# A new approach of Mickens' extended iteration method for solving some nonlinear jerk equations

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## Abstract

A new approach of the Mickens' extended iteration method has been presented to obtain approximate analytic solutions of some nonlinear jerk equations. There has been shown that the partial derivatives of integral functions are valid for modified extended iteration method in each step of iteration. Also the solutions give more accurate result than other existing methods.

*Keywords:* Jerk equations; Nonlinear oscillator; Iteration method; Extended Iteration method

**Mathematics subject classification**: 34A34, 34B99.

## 1. Introduction

Most phenomena in our world are essentially nonlinear and are described by nonlinear equations. A vast body of scientific knowledge has developed over a long period of time, devoted to a description of natural phenomena. Practically, numerous differential equations involving physical phenomena are nonlinear. These equations have also demonstrated their usefulness in ecology, business cycle and biology. Therefore the solution of such problems lies essentially in solving the corresponding differential equations. In many cases it is possible to replace such a nonlinear equation by a related linear equation, which approximates the actual problem closely enough to give useful results. The method of small oscillations is a well-known example of the linearization of problems, which are essentially

nonlinear. However, such a “linearization” is not always feasible or possible; and when it is not, the original nonlinear equation itself must be considered.

Many methods exist for constructing analytical approximations to the solution of the oscillatory system. Such as perturbation method [13,22,23], standard and modified Linstedt-Poincaré [24-26], homotopy [5,17], harmonic balance method [1-4,6-9,15,16,18,19,27-32], iteration method [10-14,20,21], etc. Perturbation method is mainly used for the small nonlinear problems. On the other hand, harmonic balance and iteration methods are mostly used for the strong nonlinear problems.

We know the governing equations of most of the dynamical systems are second-order differential equations, but some dynamical systems can be described by third-order nonlinear differential equations named jerk [30] equations. The oscillations in a nonlinear vacuum tube circuit [27] and third order mechanical oscillators [6] are two examples of jerk equations. Originally jerk equations

$$\dddot{x} = J(x, \dot{x}, \ddot{x}) . \tag{1}$$

being of some interest in mechanics [4,29], these equations are finding increasing importance in the study of chaos [4].

The main intention of this article is to investigate the approximate analytic solutions and corresponding periods, not chaotic solutions of some nonlinear jerk equations using the modified extended iteration method.

Some authors presented the analytic solutions of these oscillators. Of these researchers, Gottlieb [8] used the lowest order harmonic balance method to determine the approximations periodic solutions and corresponding angular frequencies. The solutions and the angular frequencies obtained by Gottlieb were not accurate enough. It is very difficult to construct the higher order approximations by harmonic balance method for the reason that the method requires analytical solutions of a set of complicated nonlinear algebraic equations. Wu et al. [32] and Leung et al. [15] applied, respectively, an improved harmonic balance approach and a residue harmonic balance approach to solve nonlinear jerk equations, and their higher order approximations give accurate results to large amplitudes of oscillations. Ma et al. [17] and Hu [13] used, respectively, homotopy perturbation method

and parameter perturbation method to determine the high order approximate solutions of nonlinear jerk equations, and their results obtained are more accurate than those obtained by the low order harmonic balance method. Recently, Ramos [24-26] presented an order reduction method, two-level iterative procedure and a volterra integral formulation, respectively, to solve nonlinear jerk equations. He found that the second reduction method provides accurate solutions only for initial velocities close to unity, the third reduction method produces very accurate for the first and second differential equation in [24], the fourth reduction method provides as accurate results as or more accurate results obtained by parameter perturbation method. Hu et al. [14] presented the modified Mickens iteration procedure for a nonlinear jerk equation, and the second order approximate angular frequency was obtained by Newton's method. Leung and Guo [15] has established a residue harmonic balance approach for determining limit cycles to parity- and time-reversal invariant general non-linear jerk equations with cubic nonlinearities. Recently the author [11] presented a new approach of direct iteration [21] method for solving the nonlinear jerk oscillator containing displacement time's velocity time's acceleration, and velocity.

