0.3])	([0.1429, 0.5714, 0.8571, 1.0000]; [0.8, 0.9], [0.1, 0.1])
x_5	([0.4286, 0.5714, 0.7143, 0.8571]; [0.5, 0.7], [0.2, 0.3])	([0, 0.1429, 0.2857, 0.5714]; [0.5, 0.6], [0.3, 0.4])	([0.1429, 0.2857, 0.5714, 1.0000]; [0.4, 0.6], [0.3, 0.4])	([0.4286, 0.5714, 0.7143, 1.0000]; [0.7, 0.8], [0.1, 0.2])

Table 14. The interval-valued intuitionistic trapezoidal fuzzy decision matrix $\hat{R}^{(3)}$ provided by d_3 .

14	c_1	c_2	c_3	c_4
x_1	([0, 0.1667, 0.5000, 0.8333]; [0.5, 0.6], [0.4, 0.4])	([0.3333, 0.5000, 0.8333, 1.0000]; [0.5, 0.5], [0.4, 0.5])	([0, 0.1667, 0.6667, 1.0000]; [0.6, 0.7], [0.3, 0.3])	([0.3333, 0.5000, 0.6667, 1.0000]; [0.8, 0.8], [0.1, 0.2])
x_2	([0.2857, 0.5714, 0.7143, 1.0000]; [0.6, 0.7], [0.1, 0.3])	([0, 0.1429, 0.4286, 0.5714]; [0.7, 0.8], [0.1, 0.2])	([0, 0.1429, 0.4286, 0.5714]; [0.5, 0.7], [0.2, 0.3])	([0.5714, 0.7143, 0.8571, 1.0000]; [0.4, 0.5], [0.5, 0.5])
x_3	([0, 0.1667, 0.5000, 0.6667]; [0.5, 0.6], [0.3, 0.4])	([0.3333, 0.6667, 0.8333, 1.0000]; [0.5, 0.6], [0.3, 0.4])	([0, 0.1667, 0.6667, 1.0000]; [0.4, 0.6], [0.3, 0.4])	([0.3333, 0.5000, 0.6667, 0.8333]; [0.7, 0.8], [0.1, 0.2])
x_4	([0.2857, 0.5714, 0.7143, 0.8571]; [0.6, 0.6], [0.3, 0.4])	([0, 0.1429, 0.2857, 0.4286]; [0.7, 0.8], [0.1, 0.2])	([0, 0.1429, 0.2857, 0.5714]; [0.6, 0.7], [0.2, 0.3])	([0.5714, 0.7143, 0.8571, 1.0000]; [0.3, 0.4], [0.5, 0.6])
x_5	([0, 0.3333, 0.5000, 0.6667]; [0.4, 0.5], [0.3, 0.5])	([0.3333, 0.6667, 0.8333, 1.0000]; [0.4, 0.6], [0.3, 0.4])	([0, 0.5000, 0.6667, 1.0000]; [0.5, 0.6], [0.3, 0.4])	([0.3333, 0.5000, 0.6667, 1.0000]; [0.5, 0.8], [0.1, 0.2])

Step 2: We first utilize Eqs. (46) and (47) to calculate the matrices $T^{(1)}$, $T^{(2)}$, and $T^{(3)}$ as follows:

$$\begin{aligned}
889 \quad T^{(1)} &= \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}, & T^{(2)} &= \begin{pmatrix} 0.0591 & 0.1060 & 0.0300 & 0.2571 \\ 0.0300 & 0.3900 & 0.2700 & 0.1698 \\ 0.0257 & 0.1323 & 0.0675 & 0.2400 \\ 0.0342 & 0.4357 & 0.3281 & 0.1074 \\ 0.0506 & 0.1800 & 0.0928 & 0.1097 \end{pmatrix}, \\
890 \quad T^{(3)} &= \begin{pmatrix} 0.0215 & 0.0099 & 0.0004 & 0.0412 \\ 0.0017 & 0.0156 & 0.0251 & 0.0491 \\ 0.0060 & 0.0055 & 0.0035 & 0.0365 \\ 0.0015 & 0.0057 & 0.0328 & 0.0453 \\ 0.0077 & 0.0054 & 0.0040 & 0.0357 \end{pmatrix}.
\end{aligned}$$

891 **Step 3:** Utilize the IVITFPWA operator (Eq. (48)) to aggregate all the individual interval-
892 valued intuitionistic trapezoidal fuzzy decision matrices $\hat{R}^{(k)} = (\hat{r}_{ij}^{(k)})_{5 \times 4}$ ($k=1,2,3$) into the
893 collective interval-valued intuitionistic trapezoidal fuzzy decision matrix $\hat{R} = (\hat{r}_{ij})_{5 \times 4}$ (see Table
894 15).
895

