

THE MULTI-FRACTAL SPECTRUM MODEL FOR THE MEASUREMENT OF RANDOM BEHAVIOUR OF ASSETS PRICE RETURNS.

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Abstract

To forecast the market risk, assessing the stock price indices is the foundation. Multi-fractal has lots of advantage when explaining the volatility of the stock prices. The asset price returns is a multi-period (multi-fractal dimension) market depending on market scenarios which are the measure points. This paper considers the multi-fractal spectrum model (MSM) to measure the random character of asset price returns, aimed at deriving the MSM version of the random behaviour of equity returns of the existing ones in literature. We investigate the rate of returns prior to market signals corresponding to the value for packing dimension in fractal dispersion of Hausdorff measure. Furthermore, we give some conditions which determine the equilibrium price, the future market price and the optimal trading strategy.

Introduction

The multifractal formalism was introduced in the context of fully-developed turbulence data analysis and modeling to account for the experimental observation of some deviation to Kolmogorov theory (K41) of homogenous and isotropic turbulence (Frisch, 1995). The predictions of various multiplicative cascade models, including the weighted curdling (binomial) model proposed by Mandelbrot (1974), were tested using box-counting (BC) estimates of the so-called $f(\alpha)$ singularity spectrum of the dissipation field (Meneveau & Sreenivasan, 1991). Alternatively, the intermittent nature of the velocity fluctuations were investigated via the computation of the $D(h)$ singularity spectrum using the structure function (SF) method. In financial economics, multi-fractal spectrum model (MSM) has been used to analyze the pricing implications of multi-frequency risk. The models have had some success in explaining the excess

volatility of stock returns compared to fundamentals and the negative skewness of equity returns. They have also been used to generate multi-fractal jump–diffusion (Calvet et al, 2008).

The multifractal formalism of multi-affine functions amounts to compute the so-called singularity spectrum $D(h)$ defined as the Hausdorff dimension of the set where the Hölder exponent is equal to h . Once one has the function, $h(t)$, characterizing the Holder exponent, one can construct the singularity spectrum. It is believed that the singularity spectrum is a synonym to the multi- fractal spectrum.

The specifics of how to calculate $h(t)$ and $D(\alpha)$ in practice vary from approximating them outright by their definition or by using wavelets to approximate the Holder exponent and then using a Legendre transform to approximate the multi-fractal spectrum.

MSM is a stochastic volatility model (Taylor and Steven, 2008 and Wiggins, 1987) with arbitrarily many frequencies. MSM builds on the convenience of regime –switching models, which were advanced in economics and finance (Hamilton, 2008). MSM is closely related to the multi-fractal model of Asset Returns (MMAR) (Mandelbrot et al, 1997). MSM improves on the MMAR’s combinatorial construction by randomizing arrival times, guaranteeing a strictly stationary process. MSM provides a pure regime-switching formulation of multi-fractal measures, which were pioneered by Benoit Mandelbrot (Mandelbrot, 2006, 1983 and Mandelbrot et al, 1999).

Extensions of MSM to multiple assets provide reliable estimates of the value-at-risk in a portfolio of securities (Calvet et al, 2006). MSM often provides better volatility forecasts than some of the best traditional models both in and out of sample. Calvet and Fisher, (2004) report considerable gains in exchange rate volatility forecasts at horizons of 10 to 50 days as compared with GARCH(1,1), Markov-switching GARCH, (Gray, 1996 and Klaassen, 2002) and Fractionally Integrated GARCH (Bollerslev et al, 1996). Lux, (2008) obtains similar results using linear predictions. Building on the multifractal formalism presented earlier, MMAR represents a simple compound process that has a closed-form multifractal spectrum. MMAR produces volatility clustering, heavy tails and long memory in returns.

Our aim in this paper is to apply some tools of multifractal analysis to the measurement of random behaviour of the stock market price returns. The problem associated with random behaviour of stock exchange has been addressed extensively by many authors (see for example, Black and Scholes, 1973 and Black and Karasinski, 1991). Hull and White (1987) among others followed the traditional approach to pricing options on stocks with stochastic volatility which starts by specifying the joint process for the stock price and its volatility risk. Their models are typically calibrated to the prices of a few options or estimated from the time series of stock prices. On the other hand, Ugbebor et al (2001) considered a stochastic model of price changes at the floor of stock market. Here the equilibrium price and the market growth rate of shares were determined. There have been some works with considerable extensions and constraints

subsequently (see Osu and Okoroafor, 2007 and Osu et al, 2010]). In this paper we present a dynamic stochastic multi-fractal spectrum model of variation of the stock returns aimed at deriving the MSM version of the random behaviour of equity returns of the existing ones in literature. The equilibrium price and growth rate of an asset, using a linear rate of return r_t are determined. We first developed a method of computing the so called singularity spectrum which is equivalent to computing the Holder exponent. Furthermore, the optimal trading strategy is determined.