To obtain periodic solutions for jerk-oscillators certain restrictions must be placed on the mathematical structure of the oscillator. We only consider the following cubic nonlinear functions investigated by Gottlieb [8]:

$$\text{(i) } x\,\dot{x}\,\ddot{x}\text{, (ii) } \dot{x}^3 \text{ and (iii) } x^2\,\dot{x}\,. \tag{2}$$

The most general jerk function with invariance of time- and space-reversal and which has only cubic nonlinearities may be written in the form [8]

$$\dddot{x} = \alpha\, x\,\dot{x}\,\ddot{x} - \beta\,\dot{x}\,\ddot{x}^2 - \gamma\,\dot{x} - \delta\, x^2\,\dot{x} - \varepsilon\,\dot{x}^3\,. \tag{3}$$

where an over-dot denotes the time derivative and the parameters $\alpha$, $\beta$, $\gamma$, $\delta$ and $\varepsilon$ are given real constants. The corresponding initial conditions are

$$x(0) = 0,\ \ \dot{x}(0) = A \ \text{ and } \ \ddot{x}(0) = 0\,. \tag{4}$$

These three initial conditions in Eq. (4) are to satisfy the periodic requirement. Here, at least one of $\alpha$, $\beta$, $\delta$ and $\varepsilon$ should be non-zero.

For $\alpha = 1$ with cubic nonlinearity includes $x\dot{x}\ddot{x}$ only. The resulting standard jerk equation, after rescaling of both $x$ and $t$, is taken to be

$$\dddot{x} + \dot{x} - x\dot{x}\ddot{x} = 0, x(0) = 0,\ \dot{x}(0) = A,\ \ddot{x}(0) = 0\,. \tag{5}$$

For $\delta = 1$ and $\varepsilon = 1$, the resulting standard jerk equation, after rescaling of both $x$ and $t$, is taken to be

$$\dddot{x} + \dot{x}^3 + x^2\dot{x} = 0, x(0) = 0,\ \dot{x}(0) = A,\ \ddot{x}(0) = 0\,. \tag{6}$$

In this article, I present a new approach of the Mickens' extended iteration method [21] for the determination of approximate solutions and corresponding periods of nonlinear jerk Eqs. (5) & (6). Here I have defined the function which contains dependent variable, its derivatives of first and second order also the integrals of the dependent variable. Previously the function contains only dependent variable and its derivatives of first and second order in the extended iteration method [21]. It is mentioned that our method is valid for second and higher order period of oscillations and show a good agreement compared to other existing solutions.

## 2. The method

### 2.1 The direct Mickens iteration procedure

Let us consider a nonlinear oscillator modeled by

$$\ddot{x} + f(\ddot{x}, \dot{x}, x) = 0, \quad x(0) = A, \quad \dot{x}(0) = 0\ , \tag{7}$$

where over dots denote differentiation with respect to time, $t$. We choose the frequency $\Omega$ of this system. Then adding $\Omega^2 x$ to both sides of Eq. (7), we obtain

$$\ddot{x} + \Omega^2 x = \Omega^2 x - f(x, \dot{x}, \ddot{x}) \equiv G(x, \dot{x}, \ddot{x})\,. \tag{8}$$

Following [20], we formulate the iteration scheme as

$$\ddot{x}_{k+1} + \Omega_k^2\, x_{k+1} = G(x_k, \dot{x}_k, \ddot{x}_k); \quad k = 0, 1, 2, \cdots, \tag{9}$$

together with

$$x_0(t) = A\cos(\Omega_0 t)\,. \tag{10}$$

Herein $x_{k+1}$ satisfies the conditions

$$x_{k+1}(0) = A, \quad \dot{x}_{k+1}(0) = 0. \tag{11}$$

At each stage of the iteration, $\Omega_k$ is determined by the requirement that secular terms (see [22] for details) should not occur in the solution. This procedure gives the sequence of solutions: $x_0(t), x_1(t), \cdots$. The method can be proceed to any order of approximation; but due to growing algebraic complexity the solution is confined to a lower order usually the second [20].

### 2.2 The modified Mickens iteration procedure

Here $x_1$ can easily be obtained from Eq. (9) by demanding that secular terms should not occur in the solution.

The proposed iteration scheme of Eq. (8) is

$$\begin{aligned} \ddot{x}_{k+1} + \Omega_k^2 x_{k+1} &= G(\Omega_k, x_{k-1}, \dot{x}_{k-1}, \ddot{x}_{k-1}, x'_{k-1}) + G_x(\Omega_k, x_{k-1}, \dot{x}_{k-1}, \ddot{x}_{k-1}, x'_{k-1})(x_k - x_{k-1}) \\ &+ G_{\dot{x}}(\Omega_k, x_{k-1}, \dot{x}_{k-1}, \ddot{x}_{k-1}, x'_{k-1})(\dot{x}_k - \dot{x}_{k-1}) + G_{\ddot{x}}(\Omega_k, x_{k-1}, \dot{x}_{k-1}, \ddot{x}_{k-1}, x'_{k-1})(\ddot{x}_k - \ddot{x}_{k-1}) \\ &+ G_{x'}(\Omega_k, x_{k-1}, \dot{x}_{k-1}, \ddot{x}_{k-1}, x'_{k-1})(x'_k - x'_{k-1}); \qquad k = 1, 2, 3, \cdots \end{aligned} \tag{12}$$

