

On Orthogonal Double Covers of Complete
Bipartite Graphs by an Infinite Certain Graph-Path and Graph-Cycle

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Abstract

Let F be a certain graph, the graph F -Path denoted by $\mathbb{P}_{d+1}(F)$ path of length d with $d + 1$ vertices (i.e. Every edge of this path is one-to-one corresponding to an isomorphic to the graph F). In the same manner, we define the graph F -Cycle as $\mathbb{C}_d(F)$ cycle on d vertices. In this paper, we construct orthogonal double covers (ODCs) of complete bipartite graph $K_{n,n}$ by $\mathbb{P}_{d+1}(F)$ and $\mathbb{C}_d(F)$.

Keywords: Graph decomposition; Orthogonal double cover; Symmetric starter.

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1 Introduction

An ODC of $K_{n,n}$ is a collection $\mathcal{G} = \{G_0, G_1, \dots, G_{n-1}, F_0, F_1, \dots, F_{n-1}\}$ of $2n$ subgraphs called pages) of $K_{n,n}$ such that

(i) every edge of $K_{n,n}$ is in exactly one page of $\{G_0, G_1, \dots, G_{n-1}\}$ and in exactly one page of $\{F_0, F_1, \dots, F_{n-1}\}$;

(ii) for $i, j \in \{0, 1, \dots, n-1\}$ and $i \neq j$, $E(G_i) \cap E(G_j) = E(F_i) \cap E(F_j) = \emptyset$; and $|E(G_i) \cap E(F_j)| = 1$ for all $i, j \in \{0, 1, \dots, n-1\}$.

If all the pages are isomorphic to a given graph G , then $\mathcal{G}$ is said to be an ODC of $K_{n,n}$ by G .

The vertices of the partite sets of $K_{n,n}$ are denoted by $\{0_0, 1_0, \dots, (n-1)_0\}$ and $\{0_1, 1_1, \dots, (n-1)_1\}$. The length of an edge x_0y_1 of $K_{n,n}$ is defined to be the difference $y - x$, where $x, y \in \mathbb{Z}_n = \{0, 1, 2, \dots, n-1\}$. Note that sums and differences are carried over in $\mathbb{Z}_n$ (that is, sums and differences are carried modulo n).

Let G be a subgraph of $K_{n,n}$ and $a \in \mathbb{Z}_n$, then the a -translate of G , denoted by $G + a$ is the edge-induced subgraph of $K_{n,n}$ induced by $\{(x + a)_0(y + a)_1 : x_0y_1 \in E(G)\}$.

A subgraph G of $K_{n,n}$ is called half-starter if $|E(G)| = n$ and the lengths of all edges in G are mutually different. We denote a half-starter G by the vector $v(G) = (v_0, v_1, \dots, v_{n-1})$, where $v_0, v_1, \dots, v_{n-1} \in \mathbb{Z}_n$ and v_i can be obtained from the unique edge $(v_i)_0(v_i + i)_1$ of length i in G .

Two half-starters $v(G) = (v_0, v_1, \dots, v_{n-1})$ and $v(G') = (u_0, u_1, \dots, u_{n-1})$ are said to be orthogonal if $\{v_i - u_i : i \in \mathbb{Z}_n\} = \mathbb{Z}_n$.

For a subgraph G of $K_{n,n}$ with n edges, the edge-induced subgraph G_s with $E(G_s) = \{y_0x_1 : x_0y_1 \in E(G)\}$ is called the symmetric graph of G . Following three results were established in [1].

- I. If G is a half-starter, then the collection of all translates of G forms an edge-decomposition of $K_{n,n}$.
- II. If two half-starter $v(G)$ and $v(F)$ are orthogonal, then the union of the set of translates of G and the set of translates of F forms an ODC of $K_{n,n}$.
- III. G is a symmetric starter if and only if

$$\{v_i - v_{n-i} + i : i \in \mathbb{Z}_n\} = \mathbb{Z}_n.$$

In this paper, we use the usual notation: P_{m+1} for the path on $m+1$ vertices, C_n for the cycle on n vertices, $D \overset{L_v}{\cup} F$ for the union of D and F with all the vertices of the set L_v that belongs to each other. In [2], other definitions not introduced here can be found.

Much of research on this subject focused with the detection of ODCs with pages isomorphic to a given graph G . For a summary of results on ODCs, see [3]. In [3, 4], this concept has been generalized to ODC of any graph H by a graph G .

While in principle any regular graph is worth considering (e.g., the remarkable case of hypercubes has been investigated in [4]), the choice of $H = K_{n,n}$ is quite natural, also in view of a technical motivation: ODCs of such graphs are of help in order to construct ODCs of K_n (see [1], p. 48).