896 **Table 15.** The collective interval-valued intuitionistic trapezoidal fuzzy decision matrix $\hat{R}$.
897

15	c_1	c_2	c_3	c_4
x_1	([0.0234, 0.1668, 0.3134, 0.6001]; [0.6132, 0.7049], [0.2841, 0.2954])	([0.2590, 0.5301, 0.8026, 0.9729]; [0.6100, 0.6100], [0.2894, 0.3900])	([0.0042, 0.1470, 0.2942, 0.5799]; [0.5974, 0.5974], [0.4000, 0.4026])	([0.2055, 0.4591, 0.6027, 1.0000]; [0.6865, 0.7833], [0.1243, 0.2167])
x_2	([0.0005, 0.1346, 0.2601, 0.6324]; [0.5974, 0.6002], [0.3958, 0.3998])	([0.4843, 0.6541, 0.7858, 0.9160]; [0.6543, 0.7576], [0.1469, 0.2424])	([0.2228, 0.4484, 0.5802, 0.8356]; [0.6630, 0.7806], [0.1171, 0.2194])	([0.3506, 0.6410, 0.8718, 1.0000]; [0.5117, 0.6335], [0.1067, 0.2894])
x_3	([0.0107, 0.1537, 0.3012, 0.4442]; [0.5000, 0.6029], [0.2919, 0.3931])	([0.3783, 0.5221, 0.6650, 0.9335]; [0.5128, 0.6000], [0.2101, 0.3004])	([0.0090, 0.1519, 0.4293, 0.5908]; [0.4069, 0.6942], [0.1076, 0.3058])	([0.2020, 0.4575, 0.6010, 0.8565]; [0.6698, 0.7842], [0.1139, 0.2158])
x_4	([0.0004, 0.1357, 0.2613, 0.5170]; [0.5970, 0.6000], [0.3029, 0.4000])	([0.4768, 0.6504, 0.7809, 0.9546]; [0.7357, 0.7362], [0.1521, 0.2638])	([0.2181, 0.3479, 0.5695, 0.7026]; [0.7596, 0.8662], [0.1202, 0.1338])	([0.3611, 0.6235, 0.7642, 1.0000]; [0.6167, 0.6428], [0.2763, 0.3572])
x_5	([0.0205, 0.1647, 0.3078, 0.4508]; [0.5945, 0.6989], [0.2942, 0.3011])	([0.3631, 0.5068, 0.6498, 0.8144]; [0.5854, 0.6862], [0.2131, 0.3138])	([0.0121, 0.1562, 0.4415, 1.0000]; [0.5857, 0.6923], [0.3000, 0.3077])	([0.1762, 0.3198, 0.5881, 0.8753]; [0.5239, 0.7151], [0.1832, 0.2849])

898
899 **Step 4:** Calculate the matrix $T = (T_{ij})_{5 \times 4}$ based on Eqs. (50) and (51):

$$T = \begin{pmatrix} 1 & 0.0698 & 0.0077 & 0.0002 \\ 1 & 0.0310 & 0.0085 & 0.0019 \\ 1 & 0.0287 & 0.0035 & 0.0002 \\ 1 & 0.0352 & 0.0102 & 0.0027 \\ 1 & 0.0555 & 0.0083 & 0.0007 \end{pmatrix}.$$

Step 5: Utilize the IVITFPWA operator (Eq. (52)) to aggregate all the preference values $\hat{r}_{ij}$ ($j=1,2,3,4$) in the i th line of $\hat{R}$, and derive the collective overall preference value $\hat{r}_i$ ($i=1,2,3,4,5$) of the alternative x_i ($i=1,2,3,4,5$).

$$\begin{aligned} \hat{r}_1 &= ([0.0386, 0.1902, 0.3450, 0.6241]; [0.6129, 0.6989], [0.2851, 0.3011]), \\ \hat{r}_2 &= ([0.0173, 0.1535, 0.2795, 0.6431]; [0.5997, 0.6081], [0.3796, 0.3917]), \\ \hat{r}_3 &= ([0.0209, 0.1640, 0.3118, 0.4584]; [0.5001, 0.6032], [0.2882, 0.3898]), \\ \hat{r}_4 &= ([0.0195, 0.1563, 0.2830, 0.5348]; [0.6048, 0.6098], [0.2932, 0.3902]), \\ \hat{r}_5 &= ([0.0384, 0.1826, 0.3269, 0.4743]; [0.5939, 0.6982], [0.2893, 0.3018]). \end{aligned}$$

Step 6: Calculate the score functions and the accuracy functions of $\hat{r}_i$ ($i=1,2,3,4,5$) as follows:

$$S(\hat{r}_1) = 0.0740, S(\hat{r}_2) = 0.0363, S(\hat{r}_3) = 0.0308, S(\hat{r}_4) = 0.0417, S(\hat{r}_5) = 0.0605.$$

Step 7: Because $S(\hat{r}_1) > S(\hat{r}_5) > S(\hat{r}_4) > S(\hat{r}_2) > S(\hat{r}_3)$, we have $x_1 \succ x_5 \succ x_4 \succ x_2 \succ x_3$. Therefore, the best candidate is x_1 .