where over dots denote differentiation with respect to time, $t$ over dash denote integration with respect to time, $t$ and

$$G_x = \frac{\partial G}{\partial x}, \quad G_{\dot{x}} = \frac{\partial G}{\partial \dot{x}}, \quad G_{\ddot{x}} = \frac{\partial G}{\partial \ddot{x}}, \; G_{x'} = \frac{\partial G}{\partial x'} \tag{13}$$

At each stage of the iteration, $\Omega_k$; $k = 1, 2, 3, \cdots$ is determined by the requirement that secular terms should not occur in the solution. This procedure gives the sequence of solutions: $x_2(t), x_3(t), x_4(t), \cdots$.

## 3. Example: Here we have considered two particular Jerk functions

(i) $x\,\dot{x}\,\ddot{x} - \dot{x}$ and (ii) $\dot{x}^3 + x^2\,\dot{x}$

### 3.1 Solution of the Jerk oscillator containing displacement times velocity time's acceleration, and velocity [8] $\dddot{x} + \dot{x} = x\,\dot{x}\,\ddot{x}$

Let us consider the nonlinear jerk oscillator [8]

$$\dddot{x} + \dot{x} = x\,\dot{x}\,\ddot{x}\,. \tag{14}$$

Introducing the phase space variable $(y,t)$ by the relation $\dot{x} = y$ then Eq. (14) becomes

$$\ddot{y} + y = y\,\dot{y}\,y'\,. \tag{15}$$

Obviously, Eq. (15) can be written as

$$\ddot{y} + \Omega^2 y = (\Omega^2 - 1 + \dot{y}\,y')y\,. \tag{16}$$

Now the iteration scheme is (according to Eq. (9))

$$\ddot{y}_{k+1} + \Omega_k^2\, y_{k+1} = (\Omega_k^2 \;-1 - \dot{y}_k\, y_k')\, y_k\,. \tag{17}$$

Equation (10) is rewritten as

$$y_0 = y_0(t) = A\cos\theta\ , \tag{18}$$

where $\theta = \Omega t$. For $k = 0$, the Eq. (17) becomes

$$\ddot{y}_1 + \Omega_0{}^2\, y_1 = (\Omega_0{}^2 - 1 - \dot{y}_0\, y_0')\, y_0\,. \tag{19}$$

Substituting the initial function Eq. (18) into the right hand side of Eq. (19) and expanding in a Cosine series, we obtain

$$\ddot{y}_1 + \Omega_0{}^2\, y_1 = a_1 \cos\theta + a_3 \cos 3\theta\,. \tag{20}$$

where,

$$a_1 = \frac{A}{4}(-4 - A^2 + 4\,\Omega_0{}^2), \quad a_3 = \frac{1}{4}A^3\ . \tag{21}$$

To avoid secular terms in the solution, we have to remove $\cos\theta$ from the right hand side of Eq. (20). Thus we have

$$\Omega_0{}^2 = 1 + \frac{A^2}{4}\,. \tag{22}$$

Then solving Eq. (20) and satisfying the initial condition $x_1(0) = A$, we obtain

$$y_1(t) = \left( A + \frac{a_3}{8\,\Omega_0{}^2} \right) \cos\theta - \frac{a_3}{8\,\Omega_0{}^2} \cos 3\theta \,. \tag{23}$$

This is the first approximate solution of Eq. (15) and the related $\Omega_1$ is to be determined.

According to Eq. (12), we have

$$\begin{aligned} \ddot{y}_{k+1} + \Omega_k^2\, y_{k+1} &= G(\Omega_k, y_{k-1}, \dot{y}_{k-1}, y'_{k-1}) + G_y(\Omega_k, y_{k-1}, \dot{y}_{k-1}, y'_{k-1})(y_k - y_{k-1}) \\ &\quad + G_{\dot{y}}(\Omega_k, y_{k-1}, \dot{y}_{k-1}, y'_{k-1})(\dot{y}_k - \dot{y}_{k-1}) \quad ; \quad k = 1, 2, 3, \cdots \,. \end{aligned} \tag{24}$$