BASIC TOOLS AND PRELIMINARIES

Let $(P_t)_{t \geq 0}$ denote the price process of a security, in particular of a stock. To allow comparison of investments in different securities, it is necessary to investigate the rates of return defined by

$$Z_t = \ln P_t - \ln P_{t-1} \quad (1)$$

Most authors prefer these rates, which correspond to continuous compounding, to the alternative

$$Y_t = \frac{(P_t - P_{t-1})}{P_{t-1}} \quad (2)$$

The reason for this is that the return over n periods, for example n days, is then just the sum (Eberlein and Keller, 1995).

$$Z_t + \dots + Z_{t-n-1} = \ln P_{t+n-1} - \ln P_{t-1}. \quad (3)$$

As a model for stock prices the natural candidate is now the multi-fractal process $(X_t^{d,n})_{t \geq 0}$

$$dY_t = pY_t dt + Y_t dZ_t^{d,n} \quad (4)$$

This can also be written in the form $dY_t = Y_t dZ_t$ with $Z_t = X_t^{d,n}$. The solution of this equation is the Doleans-Dade exponential (Eberlein and Keller, 1998).

$$Y_t = Y_0 \exp(X_t^{d,n}) \prod_{0 \leq j \leq t} (1 + \Delta X_j^{d,n}) e^{-\Delta X_j^{d,n}} \quad (5)$$

The MSM model can be specified in both discrete time and continuous time.

Discrete time

Let P_t denote the price of a financial assets, and let

$$Z_t = \ln(P_t/P_{t-1}) \quad (6)$$

denote the returns over two consecutive periods .In MSM, returns are specified as

$$r_t = \mu + \bar{\sigma}(M_{1,t}M_{2,t} \dots M_{\bar{k},t})^{1/2}\xi_t, \quad (7)$$

where μ and σ are constants and $\{\xi_t\}$ are independent standard Gaussians. Volatility is driven by the first-order latent Markov state vector:

$$M_t = (M_{1,t}M_{2,t} \dots M_{\bar{k},t}) \in R_+^{\bar{k}}. \quad (8)$$

Given the volatility state M_t the next-period multiplier $M_{k,t+1}$ is drawn from a fixed distribution M with probability γ_k and is otherwise left unchanged $M_{k,t}$ drawn from distribution M with probability γ_k , with probability $1 - \gamma_k$

$$M_{k,t} = M_{k,t-1}. \quad (9)$$

The transition probability is specified by

$$\gamma_k = 1 - (1 - \gamma_1)^{(b^{k-1})} \quad (10)$$

The sequence γ_k is approximately geometric with $\gamma_k \approx \gamma_1^{b^{k-1}}$ at low frequency. The marginal distribution M has a unit mean, has a positive support, and is independent of K . In empirical applications, the distribution M is often a discrete distribution that can take the values m_0 or $2 - m_0$ with equal probability. The return process r_t is then specified by the parameters $\theta = (m_0, \mu, \bar{\sigma}, b, \gamma_1)$. Note that the number of parameters is the same for all $\bar{k} > 1$.

Continuous time

MSM is similarly defined in continuous time. The price process follows the diffusion:

$$\frac{dp_t}{p_t} = \mu dt + \sigma(M_t) dW_t, \quad (11)$$

where $\sigma(M_t) = \bar{\sigma}(M_{1,t} \dots M_{\bar{k},t})^{1/2}$, W_t is a standard Brownian motion, and μ and $\bar{\sigma}$ are constants. Each component follows the dynamics: $M_{k,t}$ drawn from distribution M with probability $\gamma_k dt$. $M_{k,t+dt} = M_{k,t}$ with probability $1 - \gamma_k dt$. The intensities vary geometrically with K : $\gamma_k = \gamma_1^{b^{k-1}}$. When the number of components $\bar{k}$ goes to infinity, continuous-time MSM converges to a multi-fractal diffusion, whose sample paths take a continuum of local Holder exponents on any finite time interval.

When M has a discrete distribution, the Markov state vector M_t takes finitely many values $m^1, \dots, m^d \in R_+^{\bar{k}}$. For instance, there are $d = 2^{\bar{k}}$ possible states in binomial MSM.

126 The Markov dynamics are characterized by the transition matrix $A = (a_{i,j})_{1 \leq i,j \leq d}$ with
 127 components;

$$128 \quad a_{i,j} = P(M_{t+1} = m^j | M_{tt} = m^i). \quad (12)$$

129 Conditional on the volatility state, the return r_t has Gaussian density

$$130 \quad f(r_t | M_t = m^i) = \frac{1}{\sqrt{2\pi\sigma^2(m^i)}} \exp \left[-\frac{(r_t - \mu)^2}{2\sigma^2(m^i)} \right]. \quad (13)$$

131 Furthermore, $M_t = M_{1t}, M_{2t}, \dots, M_{kt}$ is the multiplicative measure defined at different stages. In
 132 fact, $M_b (0 \leq b \leq b-1)$ are the non-negative multipliers with arbitrary distribution. The product
 133 of k multiplier is defined by (Calvet and Fisher, 1996);

$$134 \quad \mu(\Delta t) = M_{1t} M_{2t} \dots M_{kt}, \quad (14)$$

135 with the scaling relationship $\mathbb{E}[(\mu(\Delta t))^q] = [\mathbb{E}(M)^q]$ or

$$136 \quad [\mathbb{E}(M)^q] = (\Delta t)^{\tau(q)+1}, \quad (15)$$

137 where $\tau(q) = -\log_b \mathbb{E}(M)^q - 1$.