If we deal with Example 4.1 using the IVITFPWG operator instead of the IVITFPWA operator, then the main steps are shown as follows:

Step 1': See Step 1.

Step 2': See Step 2.

Step 3': Utilize the IVITFPWG operator (Eq. (49)) to aggregate all the individual interval-valued intuitionistic trapezoidal fuzzy decision matrices $\hat{R}^{(k)} = (\hat{r}_{ij}^{(k)})_{5 \times 4}$ ($k=1,2,3$) into the collective interval-valued intuitionistic trapezoidal fuzzy decision matrix $\hat{R}' = (\hat{r}'_{ij})_{5 \times 4}$ (see Table 16).

Table 16. The collective interval-valued intuitionistic trapezoidal fuzzy decision matrix $\hat{R}'$.

16	c_1	c_2	c_3	c_4
x_1	$([0, 0.1546, 0.3038, 0.5936]; [0.6073, 0.7030], [0.2925, 0.2970])$	$([0, 0.5003, 0.7720, 0.9685]; [0.6079, 0.6079], [0.2920, 0.3921])$	$([0, 0.1458, 0.2916, 0.5783]; [0.5968, 0.5969], [0.4000, 0.4031])$	$([0.1824, 0.4559, 0.6002, 1.0000]; [0.6818, 0.7791], [0.1437, 0.2209])$
x_2	$([0, 0.1299, 0.2565, 0.6312]; [0.5968, 0.6001], [0.3969, 0.3999])$	$([0, 0.6304, 0.7713, 0.9052]; [0.6376, 0.7386], [0.1958, 0.2614])$	$([0, 0.4343, 0.5735, 0.8319]; [0.6483, 0.7760], [0.1238, 0.2240])$	$([0.3334, 0.6402, 0.8718, 1.0000]; [0.5083, 0.6200], [0.1211, 0.2965])$
x_3	$([0, 0.1480, 0.2946, 0.4388]; [0.5000, 0.6023], [0.2956, 0.3957])$	$([0, 0.4867, 0.6426, 0.9062]; [0.5107, 0.6000], [0.2128, 0.3005])$	$([0, 0.1667, 0.6667, 1.0000]; [0.4, 0.6], [0.3, 0.4])$	$([0.3333, 0.5000, 0.6667, 0.8333]; [0.7, 0.8], [0.1, 0.2])$
x_4	$([0, 0.1305, 0.2573, 0.5120];$	$([0, 0.6291, 0.7659, 0.9513];$	$([0, 0.3431, 0.5600, 0.6978];$	$([0.3485, 0.6231, 0.7634, 1.0000];$

	[0.5964, 0.6000], [0.3036, 0.4000])	[0.6937, 0.6940], [0.2038, 0.3060])	[0.7412, 0.8420], [0.1277, 0.1580])	[0.5997, 0.6133], [0.2928, 0.3867])
x_5	([0, 0.1536, 0.2997, 0.4444]; [0.5930, 0.6983], [0.2955, 0.3017])	([0, 0.4633, 0.6219, 0.8065]; [0.5825, 0.6833], [0.2165, 0.3167])	([0, 0.1522, 0.4398, 1.0000]; [0.5794, 0.6905], [0.3000, 0.3095])	([0.1630, 0.3107, 0.5866, 0.8741]; [0.5164, 0.7120], [0.1879, 0.2880])

923

924 **Step 4'**: Calculate the matrix $T' = (T'_{ij})_{5 \times 4}$ based on Eqs. (50) and (51):

$$T' = \begin{pmatrix} 1 & 0.0645 & 0.0061 & 0.0002 \\ 1 & 0.0305 & 0.0059 & 0.0011 \\ 1 & 0.0273 & 0.0027 & 0.0002 \\ 1 & 0.0345 & 0.0064 & 0.0014 \\ 1 & 0.0525 & 0.0062 & 0.0005 \end{pmatrix}$$

925

926 **Step 5'**: Utilize the IVITFPWG operator (Eq. (53)) to aggregate all the preference values $\hat{r}'_{ij}$
 927 ($j=1,2,3,4$) in the i th line of $\hat{R}'$, and derive the collective overall preference value $\hat{r}'_i$
 928 ($i=1,2,3,4,5$) of the alternatives x_i ($i=1,2,3,4,5$).