Here

$$G = y(\Omega^2 - 1 + \dot{y}\, y'),\ G_y = \Omega^2 - 1 + \dot{y}\, y',\ G_{\dot{y}} = y\, y',\ G_{y'} = y\, \dot{y} \,. \tag{25}$$

The value of $\Omega_1$ will be obtained from the solution of

$$\begin{aligned} \ddot{y}_2 + \Omega_1^2\, y_2 &= G(\Omega_1, y_0, \dot{y}_0, y'_0) + G_y(\Omega_1, y_0, \dot{y}_0, y'_0)(y_1 - y_0) + \\ &\quad G_{\dot{y}}(\Omega_1, y_0, \dot{y}_0, y'_0)(\dot{y}_1 - \dot{y}_0) + G_{y'}(\Omega_1, y_0, \dot{y}_0, y'_0)(y'_1 - y'_0) \,. \end{aligned} \tag{26}$$

Substituting $y_1(t)$ and $y_0(t)$ from Eq. (23) and Eq. (18) into the right hand side of Eq. (26), we obtain

$$\ddot{y}_2 + \Omega_1^2\, y_2 = b_1 \cos\theta + b_3 \cos 3\theta + b_5 \cos 5\theta \,. \tag{27}$$

where,

$$\begin{aligned} b_1 &= -\frac{A}{192\Omega_0^3\,\Omega_1}(48A^2\,\Omega_0^4 - 3A^4\,\Omega_1^2 + 3A^2\,\Omega_0\,\Omega_1(2 + A^2 - 2\Omega_1^2) - 48\Omega_0^3\,\Omega_1(-4 + A^2 + 4\Omega_1^2) \\ &\quad + \Omega_0^2\,(A^4 + 48A^2\,\Omega_1^2)) \\ b_3 &= \frac{A^3}{128\Omega_0^3\,\Omega_1}(32\,\Omega_0^4 - 32\,\Omega_0^3\,\Omega_1 + A^2\,\Omega_1^2 + \Omega_0\,\Omega_1(4 + 3A^2 - 4\Omega_1^2) + \Omega_0^2\,(A^2 + 32\Omega_1^2)) \\ b_5 &= -\frac{A^5}{384\Omega_0^3\,\Omega_1}(\Omega_0^2 + 3\Omega_0\,\Omega_1 + 9\Omega_1^2) \end{aligned} \tag{28}$$

Again avoiding secular terms in the solution of Eq. (27), we obtain

$$\begin{aligned} \Omega_1^2 &= (49152 - 732\,A^6 - 126\,A^8 + P + 32\,A^2(864 + P) + A^4(2992 + 6\,P))^2 / (36(4 + A^2) \\ &\quad (32 + 9\,A^2)^2\,P) \end{aligned} \tag{29}$$

where,

$$P=(9437184A^2+11206656A^4+5405696A^6+1321920A^8+163836A^{10}+8235A^{12}+$$
$$3\sqrt{((128+68A^2+9A^4)^2(-805306368-503316480A^2+567803904A^4+831356928A^6}$$
$$+452990208A^8+135794368A^{10}+23789664A^{12}+2302128A^{14}+95769A^{16})))^{2/3} \quad (30)$$

Then solving Eq. (27) and satisfying initial condition, we obtain the second approximate solution,

$$y_2(t)=\left(A+\frac{1}{\Omega_1{}^2}\left(\frac{b_3}{8}+\frac{b_5}{24}\right)\right)\cos\theta-\frac{1}{\Omega_1{}^2}\left(\frac{b_3}{8}\cos 3\theta+\frac{b_5}{24}\cos 5\theta\right). \quad (31)$$

This is the second approximate solution of Eq. (15).

The third approximation $y_3$ and the value of $\Omega_2$ will be obtained from the solution of

$$\ddot{y}_3+\Omega_2^2\,y_3=G(\Omega_2,y_1,\dot{y}_1,y_1')+G_y(\Omega_2,y_1,\dot{y}_1,y_1')(y_2-y_1)+G_{\dot{y}}(\Omega_2,y_1,\dot{y}_1,y_1')(\dot{y}_2-\dot{y}_1)$$
$$+G_{y'}(\Omega_2,y_1,\dot{y}_1,y_1')(y_2'-y_1')\,. \quad (32)$$

In a similar way the method can be proceeded higher order approximations and $\Omega_0$, $\Omega_1$, … respectively obtained by Eqs. (22), (29), … represent the approximation of frequencies of oscillator (14). However due to growing algebraic complexity, most of the approximate techniques are applied to second or third approximations.