138 If the latest observation P_t of the designated process at time $t > d$ is conditioned on the
 139 information;

$$140 \quad \varphi_{t-1} = \{P_u - \mu_u; u = 1, 2, \dots, t-1\} \quad (16)$$

141 available up to time $t-1$, then

$$142 \quad P_t | \varphi_{t-1} \sim (\mu_t, \sigma^2(M_t)) \quad (17)$$

143 where (using (10)),

$$144 \quad \sigma^2(M_t) = (\Delta t)^{\tau(q)+1} h(\xi_{t-1}, \xi_{t-2}, \dots, \xi_{t-d}, \alpha) \quad (18)$$

145 and

$$146 \quad \xi_u = P_u - \mu_u, u = 1, 2, \dots, t-1. \quad (19)$$

147 The function $h(\cdot)$ with parameters $\alpha = \{\alpha_k; k = 0, 1, \dots, d\}$ is defined as

$$148 \quad h(\xi_{t-1}, \xi_{t-2}, \dots, \xi_{t-d}, \alpha) = \alpha_0 + \sum_{k=1}^d \alpha_k \xi_{t-k}^2, \quad (20)$$

149 so that given the history (18) of the process up to time $t-1$, the conditional distribution of P_t is
 150 normal with mean μ_t and variance

$$\sigma^2(M_t) = (\Delta t)^{\tau(q)+1}(\alpha_0 + \sum_{k=1}^d \alpha_k (P_{t-k} - \mu_{t-k})^2). \quad (21)$$

Formally, if the price of an asset at time t is P_t for $t \in [0, T]$ and $T < \infty$, then the associated log-price process is given by

$$Z_t = \ln P_t - \ln P_0, \quad t \in [0, T].$$

The log-price process is then modeled as a compound process

$$Z_t = W[\theta(t)], \quad t \in [0, T],$$

where $W(t)$ is Brownian motion and $\theta(t)$ is a stochastic process termed trading time (which is the cumulative distribution function (c.d.f.) of a random multifractal measure, and the processes $W(t)$ and $\theta(t)$ are assumed to be independent.

The fractal dimension is the basic notion for describing structures that have a scaling symmetry. Scaling symmetry means self-similarity of the considered object on varying scale of magnification.

Hausdorff (1919) introduced the first notion of dimension, providing a measure for filling space which allows for the possibility of non-integral dimensions. Then D -dimensional Hausdorff measure on a set A is given by

$$M_D(A) = \lim_{r_k \rightarrow 0} \sum_k r_k^D, \quad D > 0. \quad (22)$$

Define the optimal covering of this set using spheres of variable radius r_k . The Hausdorff-dimension D_H is the value of D at which M_D jumps from ∞ to 0, while the dimension D can be calculated as

$$D = \frac{\log N}{\log S}, \quad (23a)$$

(N is the number of small pieces that go into the larger one and S is the scale to which the smaller pieces compare to the larger one). Equivalently for a given precision level $\varepsilon > 0$, $N(\varepsilon)$ satisfies a power law as $\varepsilon \rightarrow 0$ so that

$$N(\varepsilon) \sim \varepsilon^{-D}. \quad (23b)$$

In equation (23b), D is a constant called the fractal dimension, which helps to analyze the structure of a fixed multifractal. For a large class of multifractals, the dimension $D(\alpha)$ coincides with the multifractal spectrum. For any $\alpha \geq 0$, the set $T(\alpha)$ can be defined as the Hölder exponent α with a fractal dimension $D(\alpha)$ satisfying $0 \leq D(\alpha) \leq 1$.

179 Consider the set A given by $A = \{a: 0 \leq D \leq b - 1\}$. That is we can as well consider non-
 180 negative multipliers $M_D (0 \leq D \leq b - 1)$ with arbitrary distribution (Calvet and Fisher, 1996). If
 181 the multipliers are identically distributed ($M_D \underline{d} M \setminus D$), and independent at different stages of the
 182 construction, the limit multiplicative measure is so called conservative when mass is conserved
 183 exactly at each stage. Thus $\mathbb{E}(\Sigma M_D) = 1$ or equivalently $M = 1/b$.

184 If A is any set in $\mathbb{R}^n$, the Hausdorff measure of a subset A of $\mathbb{R}^n$ is defined by

$$185 \quad h - m(A) = \lim_{\delta \rightarrow 0} \left\{ \sum_{A \sqsubset C_i} \inf h(d(c_i)) \right\}, \quad (24)$$

186 where the infimum extends over all countable covers of A by sets c_i of diameter $d(c_i) < \delta$, h is
 187 monotone, right continuous and $h(0^+) = 0$.

188 Let μ be a finite Borel measure in $\mathbb{R}^n$, consider the spherical density of U at x given by

$$189 \quad D_h(u, x) = \lim_{r \rightarrow 0} \sup \frac{u(B(x, r))}{h(r)}, \quad (25)$$

190 where $B(x, r)$ is a ball of radius r and center x . The behavior of this density on a set $E \subset \mathbb{R}^n$ as a
 191 function of x connects $h(E)$ with the size of E in some sense. For example, in (25), we have that
 192 if

$$193 \quad \lim_{r \rightarrow 0} \frac{u(E \cap B(x, r))}{h(r)} \leq K, \quad \forall B(x, r) \in E, \quad (26)$$

194 then

$$195 \quad h - m(E) \geq \lambda k^{-1} \mu(E), \quad (27)$$

196 where λ is a constant (Taylor, 1953, 1961).