An algebraic construction of ODCs via “symmetric starters” has been exploited to get a complete classification of ODCs of $K_{n,n}$ by G for $n \leq 9$: a few exceptions apart, all graphs G are found this way (see Table 1 in [1]). This method has been applied in [1, 5] to detect some infinite classes of graphs G for which there are ODCs of $K_{n,n}$ by G . In [6], El-Shanawany et al. studied the orthogonal double covers of $K_{n,n}$ by $P_{m+1} \cup^* S_{n-m}$, where n and m are integers, $2 \leq m \leq 10$, $m \leq n$ and $P_{m+1} \cup^* S_{n-m}$ is a tree obtained from the path P_{m+1} with m edges and a star S_{n-m} with $n-m$ edges by identifying an end-vertex of P_{m+1} with the center of S_{n-m} .

Let F be a certain graph, the graph F -Path denoted by $\mathbb{P}_{d+1}(F)$, is a path of set of vertices $\mathbb{V} = \{V_i : 0 \leq i \leq d\}$ and a set of edges $\mathbb{E} = \{E_i : 0 \leq i \leq d-1\}$ if and only if there exists the following two bijective mappings:

1. $\varphi : \mathbb{E} \rightarrow \mathcal{F}$ defined by $\varphi(E_i) = F_i$, where $\mathcal{F} = \{F_0, F_1, \dots, F_{d-1}\}$ is a collection of d graphs each one is isomorphic to the graph F .
2. $\psi : \mathbb{V} \rightarrow \mathcal{A}$ defined by $\psi(V_i) = X_i$, where $\mathcal{A} = \{X_i : 0 \leq i \leq d : \cap_i X_i = \phi\}$ a class of disjoint sets of vertices.

In the same manner, we define the graph F -Cycle as $\mathbb{C}_d(F)$ cycle on d vertices.

Let $F = K_{r,s}$ be a subgraph of $K_{n,n}$, and for all $0 \leq i \leq d$,

$X_i = \{Y_{\frac{i}{2}} : i \equiv 0 \pmod{2}\} \cup \{Z_{\lfloor \frac{i}{2} \rfloor} : i \equiv 1 \pmod{2}\}$, satisfying the following conditions:

- i. $|Y_{\frac{i}{2}}| = r$ and $|Z_{\lfloor \frac{i}{2} \rfloor}| = s$, for all $0 \leq i \leq d$.
- ii. $Y_{\frac{i}{2}} \subset \mathbb{Z}_n \times \{0\}$, $Z_{\lfloor \frac{i}{2} \rfloor} \subset \mathbb{Z}_n \times \{1\}$, and $\cap_i Y_{\frac{i}{2}} = \cap_i Z_{\lfloor \frac{i}{2} \rfloor} = \phi$.
- iii. For $0 \leq i \leq d-1$, $F_i \cong K_{r,s}$ has the following edges:

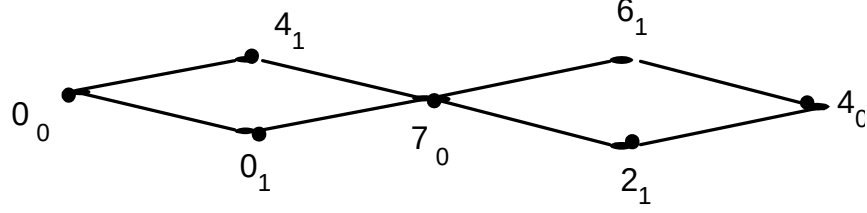


Figure 1: Symmetric starter of an ODC of $K_{8,8}$ by $G = \mathbb{P}_3(K_{2,2})$ w.r.t. $\mathbb{Z}_8$

$$E(F_i) = \{Y_{\frac{i}{2}}Z_{\frac{i}{2}} : i \equiv 0 \pmod{2}\} \cup \{Y_{\lfloor \frac{i}{2} \rfloor}Z_{\lceil \frac{i}{2} \rceil} : i \equiv 1 \pmod{2}\}.$$

Consider $s \geq 0$ paths of length $k \geq 1$, all attached to the same vertex (root vertex). This tree will be called $T(s, k)$. Clearly, $T(s, 1)$ is the star with s edges and $T(1, k)$ is the path with k edges. Define L_s is a set of all leaves of $T(s, k)$ i.e. $L_s = \{v : v \text{ is a leaf in } T(s, k)\}$ and $|L_s| = s$.

In this paper, we are concerned with an ODC of $K_{n,n}$ by $G = \mathbb{P}_{d+1}(F)$, Where $F = T(s, k) \overset{L_s}{\cup} K_1$ and $F = \mathbb{C}_d(K_{2,2})$.

2 Main result

El-Shanawany et al. in [7] proved that the vector

$$v(G) = (0, n-1, n-2^2, n-3^2, \dots, n-3^2, n-2^2, n-1) \quad (1)$$

is a symmetric starter of an ODC of $K_{n,n}$ by G , where n is a positive integer. Moreover, for any $i \in \mathbb{Z}_n$, the i^{th} graph isomorphic to G has the edges:

$$E(G_i) = \{(n-i^2)_0 (n-i(i-1))_1 : 0 \leq i \leq n-1\} \quad (2)$$

As a direct application of the vector in equation 1 and its edges in equation 2 for special case $n = 2^3$, see figure 1.