### 3.2 Jerk oscillator containing velocity-cube and velocity times displacement-squared [8]: $\dddot{x}+\dot{x}^3+x^2\,\dot{x}=0$

Let us consider the nonlinear jerk oscillator [8]

$$\dddot{x}+\dot{x}^3+x^2\,\dot{x}=0\,. \quad (33)$$

Introducing the phase space variable $(y,t)$ by the relation $\dot{x}=y$ then Eq. (33) becomes

$$\ddot{y}+y^3+y\,y'^2=0\,. \quad (34)$$

The Eq. (34) can be written as

$$\ddot{y}+\Omega^2\,y=(\Omega^2-y^2-y'^2)y\,. \quad (35)$$

According to Eq. (9), the iteration scheme of Eq. (35) will be

$$\ddot{y}_{k+1} + \Omega_k^2\, y_{k+1} = (\Omega_k^2 \;\; - y_k^2 - y_k'^2)\, y_k\,. \tag{36}$$

The first approximation $y_1(t)$ and the frequency $\Omega_0$ will be obtained from the solution (putting $k = 0$ in Eq. (36))

$$\ddot{y}_1 + \Omega_0{}^2\, y_1 = (\Omega_0{}^2 - y_0^2 - y_0'^2)\, y_0\,. \tag{37}$$

Utilizing the initial function Eq. (18) into the right hand side of Eq. (37) and expanding in a Fourier Cosine series, we obtain

$$\ddot{y}_1 + \Omega_0{}^2\, y_1 = c_1 \cos\theta + c_3 \cos 3\theta\,. \tag{38}$$

where,

$$c_1 = -\frac{A}{4\Omega_0{}^2}(A^2 + 3\,A^2\;\Omega_0{}^2 - 4\,\Omega_0{}^4), \quad c_3 = -\frac{A^3}{4\Omega_0{}^2}(-1 + \Omega_0{}^2)\,. \tag{39}$$

To avoid secular terms in the solution, we have to remove $\cos\theta$ from the right hand side of Eq. (38). Thus we have

$$\Omega_0{}^2 = \frac{A}{8}(3A + \sqrt{16 + 9A^2}\,)\,. \tag{40}$$

Then solving Eq. (38) and satisfying the initial condition $x_1(0) = A$, we obtain

$$y_1(t) = \left(A + \frac{c_3}{8\,\Omega_0{}^2}\right)\cos\theta - \frac{c_3}{8\,\Omega_0{}^2}\cos 3\theta\,. \tag{41}$$

This is the first approximate solution of Eq. (34) and the related $\Omega_1$ is to be determined.

According to Eq. (12), the second approximation $y_2(t)$ and the value of $\Omega_1$ will be obtained from the solution of

$$\ddot{y}_2 + \Omega_1^2\, y_2 = G(\Omega_1, y_0, y_0') + G_y(\Omega_1, y_0, y_0')(y_1 - y_0) + G_{y'}(\Omega_1, y_0, y_0')(y_1' - y_0')\,. \tag{42}$$

where

$$G = (\Omega^2 - y^2 - y'^2)y,\;\; G_y = \Omega^2 - 3y^2 - y'^2,\;\; G_{y'} = -2\,y\,y'\,. \tag{43}$$

Substituting $y_1(t)$ and $y_0(t)$ from Eq. (41) and Eq. (18) into the right hand side of Eq. (42), we obtain

$$\ddot{y}_2 + \Omega_1^2\, y_2 = d_1 \cos\theta + d_3 \cos 3\theta + d_5 \cos 5\theta\,. \tag{44}$$

where,

$$\begin{aligned}
d_1 &= \frac{1}{192\Omega_0^6\,\Omega_1}(2A^3\Omega_0(-48\,\Omega_0^4 + A^2(-1+\Omega_0^2)) + 3A^3(-A^2 - 2A^2\,\Omega_0^2 + (16+3A^2)\Omega_0^4 \\
&\quad - 48\Omega_0^6)\Omega_1 + 6A\,\Omega_0^2\,(A^2 - A^2\,\Omega_0^2 + 32\Omega_0^4)\Omega_1^3) \\
d_3 &= \frac{A^3}{128\Omega_0^6\,\Omega_1}(2\Omega_0(A^2 - A^2\,\Omega_0^2 + 32\Omega_0^4) - (1+\Omega_0^2)(32\Omega_0^4 + 3A^2(-1+\Omega_0^2))\Omega_1 \\
&\quad + 4\Omega_0^2(-1+\Omega_0^2)\Omega_1^3) \\
d_5 &= -\frac{A^5}{384\Omega_0^6\,\Omega_1}((-1+\Omega_0^2)(-2\Omega_0 + (-3+9\Omega_0^2)\Omega_1))
\end{aligned} \tag{45}$$