197 In order to apply the density on random sets, we need to construct a suitable Borel finite
 198 measure U on $\mathbb{R}^n$ with support in E using the sample path of a process. It is natural to try a
 199 measure which is uniformly spread over E , so an obvious candidate for $E(w) = X_t^{d,n}$ where $X_t^{d,n}$
 200 is a stochastic process in $\mathbb{R}^n$ of index d , is the occupation measure defined by

$$\begin{aligned} u(B(0, r)) &= |\{E \in \mathbb{R}^n: X_t^{d,n} \in B(0, r)\}| \\ &= \int_0^\infty I_{B(0, r)} X_t^{d,n} dt \\ &= T(r) \end{aligned} \quad (28)$$

203 Equation (28) is called the sojourn time and I is the indicator function.

This density theorem is necessary because the definition of Hausdorff measure (25) requires us to consider all possible coverings of E by small sets in order to show that $h - m(E) > 0$.

Hence to examine random sets using Hausdorff measure, then important tool is $T(r)$ process, i.e. time spent in a ball of radius r . Thus (25) which becomes

$$\limsup \frac{T(r)}{h(r)} \quad (29)$$

is relevant to Hausdorff measure.

Let $(\mathbb{R}^n, \beta(\mathbb{R}^n))$ be a measurable space and let $f: \beta(\mathbb{R}^n) \rightarrow \mathbb{R}$ be a measurable function. Let λ be a real valued function defined on $\beta(\mathbb{R}^n)$. Then the multi-fractal spectrum with respect to the function f and λ is defined by

$$D(\alpha) = \lambda\{x \in \beta(\mathbb{R}^n): f(x) = \alpha\}. \quad (30)$$

In multi-fractal analysis, the function $D(\cdot)$ is usually taken to be the Hausdorff dimension (Jaffard, 1999 and xiao, 2004). The basic problem is to calculate the function $D(a)$. To do this, we need to know what the function $f(x)$ is.

Xiao (2004) defined

$$f(x) = \limsup \frac{\cup(B(x,r))}{r^\alpha (\log r)} \quad (31)$$

and show that

$$\dim E = \left\{ x \in \mathbb{R}^n: \limsup_{t \rightarrow 0} \frac{\cup(B(x,r))}{r^2 |\log r|} = \alpha \right\} = 2 - \frac{aq_n^2}{2}$$

for the Brownian motion X_t in $\mathbb{R}^n, n \geq 3$ for all $0 \leq \alpha \leq \frac{4}{q_n^2}$ and q_n is the first position zero of the Bessel function $J_{n/2-2}(x)$.

On the other hand, Uzoma (2006) defined another gauge function given by

$$f(x) = \limsup \frac{\cup(B(x,r))}{r^{2(\log r)^\lambda}}, \quad (32)$$

and show that if $\lambda > 1$ then

$$\dim E = \left\{ x \in \mathbb{R}^n: \limsup \frac{\cup(B(x,r))}{r^{2|\log r|^\lambda}} = 0 \right\} = 2, a.s. \quad n > 3. \quad (33)$$

226 The multifractal formalism of multi-affine functions amounts to compute the so-called
 227 *singularity spectrum* $D(\alpha)$ defined as the Hausdorff dimension of the set where the Hölder
 228 exponent is equal to α (Arneodo et al., 1995).

229 To obtain the function $f(x) = \limsup \frac{T(r)}{h(r)}$ in our case, we require the local asymptotic behavior
 230 of the sample path of the process. And what comes to mind is the subject of the law of iterated
 231 logarithm (LIL).

232 To this end we assume a double stochastic integrals by a direct adaptation of the case of the
 233 Brownian motion and set

$$234 \quad h(t) = 2t \log \log \frac{1}{t} \text{ for } t > 0. \quad (34)$$

235 In what follows, we now state;

236

237 **Theorem 1:**

238 For $t > 0$ and $h(t) = 2t \log \log \frac{1}{t}$, the so-called *singularity spectrum* $D(\alpha)$ defined as the
 239 Hausdorff dimension of the set where the Hölder exponent is equal to α is given by

$$240 \quad \lim_{t \rightarrow 0} \sup \frac{X_t^d}{h(t)} = \frac{(1+\gamma)^2}{2\theta}, \theta = 1, \gamma \in [0, 1]. \quad (35)$$

241 **Proof**

242 Let d be a predictable process valued in a bounded interval $[\alpha_0, \alpha_1]$ for some real
 243 parameters $0 \leq \alpha_0 \leq \alpha_1$, and $X_t^d := \int_0^t \int_0^u a_r dW_r dW_u$.

$$244 \quad \text{Then} \quad \alpha_0 \leq \lim \frac{2X_t^d}{h(t)} \leq \alpha_1 \text{ a.s.}$$

$$245 \quad \text{Now set} \quad \bar{\alpha} = \frac{(\alpha_0 + \alpha_1)}{2} \geq 0$$

246 and

$$247 \quad \delta := \frac{(\alpha_1 + \alpha_0)}{2}.$$

248 For the first inequality, we have by the law of the iterated logarithm for the Brownian motion.

$$\bar{\alpha} = \lim_{t \rightarrow 0} \sup \frac{2X_t^d}{h(t)} \leq \delta \lim_{t \rightarrow 0} \sup \frac{2X_t^d}{h(t)} + \lim_{t \rightarrow 0} \sup \frac{2X_t^d}{h(t)},$$