Theorem 1 Let $m \geq 2$ be a positive integer. Then there exists a symmetric starter of an ODC of $K_{3^m, 3^m}$ by $G = \mathbb{P}_{3^{m-2}+1} \left(T(3, 3) \overset{L_3}{\cup} K_1 \right)$.

Proof. The result follows from the vector in equation 1 and its edges in equation 2 with $|E(G)| = 3^m = 9d$ imply that $d = 3^{m-2}$ define the number of graphs isomorphic to $F = T(3, 3) \overset{L_3}{\cup} K_1$. As a direct application of this Theorem, see figure 2. ■

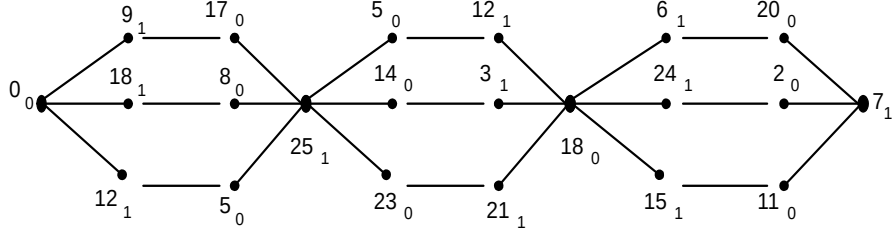


Figure 2: Symmetric starter of an ODC of $K_{27,27}$ by $G = \mathbb{P}_4 \left(T(3, 3) \overset{L_3}{\cup} K_1 \right)$ w.r.t. $\mathbb{Z}_{27}$

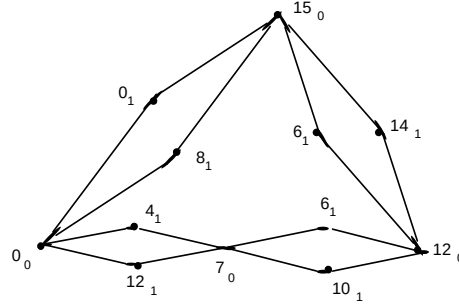


Figure 3: Symmetric starter of an ODC of $K_{16,16}$ by $G = \mathbb{P}_2 (\mathbb{C}_4 (K_{2,2})) = \mathbb{C}_4 (K_{2,2})$ w.r.t. $\mathbb{Z}_{16}$

Theorem 2 Let $m \geq 4$ be a positive integer. Then there exists a symmetric starter of an ODC of $K_{2^m, 2^m}$ by $G = \mathbb{P}_{2^{m-4}+1} (\mathbb{C}_4 (K_{2,2}))$.

Proof. The result follows from the vector in equation 1 and its edges in equation 2 with $|E(G)| = 2^m = 16d$ imply that $d = 2^{m-4}$ define the number of graphs isomorphic to $F = \mathbb{C}_4 (K_{2,2})$ (i.e. the $K_{2,2}$ -Cycle with length 4). As a direct applications of this Theorem, see figure 3 and 4. ■

3 Conclusion

In this paper we are concerned with a symmetric starter vector of an ODC of $K_{n,n}$ by certain new classes of graphs such as graph-Path and graph-Cycle. In conclusion, we pose the following conjectures:

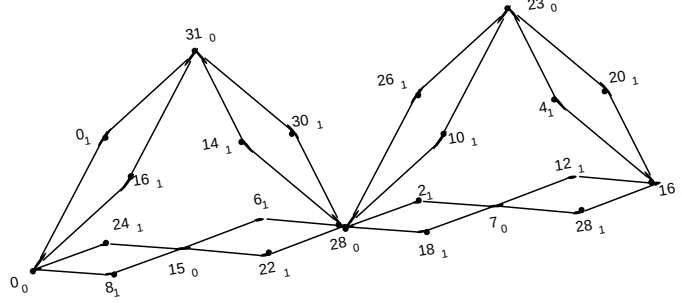


Figure 4: Symmetric starter of an ODC of $K_{32,32}$ by $G = \mathbb{P}_3(\mathbb{C}_4(K_{2,2}))$ w.r.t. $\mathbb{Z}_{32}$

Conjecture 3 Let n, d, s, k be positive integers. Then there exists a symmetric starter vector of an ODC of $K_{n,n}$ by $G = \mathbb{P}_{d+1}\left(T(s, k) \cup^{L_s} K_1\right)$.

Conjecture 4 Let n, d, r, s be positive integers, then there is a symmetric starter vector of an ODC of the complete bipartite graph $K_{n,n}$ by $G = \mathbb{P}_{d+1}(\mathbb{C}_d(K_{r,s}))$.

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