

COEFFICIENTS BOUNDS FOR CERTAIN CLASSES OF ANALYTIC FUNCTIONS OF COMPLEX ORDER

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ABSTRACT. In this paper, we determine coefficients bounds for functions in certain sub-classes of analytic functions of complex order, which are introduced here by means of the nonhomogeneous Cauchy-Euler differential equation of order m . Our main result contain some corollaries as special cases.

1. INTRODUCTION AND DEFINITIONS

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (1.1)$$

which are analytic and univalent in the open disk $\mathcal{U} = \{z : |z| < 1\}$. A function $f(z) \in \mathcal{A}$

is said to be starlike of complex order $\gamma (\gamma \in \mathbb{C}^* := \mathbb{C} \setminus \{0\})$ and type $\beta (0 \leq \beta < 1)$, that is $f(z) \in \mathcal{S}_\gamma^*(\beta)$, if and only if

$$\operatorname{Re} \left\{ 1 + \frac{1}{\gamma} \left(\frac{zf'(z)}{f(z)} - 1 \right) \right\} > \beta \quad (z \in \mathcal{U}; \gamma \in \mathbb{C}^*), \quad (1.2)$$

and is said to be convex of complex order $\gamma (\gamma \in \mathbb{C}^*)$ and type $\beta (0 \leq \beta < 1)$, denoted by $\mathcal{C}_\gamma(\beta)$ if and only if

$$\operatorname{Re} \left\{ 1 + \frac{1}{\gamma} \frac{zf''(z)}{f'(z)} \right\} > \beta \quad (z \in \mathcal{U}; \gamma \in \mathbb{C}^*). \quad (1.3)$$

The classes $\mathcal{S}_\gamma^*(\beta)$ and $\mathcal{C}_\gamma(\beta)$ were introduced by the first author in [3]. Note that $\mathcal{S}_\gamma^*(0) = \mathcal{S}_\gamma^*$ and $\mathcal{C}_\gamma(0) = \mathcal{C}_\gamma$ the classes considered earlier by Nasr and Aouf [4] and Wiatrowski [5]. Also, $\mathcal{S}_1^*(\beta) = \mathcal{S}^*(\beta)$ and $\mathcal{C}_1(\beta) = \mathcal{C}(\beta)$ which are, respectively, the familiar classes of starlike functions of order $\beta (0 \leq \beta < 1)$ and convex functions of order $\beta (0 \leq \beta < 1)$.

Let $\mathcal{Q}(\gamma, \lambda, \mu, \beta)$ denote the subclass of $\mathcal{A}$ consisting of functions $f(z)$ which satisfy the following condition

$$\operatorname{Re} \left[1 + \frac{1}{\gamma} \left(\frac{z[\lambda\mu z^2 f''(z) + (\lambda - \mu)zf'(z) + (1 - \lambda + \mu)f(z)]'}{\lambda\mu z^2 f''(z) + (\lambda - \mu)zf'(z) + (1 - \lambda + \mu)f(z)} - 1 \right) \right] > \beta \quad (1.4)$$

where $0 \leq \mu \leq \lambda \leq 1; 0 \leq \beta < 1; \gamma \in \mathbb{C}^*$ and $z \in \mathcal{U}$.

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For $\mu = 0$, the class $\mathcal{Q}(\gamma, \lambda, \mu, \beta)$ reduces to the class $\mathcal{SC}(\gamma, \lambda, \beta)$ introduced by Altıntaş et al. [1]. Clearly, we have $\mathcal{Q}(\gamma, 0, 0, \beta) = \mathcal{S}_\gamma^*(\beta)$ and $\mathcal{Q}(\gamma, 1, 0, \beta) = \mathcal{C}_\gamma(\beta)$.

In the present paper, we propose to derive some coefficient bounds for the class $\mathcal{Q}(\gamma, \lambda, \mu, \beta)$ and also for functions in the subclass $\mathcal{H}(\gamma, \lambda, \mu, \beta, m; \zeta)$ of $\mathcal{A}$, which consists of functions $f(z) \in \mathcal{A}$ satisfying the following nonhomogeneous Cauchy-Euler differential equation of order m :

$$z^m \frac{d^m w}{dz^m} + \binom{m}{1} (\zeta + m - 1) z^{m-1} \frac{d^{m-1} w}{dz^{m-1}} + \cdots + \binom{m}{m} w \prod_{j=0}^{m-1} (\zeta + j) = g(z) \prod_{j=0}^{m-1} (\zeta + j + 1) \quad (1.5)$$

$$(w = f(z); g(z) \in \mathcal{Q}(\gamma, \lambda, \mu, \beta), \zeta \in \mathbb{R} \setminus (-\infty, -1]; m \in \mathbb{N}^* := \mathbb{N} \setminus \{1\} = \{2, 3, \dots\}).$$

2. COEFFICIENT ESTIMATES

We begin by obtaining coefficient bounds for functions in the class $\mathcal{Q}(\gamma, \lambda, \mu, \beta)$.