249 where $\check{d} = \delta^{-1}(\bar{\alpha} - d)$ is the value in $[-1, 1]$. It then follows from the second inequality that;

$$\limsup_{t \rightarrow 0} \frac{2X_t^d}{h(t)} \geq \bar{\alpha} - \delta = \beta_0.$$

250 For the prove of the second inequality, we assume without loss of generality that $\|d\|_\infty \leq 1$.

251 Let $T > 0$ and $\lambda > 0$ be such that $2\lambda T < 1$. Then from Doob's maximal inequality for sub-
252 martingales for all $\beta \geq 0$, we have

$$\begin{aligned} 253 \quad P[\max_{0 \leq t \leq T} 2X_t^d \geq \beta] &= P[\max_{0 \leq t \leq T} \exp(2\lambda X_t^d) \geq \exp(\lambda\beta)] \\ &\leq e^{-\lambda\beta} E[e^{2\lambda X_T^d}] \end{aligned}$$

$$254 \quad = e^{-\lambda(\beta+T)}(1 - 2\lambda T)^{-1/2}. \quad (36)$$

255 Take $\theta, \gamma \in (0, 1)$ and set all $K \in \mathbb{N}$

$$\beta_k = (1 + \gamma)^2 h(\theta^k)$$

256 and

$$257 \quad \lambda_k = [2\theta^k(1 + \gamma)]^{-1}.$$

258 Applying (22), we see that for all $K \in \mathbb{N}$,

$$259 \quad P\left[\max_{0 \leq t \leq \theta^k} 2X_t^d \geq (1 + \gamma)^2 h(\theta^k)\right] \leq e^{-1/2(1+\gamma)}(1 + \gamma^{-1})(-k \log \theta)^{-(1+\gamma)}.$$

260 It follows from the Borel-Cantelli lemma that for almost all $w \in \Omega$ and since

$$261 \quad \sum_{k > 0} \frac{1}{K^{(1+\gamma)}},$$

262 there exists a natural number $K^{\theta, \gamma}(w)$ such that for all $k \geq K^{\theta, \gamma}(w)$,

$$263 \quad \max_{0 \leq t \leq \theta^k} 2X_t^d(w) < (1 + \gamma)^2 h(\theta^k).$$

264 In particular, for all $t \in (\theta^{k+1}, \theta^k)$,

$$265 \quad 2X_t^d(w) < (1 + \gamma)^2 h(\theta^k) \leq (1 + \gamma)^2 \frac{h(t)}{\theta}.$$

266 Hence

$$267 \quad \lim_{t \rightarrow 0} \sup \frac{X_t^d}{h(t)} \leq \frac{(1+\gamma)^2}{2\theta} \text{ a.s.}$$

268 **The equilibrium price and growth rate**

269 **The particular case**

270 When the mean interest rate of some stocks does not depend on other stocks in the market, we
 271 consider again the following stochastic differential equation (SDE);

272
$$dP_t = P_t(\mu dt + \sigma(M_t)dW_t), \quad P_0 = \varepsilon_0. \quad (37)$$

273 Equation (35) is our so called multifractal spectrum model defined in continuous time. The most
 274 probable path $\varphi(t)$ associated with this equation satisfies

275
$$\varphi'(t) = \left(\mu - \frac{1}{2}\sigma^2(M_t)\right)\varphi, \quad \varphi(0) = \varepsilon. \quad (38)$$

276 Equation (36) means that an investor has invested his money in a stock with a linear mean return
 277 μ_t and volatility $\sigma(M_t)$ and his real return rate is most likely to be given by

278
$$c(t) = \mu - \frac{1}{2}\sigma^2(M_t), \quad (39)$$

279 Instead of the usual μ .

280 The Ito's Formula on (35) leads to the model equation

281
$$-\frac{\partial W}{\partial t} = \mu P \frac{\partial W}{\partial P} + \frac{1}{2}\sigma^2(M_t)P^2 \frac{\partial^2 W}{\partial P^2} - rW, t \geq t_0, P_0 = \varepsilon_0, \quad (40)$$

282 where $W = W(P_t, K_t)$ is the investment worth and K_t is the investment over period t . Following
 283 the assumption in Osu and Okoroafor (2007) and the references therein, we have (40) reduced to
 284 an ordinary differential equation of the form

285
$$\mu P \frac{dW}{dP} + \frac{1}{2}\sigma^2(M_t)P^2 \frac{d^2 W}{dP^2} - rW = -P. \quad (41)$$

286 The solution of (39) by the method of change of variable using Euler's substitution is;

287
$$W(P) = AP^{\lambda_1} + \frac{P}{r-\mu}, \quad (42)$$

288 with

289
$$A\lambda_1 \hat{P}^{\lambda_1} + \frac{\hat{P}}{r-\mu} = 0, \quad (43)$$

290 where

291
$$\lambda_1 = \frac{-\left(\mu - \frac{1}{2}\sigma^2(M_t)\right) + \sqrt{\left(\mu - \frac{1}{2}\sigma^2(M_t)\right)^2 + 2\sigma(M_t)r}}{\sigma^2(M_t)}, \quad (44)$$

292 is the positive characteristics root of (41).