Again avoiding secular terms in the solution of Eq. (44), we obtain

$$\begin{aligned}
\Omega_1^2 = (R/(18\times 2^{1/3}\Omega_0^2(A^2 - A^2\,\Omega_0^2 + 32\Omega_0^4)) - 3\times 2^{1/3}A^2(-A^2 - 2A^2\Omega_0^2 \\
+ (16+3A^2)\Omega_0^4 - 48\Omega_0^6)/R)^2
\end{aligned} \tag{46}$$

where,

$$\begin{aligned}
R &= 3\times 2^{2/3}\times 3^{1/3}(A^2\Omega_0^5(6A^6 - 18A^6\Omega_0^2 + 672A^4\Omega_0^4 + 18A^6\Omega_0^4 - 1344A^4\Omega_0^6 - 6A^6\Omega_0^6 \\
&\quad + 24576A^2\Omega_0^8 + 672A^4\Omega_0^8 - 24576A^2\Omega_0^{10} + 294912\Omega_0^{12} + \sqrt{6}(\Omega_0^6(6\Omega_0^4(A^2 - A^2\Omega_0^2 \\
&\quad + 48\Omega_0^4)^2(A^3 - A^3\Omega_0^2 + 32A\Omega_0^4)^4 - A^6(A^2 - A^2\Omega_0^2 + 32\Omega_0^4)^3(A^2 + 2A^2\Omega_0^2 - (16 \\
&\quad + 3A^2)\Omega_0^4 + 48\Omega_0^6))^3)^{1/2})^{1/3}
\end{aligned} \tag{47}$$

Then solving Eq. (44) and satisfying initial condition, we obtain the second approximate solution,

$$y_2(t) = \left( A + \frac{1}{\Omega_1^{\,2}}\left(\frac{d_3}{8} + \frac{d_5}{24}\right)\right)\cos\theta - \frac{1}{\Omega_1^{\,2}}\left(\frac{d_3}{8}\cos 3\theta + \frac{d_5}{24}\cos 5\theta\right). \tag{48}$$

This is the second approximate solution of Eq. (34).

The third approximation $y_3$ and the value of $\Omega_2$ will be obtained from the solution of

$$\ddot{y}_3 + \Omega_2^2\, y_3 = G(\Omega_2, y_1, y_1') + G_y(\Omega_2, y_1, y_1')(y_2 - y_1) + G_{y'}(\Omega_2, y_1, y_1')(y_2' - y_1')\,. \tag{49}$$

In a similar way the method can be proceeded higher order approximations and $\Omega_0$, $\Omega_1$, … respectively obtained by Eqs. (40), (46), … represent the approximation of frequencies of oscillator (33).

## 4. Results and Discussions

A modified iteration method is developed based on Mickens' [21] extended iteration method to solve a class of nonlinear jerk equations. In this section, I express the accuracy of the modified approach of extended iteration method by comparing with the existing results from different methods and with the exact results of the nonlinear jerk equations. To show the accuracy, we have calculated the percentage errors (denoted by Er(%)) by the definitions $\left|100(T_e - T_i)/T_e\right|$, where $T_i = 2\pi/\Omega_i\,; i = 0, 1, 2, \cdots$ represents the approximate periods obtained by the present method and $T_e$ represents the corresponding exact period of the oscillator.

**4.1** First of all we demonstrate the comparison of results for examples 3.1. *i.e., For the Jerk function containing displacement time's velocity time's acceleration, and velocity* $x\,\dot{x}\,\ddot{x} - \dot{x}$ .

Recently, Gottlieb [8], Ma *et al.* [17], Ramos [24] and Leung & Guo [15] has found approximate solutions, frequencies and as well as approximate periods of nonlinear jerk oscillator (given by Eq. (14)) by different methods other than Iteration method.

I have used a modified extended iteration procedure to obtaining approximate solutions of the oscillator. The procedure is very simple. It has been shown that, in most of the cases our solution gives significantly better result than other existing results and sometimes it is almost similar to other existing results.

Herein I have calculated the first second and third approximate frequencies which are denoted by $\Omega_1$, $\Omega_2$ and $\Omega_3$ respectively and corresponding periods are $T_1$, $T_2$ and $T_3$. All the results are given in Table 5.1a and Table 5.1b, to compare the approximate frequencies we have also given the existing results determined separately by Gottlieb [8], Ma *et al.* [17], Ramos [24] and Leung & Guo [15] respectively.