Theorem 2.1. *Let the function $f(z) \in \mathcal{A}$ be given by (1.1). If $f(z) \in \mathcal{Q}(\gamma, \lambda, \mu, \beta)$, then*

$$|a_n| \leq \frac{\prod_{j=0}^{n-2} [j + 2|\gamma| (1 - \beta)]}{(n-1)! [1 + (\lambda\mu n + \lambda - \mu)(n-1)]} \quad (n \in \mathbb{N}^*), \quad (2.1)$$

where $0 \leq \mu \leq \lambda \leq 1; 0 \leq \beta < 1$, and $\gamma \in C^*$.

Proof. Let the function $F(z)$ be defined by

$$F(z) = \lambda\mu z^2 f''(z) + (\lambda - \mu) z f'(z) + (1 - \lambda + \mu) f(z) \quad (f \in \mathcal{A}; z \in \mathcal{U}). \quad (2.2)$$

Then $F(z)$ is analytic in $\mathcal{U}$ with $F(0) = F'(0) - 1 = 0$. From (1.1) and (2.2) it is easily seen that

$$F(z) = z + \sum_{k=2}^{\infty} A_k z^k \quad (z \in \mathcal{U}).$$

where

$$A_k := [1 + (\lambda\mu k + \lambda - \mu)(k-1)] a_k \quad (k \in \mathbb{N}^*). \quad (2.3)$$

Define the function $p(z)$ by

$$p(z) = \frac{1 + \frac{1}{\gamma} \left(\frac{zF'(z)}{F(z)} - 1 \right) - \beta}{1 - \beta}$$

or, equivalently,

$$zF'(z) - F(z) = \gamma(1 - \beta)(p(z) - 1)F(z) \quad (2.4)$$

then $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$ is analytic in $\mathcal{U}$ and $\operatorname{Re}\{p(z)\} > 0$. Therefore, we have $|c_n| \leq 2$ ($n \in \mathbb{N}$). From (2.4), it follows that

$$(n-1)A_n = \gamma(1 - \beta)(c_{n-1} + c_{n-2}A_2 + \cdots + c_1 A_{n-1}).$$

In particular, for $n = 2, 3, 4$, we have

$$\begin{aligned} |A_2| &\leq 2|\gamma|(1-\beta), \\ |A_3| &\leq \frac{2|\gamma|(1-\beta)[1+2|\gamma|(1-\beta)]}{2!}, \end{aligned}$$

and

$$|A_4| \leq \frac{2|\gamma|(1-\beta)[1+2|\gamma|(1-\beta)][2+2|\gamma|(1-\beta)]}{3!},$$

respectively. Thus, by using the principle of mathematical induction, we obtain

$$|A_n| \leq \frac{\prod_{j=0}^{n-2} [j+2|\gamma|(1-\beta)]}{(n-1)!} \quad (n \in \mathbb{N}^*). \quad (2.5)$$

From (2.3) it is clear that

$$A_n = [1 + (\lambda\mu n + \lambda - \mu)(n-1)]a_n \quad (n \in \mathbb{N}^*). \quad (2.6)$$

Now the inequality (2.1) follows immediately from (2.5) and (2.6). This completes the proof of Theorem 2.1. $\square$

Putting $\mu = \lambda = 1$ in Theorem 2.1, we get the following corollary.

Corollary 2.2. *Let the function $f(z) \in \mathcal{A}$ be given by (1.1) and satisfies the condition*

$$\operatorname{Re} \left[1 + \frac{1}{\gamma} \left(\frac{z[z^2 f''(z) + f(z)]'}{z^2 f''(z) + f(z)} - 1 \right) \right] > \beta \quad (2.7)$$

then

$$|a_n| \leq \frac{\prod_{j=0}^{n-2} [j+2|\gamma|(1-\beta)]}{(n^2 - n + 1)(n-1)!} \quad (n \in \mathbb{N}^*), \quad (2.8)$$

where $0 \leq \beta < 1$, and $\gamma \in C^*$.

Putting $\mu = 0$ in Theorem 2.1, we get the following result obtained by Altıntaş et al. [2].

Corollary 2.3. *Let the function $f(z) \in \mathcal{A}$ be given by (1.1). If $f(z) \in \mathcal{SC}(\gamma, \lambda, \beta)$, then*

$$|a_n| \leq \frac{\prod_{j=0}^{n-2} [j+2|\gamma|(1-\beta)]}{(n-1)![1+\lambda(n-1)]} \quad (n \in \mathbb{N}^*), \quad (2.9)$$

where $0 \leq \lambda \leq 1; 0 \leq \beta < 1$, and $\gamma \in C^*$.

Finally, we prove the following theorem.