293 We have assumed here that $W(P)$ is twice differentiable such that

294 $W(0) = 0$ and $\frac{dW(P)}{dP} = 0$.(45)

295 Using equations (7), (15), (19) and (21), we have (40) as

296 $W(P) = AP^{\lambda_1} + \frac{(\Delta t)^{-(\tau(q)+1)}P}{(\alpha_0 + \sum_{k=1}^d \alpha_{k+1}(P_{t-k} - \mu_{t-k})^2)},$ (46)

297 with (using 45b)

298 $A\lambda_1 \hat{P}^{\lambda_1} - \frac{\mu(\Delta t)^{-(\tau(q)+1)}}{\sigma(\hat{P}_u - \mu_u)^2} = 0.$ (47)

299 Equation (46) is the worth growth rate of an investor under MSM.

300 Under equilibrium condition, the discounted profit from a unit capacity at $\hat{P}$ must be equal to the
301 expected unit cost $\bar{P}$ of the risky stock option. This therefore implies that by (47), we have

302 $W(P) = A\lambda_1 \hat{P}^{\lambda_1} - \frac{\mu(\Delta t)^{-(\tau(q)+1)}}{\sigma(\hat{P}_u - \mu_u)^2} = \bar{P}.$ (48)

303 Solving for A in (47) and (48) and equating the results gives;

304 $\hat{P} = \frac{\lambda_1 \sigma (\Delta t)^{(\tau(q)+1)} \mu \bar{P}}{(\lambda_1 - 1)(\sigma (\Delta t)^{(\tau(q)+1)} \bar{P}) - 1}.$ (49)

305 The general case

306 In the real world in general, markets are neither ideal nor complete. Therefore the model
307 equation (40) can hardly be seen as real world behaviour of a stock market price.

308 Consider a market comprising h unit of asset in long position and one unit of the option in short
309 position. At time T the market value is assumed to be $h - p(E)$ or $h(s) - v$.

310 After an elapse, Δt , the value of the market will change by an amount/rate $h(\Delta P + \vartheta \Delta t) - \Delta W$, in
311 view of the dividend received on h unit held, where ϑ is the market price of risk. By Ito's lemma
312 we have

313 $h(\mu P \Delta t + \sigma P \Delta t + \vartheta \Delta t) = \left[\left(\frac{\partial W}{\partial t} + \frac{\partial W}{\partial P} \mu P + \frac{1}{2} \frac{\partial^2 W}{\partial P^2} \sigma^2 P^2 \right) \Delta t + \frac{\partial W}{\partial P} \sigma P \Delta z \right],$

314 or

315 $\left[h\mu P + h\vartheta - \left(\frac{\partial W}{\partial t} + \frac{\partial W}{\partial P} \mu P + \frac{1}{2} \frac{\partial^2 W}{\partial P^2} \sigma^2 P^2 \right) \right] \Delta t = \left[h\sigma P - \frac{\partial W}{\partial P} \sigma P \right] \Delta t.$ (50)

316 Take $h = \frac{\partial W}{\partial P}$, then the uncertainty term disappears and the market in this case is temporarily
 317 riskless (no signal). It should therefore grow in value by the riskless rate in force i.e.

$$318 \left(h\mu P + h\vartheta - \left[\frac{\partial W}{\partial t} + \frac{\partial W}{\partial P} \mu P + \frac{1}{2} \frac{\partial^2 W}{\partial P^2} \sigma^2 P^2 \right] \right) \Delta t = (hP - W)r\Delta t. \quad (51)$$

319 Thus,

$$320 \vartheta \frac{\partial W}{\partial P} - \left(\frac{\partial W}{\partial t} + \frac{1}{2} \frac{\partial^2 W}{\partial P^2} \sigma^2 P^2 \right) = \left(\frac{\partial W}{\partial P} P - W \right) r, (52)$$

321 so that

$$322 \frac{\partial W}{\partial t} + (rP - \vartheta) \frac{\partial W}{\partial P} + \frac{1}{2} \frac{\partial^2 W}{\partial P^2} \sigma^2 P^2 = rW. \quad (53)$$

323 Under the following dynamics

$$324 dP_t = \alpha(t)P_t dt + \sigma(M_t)P_t dW(t),$$

325 we have the MSM version of the parabolic partial differential equation (53) with $\vartheta = 0$ as;

$$326 - \frac{\partial W(P,t)}{\partial t} = rP \frac{\partial W(P,t)}{\partial P} + \frac{1}{2} \sigma^2(M_t)P^2 \frac{\partial^2 W(P,t)}{\partial P^2} - rW(P,t), \forall (P,t) \in (0, \infty) \times (0, T), \quad (54)$$

327 where $\alpha(t) = \ln \left(\frac{P_{t+\Delta t} - P_t}{\Delta t} \right)$ is the rate of stock price changes at time t .