$$929 \quad \hat{r}'_1 = ([0, 0.1659, 0.3213, 0.6113]; [0.6073, 0.6962], [0.2931, 0.3038]),$$

$$930 \quad \hat{r}'_2 = ([0, 0.1372, 0.2665, 0.6393]; [0.5982, 0.6047], [0.3902, 0.3952]),$$

$$931 \quad \hat{r}'_3 = ([0, 0.1528, 0.3011, 0.4477]; [0.5000, 0.6025], [0.2931, 0.3931]),$$

$$932 \quad \hat{r}'_4 = ([0, 0.1386, 0.2684, 0.5240]; [0.6002, 0.6042], [0.2995, 0.3958]),$$

$$933 \quad \hat{r}'_5 = ([0, 0.1623, 0.3116, 0.4601]; [0.5924, 0.6975], [0.2918, 0.3025]).$$

934 **Step 6'**: Calculate the score functions and the accuracy functions of $\hat{r}'_i$ ($i=1,2,3,4,5$) as
 935 follows:

$$936 \quad S(\hat{r}'_1) = 0.0657, \quad S(\hat{r}'_2) = 0.0329, \quad S(\hat{r}'_3) = 0.0283, \quad S(\hat{r}'_4) = 0.0372, \quad S(\hat{r}'_5) = 0.0547.$$

937 **Step 7'**: Because $S(\hat{r}'_1) > S(\hat{r}'_5) > S(\hat{r}'_4) > S(\hat{r}'_2) > S(\hat{r}'_3)$, we have $x_1 \succ x_5 \succ x_4 \succ x_2 \succ x_3$.
 938 Therefore, the best candidate is x_1 .

939 In Steps 6 and 6', observe that the score functions obtained with the IVITFPWA operator
 940 always exceed those obtained with the IVITFPWG operator for the same aggregation values,
 941 as illustrated in Theorem 4.7.

942 In Examples 3.1 and 4.1, there are different priority levels among these four attributes and
 943 three decision makers. For example, it is impossible for three decision makers to select a
 944 candidate who owns bad morality, no matter how well this candidate performs on research
 945 capabilities, teaching skill, and education background. In addition, if a candidate receives the
 946 bad evaluation from university president, then he is impossible to be selected no matter how
 947 high evaluations he has received from dean of management school and human resource
 948 officer.

949 The existing intuitionistic trapezoidal fuzzy and interval-valued intuitionistic trapezoidal
 950 fuzzy operators have difficulty in dealing with such situations and these operators usually are
 951 used to solve multiple attribute group decision making where the attributes and the decision
 952 makers are at the same priority level. Though the existing prioritized operators take into
 953 account the prioritization among the attributes and the decision makers, they cannot
 954 accommodate the input arguments in the form of intuitionistic trapezoidal fuzzy and interval-
 955 valued intuitionistic trapezoidal fuzzy information. The proposed operators and approaches
 956 can effectively overcome the above two drawbacks due to the fact that they not only

accommodate intuitionistic trapezoidal fuzzy and interval-valued intuitionistic trapezoidal fuzzy information but also consider the prioritization among the attributes and the decision makers.

5. CONCLUSIONS

In some multiple attribute group decision making problems with intuitionistic trapezoidal fuzzy information or interval-valued intuitionistic trapezoidal fuzzy information, there may exist a prioritization relationship over the attributes and decision makers. To deal with such cases, we in this paper develop some intuitionistic trapezoidal fuzzy prioritized aggregation operators and interval-valued intuitionistic trapezoidal fuzzy prioritized aggregation operators in which there exists a prioritization relationship between the input arguments, including the intuitionistic trapezoidal fuzzy prioritized weighted average (ITFPWA) operator, the intuitionistic trapezoidal fuzzy prioritized weighted geometric (ITFPWG) operator, the interval-valued intuitionistic trapezoidal fuzzy prioritized weighted average (IVITFPWA) operator, and the interval-valued intuitionistic trapezoidal fuzzy prioritized weighted geometric (IVITFPWG) operator. We investigate some basic properties of these operators, such as idempotency, boundedness, and monotonicity. Furthermore, some multiple attribute group decision making methods based on the proposed operators are developed and some illustrative examples are given to verify our methods. It is worthwhile to mention that the new approaches capture an important feature for group decision making in an intuitionistic trapezoidal fuzzy and interval-valued intuitionistic trapezoidal fuzzy environment: there exists a prioritization relationship over the attributes and decision makers. Therefore, the new proposals are not only more reasonable but more efficient for some real-life applications of group decision making with intuitionistic trapezoidal fuzzy information and interval-valued intuitionistic trapezoidal fuzzy information.

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COMPETING INTERESTS

AUTHORS' CONTRIBUTIONS

'Zhiming Zhang' designed the study, performed the statistical analysis, wrote the protocol, and wrote the first draft of the manuscript. 'Guofang Zhang' managed the analyses of the study and the literature searches. All authors read and approved the final manuscript.

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