**4.2** Now, I demonstrate the comparison of results for examples 3.2.

*i.e.,* For the Jerk-function containing velocity-cube and velocity times displacement-squared

$\dot{x}^3 + x^2 \dot{x}$.

Recently, Wu *et al.* [32], Ma *et al.* [17], Ramos [24], Hu [13] and Leung & Guo [15] has found approximate solutions, frequencies and as well as approximate periods of nonlinear jerk oscillator (given by Eq. (33)) by different methods other than Iteration.

I have used a modified extended iteration procedure to obtaining approximate solutions of the oscillator. The procedure is very simple. It has been shown that, in most of the cases our solution gives significantly better result than other existing results and sometimes it is almost similar to other existing results.

Herein I have calculated the first second and third approximate frequencies which are denoted by $\Omega_1$, $\Omega_2$ and $\Omega_3$ respectively and corresponding periods are $T_1$, $T_2$ and $T_3$. All the results are given in Table 5.2a and Table 5.2b, to compare the approximate frequencies we have also given the existing results determined separately by Wu *et al.* [32], Ma *et al.* [17], Ramos [24], Hu [13] and Leung & Guo [15] respectively.

## 5. Tables

**Table 5.1a**

Comparison of the approximate periods with exact periods $T_e$ [8] of $\dddot{x} + \dot{x} = x\dot{x}\ddot{x}$ :

| $A$ | $T_e$ | Modified $T_0$<br>Er(%) | Modified $T_1$<br>Er(%) | Modified $T_2$<br>Er(%) |
|---|---|---|---|---|
| 0.1 | 6.275347 | 6.275346<br>1.56 $e^{-5}$ | 6.275347<br>2.65 $e^{-6}$ | 6.275346<br>2.60 $e^{-6}$ |
| 0.2 | 6.252016 | 6.252003<br>2.07 $e^{-4}$ | 6.252016<br>3.11 $e^{-6}$ | 6.252016<br>1.49 $e^{-7}$ |
| 0.5 | 6.096061 | 6.095585<br>7.8 $e^{-3}$ | 6.096021<br>6.5 $e^{-4}$ | 6.096059<br>3.1 $e^{-5}$ |
| 1 | 5.626007 | 5.619852<br>1.09 $e^{-1}$ | 5.624426<br>2.8 $e^{-1}$ | 5.625802<br>3.6 $e^{-3}$ |
| 2 | 4.491214 | 4.442883<br>1.08 | 4.464823<br>5.9 $e^{-1}$ | 4.482164<br>2.0 $e^{-1}$ |

$T_0$ , $T_1$ and $T_2$ respectively denote initial, first and second modified approximate periods. Er(%) denotes percentage error.

**Table 5.1b**

Comparison of the approximate periods obtained by my method and other existing result with exact periods $T_e$ [8] of $\dddot{x} + \dot{x} = x\,\dot{x}\,\ddot{x}$ :

| $A$ | $T_e$ | Modified $T_2$ Er(%) | Gottlieb $T_{G2}$ (2004) [8] Er(%) | Ma et al. $T_{M2}$ (2008) [17] Er(%) | Ramos $T_{R2}$ (2010) [24] Er(%) | Leung-Guo $T_{LG2}$ (2011) [15] Er(%) |
|---|---|---|---|---|---|---|
| 0.1 | 6.275347 | 6.275346<br>2.60 $e^{-6}$ | 6.275346<br>1.3 $e^{-5}$ | 6.275347<br>2.5 $e^{-6}$ | 6.275347<br>9.0 $e^{-8}$ | 6.275347<br>2.6 $e^{-6}$ |
| 0.2 | 6.252016 | 6.252016<br>1.49 $e^{-7}$ | 6.252003<br>2.11 $e^{-4}$ | 6.252016<br>1.6 $e^{-7}$ | 6.252016<br>6.4 $e^{-6}$ | 6.252016<br>1.6 $e^{-7}$ |
| 0.5 | 6.096061 | 6.096059<br>3.1 $e^{-5}$ | 6.095585<br>7.8 $e^{-3}$ | 6.096059<br>3.21 $e^{-5}$ | 6.096025<br>6.06 $e^{-4}$ | 6.096061<br>8.2 $e^{-6}$ |
| 1 | 5.626007 | 5.625802<br>3.6 $e^{-3}$ | 5.619852<br>1.09 $e^{-1}$ | 5.625795<br>3.8 $e^{-3}$ | 5.624539<br>2.6 $e^{-2}$ | 5.625993<br>2.5 $e^{-4}$ |
| 2 | 4.491214 | 4.482164<br>2.0 $e^{-1}$ | 4.442883<br>1.08 | 4.482081<br>2.03 | 4.466144<br>5.6 $e^{-1}$ | 4.490125<br>2.4 $e^{-2}$ |

$T_2$ denotes second modified approximate periods; $T_{G2}, T_{M2}, T_{R2}$ and $T_{LG2}$ respectively denote second approximate periods obtained by Gottlieb, Ma et al., Ramos and Leung-Guo. Er(%) denotes percentage error.