328 **Theorem 2**

329 Let $W(P, t)$ be the investment output, r the linear discount rate (as in equation (7)) and P the
 330 stock prices. The PDE

$$\frac{1}{2} \sigma^2(M_t)P^2 \frac{\partial^2 W(P,t)}{\partial P^2} + r \frac{P \partial W(P,t)}{\partial P} - rW(P,t) = \frac{\partial W(P,t)}{\partial t}$$

331 with

$$332 \bar{W}(\bar{P}, t) = \bar{W}_0, \quad (55)$$

333 has solution of the type

334

$$\bar{W}(\bar{P}, t) = \bar{W}(0) \exp \left[- \left\{ \frac{2 \ln \left(\frac{P_t + Dt - P_t}{Dt} \right) \bar{P}^{-2}}{(\Delta t)^{\tau(q)+1} h(\xi_{t-1}, \xi_{t-2}, \dots, \xi_{t-d}, \alpha) \bar{P}_u - \mu_u} \right. \right. \\ \left. \left. + \frac{\left(\mu - \frac{1}{2} \sigma^2(M_t) \right) + \sqrt{\left(\mu - \frac{1}{2} \sigma^2(M_t) \right)^2 + 2\sigma(M_t)r}}{\sigma^2(M_t)} \bar{P} \right\} \right]$$

$$= \bar{W}(0) \exp \left[- \left\{ \frac{2 \ln \left(\frac{P_t + \Delta t - P_t}{\Delta t} \right) \bar{P}^{-2} + \sqrt{\left(\mu - \frac{1}{2} \sigma^2(M_t) \right)^2 + 2\sigma(M_t)r}}{(\Delta t)^{\tau(q)+1} (\alpha_0 + \sum_{k=1}^d \alpha_t \varepsilon_{t-k})} \right\} \right]. \quad (56)$$

336

337 Furthermore,

$$338 \quad \frac{\partial \bar{W}(\bar{P}, t)}{\partial \bar{P}} = 0, \quad \square t, \quad (57)$$

339 with

$$340 \quad \left(\mu - \frac{1}{2} \sigma^2(M_t) \right) + \sqrt{\left(\mu - \frac{1}{2} \sigma^2(M_t) \right)^2 + 2\sigma(M_t)r} + \frac{4 \ln \left(\frac{P_t + Dt - P_t}{Dt} \right)}{p^3 (\Delta t)^{\tau(q)+1} h(\xi_{t-1}, \xi_{t-2}, \dots, \xi_{t-d}, \alpha) \bar{P}_u - \mu_u} = 0. \quad (58)$$

341 **Proof:**

342 To remove r from (54), set

$$343 \quad \bar{W} = e^{-rt} W \quad (59)$$

344 and

$$345 \quad \bar{P} = e^{-rt} P, \quad (60)$$

346 so that (54) becomes;

$$347 \quad \frac{1}{2} \sigma^2(M_t) \bar{P}^2 \frac{\partial^2 \bar{W}(P, t)}{\partial \bar{P}^2} + \frac{\partial \bar{W}(P, t)}{\partial t} = 0. \quad (61)$$

348 By the method of separation we obtain (see Osu, et al, 2013), we obtain a special solution of the
349 form;

$$350 \quad \bar{W} = W_0 \exp \left\{ \frac{-2\alpha t \bar{P}^{-2}}{\sigma^2(M_t)} + \lambda \bar{P} \right\}. \quad (62)$$

351 Using equations (7), (18) and (19) into (62), we have easily, (56).

352 Again, solving for $\bar{P}$ in equation (60) and equating to (58) gives(using (20) and if λ is as in
353 (44));

$$354 \quad \bar{P} = \left[\frac{4 \ln(P_t - P_{t+\Delta t})(\Delta t)^{-T(q)}}{\left(\left(\mu - \frac{1}{2} \sigma^2(M_t) \right) + \sqrt{\left(\mu - \frac{1}{2} \sigma^2(M_t) \right)^2 + 2 \sigma(M_t) r} \right) (\alpha_0 + \sum_{k=1}^d \alpha_k \varepsilon_{t-k})} \right]^{1/3}. \quad (63)$$

355 Also equating equations(63) and (59) gives

$$\begin{aligned} P &= \left[\frac{4 \ln(P_t - P_{t+\Delta t})(\Delta t)^{-T(q)}}{\lambda \left((\alpha_0 + \sum_{k=1}^d \alpha_k \xi_{t-k}) \right)} \right]^{1/3} e^{(\mu + \sigma/m_t) \xi_t} \\ 356 \quad &= \left[\frac{4 \ln(P_t - P_{t+\Delta t})(\Delta t)^{-T(q)}}{\lambda \left((\alpha_0 + \sum_{k=1}^d \alpha_k \xi_{t-k}) \right)} \right]^{1/3} \exp\{\mu + (\Delta t)^{-(T(q)+1)} (\alpha_0 + \sum_{k=1}^d \alpha_k \xi_{t-k}) P_t - \mu_t\}, \end{aligned}$$

357 (using equations (7), (18) and (19)). We therefore have the future price as

$$358 \quad P_t = \left[\frac{4 \ln(P_t - P_{t+\Delta t})(\Delta t)^{-T(q)}}{\lambda \left((\alpha_0 + \sum_{k=1}^d \alpha_k \xi_{t-k}) \right)} \right]^{1/3} \exp\{\mu + (\Delta t)^{-(T(q)+1)} (\alpha_0 + \sum_{k=0}^d \alpha_{k+1} (P_t - \mu_t)^2\}. \quad (64)$$