Theorem 2.4. *Let the function $f(z) \in \mathcal{A}$ be given by (1.1). If $f(z) \in \mathcal{H}(\gamma, \lambda, \mu, \beta, m; \zeta)$, then*

$$|a_n| \leq \frac{\prod_{j=0}^{n-2} [j + 2|\gamma|(1 - \beta)] \prod_{j=0}^{m-1} (\zeta + j + 1)}{(n-1)! [1 + (\lambda\mu n + \lambda - \mu)(n-1)] \prod_{j=0}^{m-1} (\zeta + j + n)} \quad (m, n \in \mathbb{N}^*), \quad (2.10)$$

where $0 \leq \mu \leq \lambda \leq 1; 0 \leq \beta < 1; \gamma \in C^*$ and $\zeta \in \mathbb{R} \setminus (-\infty, -1]$.

Proof. Let the function $f(z) \in \mathcal{A}$ be given by (1.1). Also let

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k \in \mathcal{Q}(\gamma, \lambda, \mu, \beta).$$

Then from (1.5), we get

$$a_n = \left(\frac{\prod_{j=0}^{m-1} (\zeta + j + 1)}{\prod_{j=0}^{m-1} (\zeta + j + n)} \right) b_n \quad (n \in \mathbb{N}^*; \zeta \in \mathbb{R} \setminus (-\infty, -1]).$$

Thus, by using **Theorem 2.1**, we readily obtain the inequality (2.10). $\square$

Putting $\mu = \lambda = 1$ in Theorem 2.4, we get the following corollary.

Corollary 2.5. *Let the function $f(z) \in \mathcal{A}$ be given by (1.1). If $f(z)$ satisfies the equation (1.5) and $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$ satisfies the condition (2.7), then*

$$|a_n| \leq \frac{\prod_{j=0}^{n-2} [j + 2|\gamma|(1 - \beta)] \prod_{j=0}^{m-1} (\zeta + j + 1)}{(n^2 - n + 1)(n-1)! \prod_{j=0}^{m-1} (\zeta + j + n)} \quad (m, n \in \mathbb{N}^*), \quad (2.11)$$

where $0 \leq \beta < 1; \gamma \in C^*$ and $\zeta \in \mathbb{R} \setminus (-\infty, -1]$.

Putting $\mu = 0$ and $m = 2$ in Theorem 2.4, we get the following result obtained by Altıntaş et al. [2].

Corollary 2.6. *Let the function $f(z) \in \mathcal{A}$ be given by (1.1). If $f(z)$ satisfies the nonhomogeneous Cauchy-Euler differential equation of order 2, given by (1.5) and $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$ satisfies the condition (2.7), then*

$$|a_n| \leq \frac{(\zeta + 1)(\zeta + 2) \prod_{j=0}^{n-2} [j + 2|\gamma|(1 - \beta)]}{(n-1)! [1 + (\lambda\mu n + \lambda - \mu)(n-1)] (\zeta + n)(\zeta + n + 1)} \quad (n \in \mathbb{N}^*), \quad (2.12)$$

where $0 \leq \lambda \leq 1; 0 \leq \beta < 1; \gamma \in C^*$ and $\zeta \in \mathbb{R} \setminus (-\infty, -1]$.

A similar work can also be referred to Eker et al.[6]. In this article they studied the Dziok-Srivastava operator.

Open problem: Is it possible to solve problems related to the Fekete-Szegő theorem as given in [7]? It is yet to be solved.

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REFERENCES

- [1] O. Altıntaş, O. Oskan , H. M. Srivastava, Neighborhoods of a class of analytic functions with negative coefficients, *Applied Mathematics Letters*, 13(3)(2000), 63-67.
- [2] O. Altıntaş, H.Irmak, S. Owa, H.M. Srivastava, Coefficient bounds for some families of starlike and convex functions of complex order, *Applied Mathematics Letters*, 20(2007), 1218-1222.
- [3] B. A. Frasin, Family of analytic functions of complex order, *Acta Mathematica Academiae Paedagogicae Nyíregyháziensis*, 22(2) (2006), 179–191.
- [4] M.A. Nasr and M.K. Aouf, Starlike functions of complex order, *J. Natur. Sci. Math.*, (1985),1-12.
- [5] P. Wiatroski, On the coefficients of some family of holomorphic functions, *Zeszyty Nauk. Uniw. Łódź Nauk. Mat.-Przyrod* 2 (39) (1970), 75-85.
- [6] S.S. Eker, B. Seker and M. Acu, Coefficient bounds for some families of starlike and convex functions of complex order, *General Mathematics* 17 (4) (2009), 195–204.
- [7] A.A. Khaled and M. Darus, On the Fekete-Szegő theorem for the generalized Owa-Srivastava operator, *Proceedings of the Romanian Academy-series A: Mathematics*, 12(3)(2011), 179-188.

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