359 In general we have the growth rate of the investor's portfolio as (using equations(59) and (56))

$$360 \quad W = \bar{W}(0) \exp \left[- \left\{ \frac{2 \ln \left(\frac{P_{t+\Delta t} - P_t}{\Delta t} \right) \bar{P}^{-2} + \sqrt{\left(\mu - \frac{1}{2} \sigma^2(M_t) \right)^2 + 2 \sigma(M_t) r}}{(\Delta t)^{T(q)+1} (\alpha_0 + \sum_{k=1}^d \alpha_k \varepsilon_{t-k})} \right\} \right] \exp \left\{ \left(\mu + \bar{\sigma}(M_{1,t} M_{2,t} \dots M_{\bar{k},t})^{1/2} \xi_t \right) t \right\}. \quad (65)$$

361 The optimal trading strategy

362 Let us denote an optimal trading strategy π_t^* for which we define (Zhang and Rong. 2013)

$$363 \quad H_{\pi_t^*}(t, r, P) = E_{\pi_t^*}[U(W(T))]. \quad (66)$$

364 Our objective here is to find the optimal value function such that

$$365 \quad H(t, r, P) = \sup_{\pi_t^* \in \pi} H_{\pi_t^*}(t, r, P). \quad (67)$$

366 Assume equation (54) for $\vartheta \neq 0$ together with the optimal π_t^* , we have

$$\frac{\partial W}{\partial t} + (rP - \vartheta) \frac{\partial W}{\partial P} + \frac{1}{2} \frac{\partial^2 W}{\partial P^2} \sigma^2(M_t) \pi_t^* P^2 = rW. \quad (68)$$

Put $z = \frac{\alpha}{P}$; $W(P) = z^\beta H(Z)$. Since W is not dependent on r , differentiating and substituting into (68), gives

$$rz^\beta W = -\frac{\sigma^2(M_t)}{2} \left[\beta(\beta+1)z^\beta w + \beta z^{\beta+1} \frac{\partial H}{\partial z} + 2(\beta+1)z^{\beta+1} \frac{\partial H}{\partial z} + z^{\beta+2} \pi_t^* \frac{\partial^2 H}{\partial z^2} \right] + \left(\frac{r\alpha}{z} - \vartheta \right) \left(-\frac{1}{\alpha} \right) \left(\beta z^{\beta+1} w + z^{\beta+2} \frac{\partial w}{\partial z} \right). \quad (69)$$

Now cancelling by z^β and collecting like terms gives

$$0 = \frac{\sigma^2(M_t) \pi_t^*}{2} z^2 \frac{\partial^2 H}{\partial z^2} + \frac{\partial H}{\partial z} z \left(\sigma^2(\beta+1) - r - \frac{D}{\alpha} z \right) + H \left(\frac{\sigma^2}{2} \beta(\beta+1) - r(\beta+1) + \beta \frac{D}{\alpha} z \right). \quad (70)$$

Set $\frac{\sigma^2(M_t)}{2} \beta = r$ and let $\frac{\vartheta/\alpha}{\frac{\sigma^2(M_t)}{2}} = -1$ to have $\beta = \frac{2r}{\sigma^2(M_t)}$, $\alpha = -\frac{2D}{\sigma^2(M_t)}$. Substitute into (70) to obtain

$$z\pi_t^* H_{zz} + (2-z)H_z = \beta(H - H_z). \quad (71)$$

Solving for optimal trading strategy, we have

$$\begin{aligned} \pi_t^* &= \frac{\beta(H - H_z)}{zH_{zz}} + \frac{(z-2)H_z}{zH_{zz}} \\ &= \frac{\frac{2r}{\sigma^2(M_t)}(H - H_z)}{zH_{zz}} + \frac{(z-2)H_z}{zH_{zz}} \\ &= \frac{\frac{2\mu + \sigma(M_t)P_t - \mu_t}{(\Delta t)^{\tau(q)+1} (\alpha_0 + \sum_{k=1}^d \alpha_k (P_{t-k} - \mu_{t-k})^2)^{\frac{1}{2}}} (H - H_z)}{zH_{zz}} + \frac{(z-2)H_z}{zH_{zz}}. \end{aligned} \quad (72)$$

Conclusion

The multi-fractal spectrum model has some successes in explaining excess volatility of stock returns compared to fundamentals and negative Skewness of equity returns as well as generating multi-fractal spectrum. By (50) there is no market signal as it tends to zero, meaning that the market is likely to crash at such point, signifying insolvency in asset returns. Equation (23a) can be represented by a renormalized probability distribution of local Holder exponents, called the multifractal spectrum by the form

$$f(\alpha) = \lim_{k \rightarrow \infty} \left(\frac{\ln N(\alpha)}{\ln s^k} \right). \quad (73a)$$

Denoting by $N(\alpha, \Delta t)$ the number of intervals $[t, t + \Delta t]$ required to cover $\tau(\alpha)$ we can write

$$N(\alpha, \Delta t) \sim (\Delta t)^{-f(\alpha)}. \quad (73b)$$

The market prices correspond to the values of α between α_{min} and $\alpha(f_{max})$ (Piacquadio and Redelico, 2009).

Furthermore, equation (71) can be solved for the value function H and replaced in (72) in order to obtain the optimal investment strategies.

We have only consider herein, the continuous case. An on-going work considers an extension where we will derive the MSM with the discrete version and establish its comparative effectiveness with the continuous case. An empirical illustration to compare both the MSM for equity returns and those in existing literature will be shown.

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