**Table 5.2a**

Comparison of the approximate periods with exact periods $T_e$ [8] of $\dddot{x} + \dot{x}^3 + x^2\,\dot{x} = 0$ :

| $A$ | $T_e$ | Modified $T_1$ Er(%) | Modified $T_2$ Er(%) | Modified $T_3$ Er(%) |
|---|---|---|---|---|
| 0.1 | 25.359725 | 25.665523<br>1.206 | 25.407289<br>1.876 $e^{-1}$ | 25.367539<br>3.081 $e^{-2}$ |
| 0.2 | 17.495410 | 17.6627<br>9.564 $e^{-1}$ | 17.52<br>1.404 $e^{-1}$ | 17.4992<br>2.184 $e^{-2}$ |
| 0.5 | 10.210761 | 10.250448<br>3.887 $e^{-1}$ | 10.215983<br>5.115 $e^{-2}$ | 10.211439<br>6.641 $e^{-3}$ |
| 1 | 6.283185 | 6.2831852<br>0.000 | 6.2831852<br>0.000 | 6.2831852<br>0.000 |
| 2 | 3.508793 | 3.50639<br>6.8 $e^{-2}$ | 3.50881<br>4.487 $e^{-4}$ | 3.50881<br>4.214 $e^{-4}$ |
| 5 | 1.468638 | 1.46862 | 1.46888 | 1.46867 |

| | | | | |
|---|---|---|---|---|
| | | 9.5 $e^{-4}$ | 1.65 $e^{-2}$ | 2.4 $e^{-3}$ |
| 10 | 0.739762 | 0.739902<br>1.886 $e^{-2}$ | 0.739913<br>2.036 $e^{-2}$ | 0.739783<br>2.89 $e^{-3}$ |
| 20 | 0.370580 | 0.370671<br>2.45 $e^{-2}$ | 0.370660<br>2.15 $e^{-2}$ | 0.370591<br>1.1 $e^{-3}$ |

$T_1$, $T_2$ and $T_3$ respectively denote first, second and third modified approximate. Er(%) denotes percentage error.

**Table 5.2b**

Comparison of the approximate periods obtained by my method and other existing result with exact periods $T_e$ [8] of $\dddot{x} + \dot{x}^3 + x^2\dot{x} = 0$:

| $A$ | $T_e$ | Modified $T_2$ Er(%) | Wu *et al.* [32] $T_{W2}$ Er(%) | Ma *et al.* [17] $T_{M2}$ Er(%) | Ramos [24] $T_{R2}$ Er(%) | Hu [13] $T_{H2}$ Er(%) | Leung& Guo [15] $T_{LG2}$ Er(%) |
|---|---|---|---|---|---|---|---|
| 0.1 | 25.35972 | 25.40729<br>1.876 $e^{-1}$ | 24.6677<br>2.72 | 24.517938<br>3.32 | 25.56407<br>8.06 $e^{-1}$ | 25.612713<br>9.98 $e^{-1}$ | 25.946122<br>2.312 |
| 0.5 | 10.21076 | 10.21598<br>5.115 $e^{-2}$ | 10.1884<br>2.2 $e^{-1}$ | 10.146603<br>2.27 | 10.24328<br>3.18 $e^{-1}$ | 10.244058<br>3.26 $e^{-1}$ | 10.213682<br>2.86 $e^{-2}$ |
| 1 | 2π | 2π<br>0 | 2π<br>0 | 2π<br>0 | 2π<br>0 | 2π<br>0 | 2π<br>0 |
| 5 | 1.468638 | 1.46888<br>1.65 $e^{-2}$ | 1.46650<br>1.45 $e^{-1}$ | 1.4679554<br>4.6 $e^{-3}$ | 1.46715<br>1.01 $e^{-1}$ | 1.466631<br>1.37 $e^{-1}$ | 1.468570<br>4.6 $e^{-3}$ |
| 10 | 0.739762 | 0.73991<br>2.036 $e^{-2}$ | 0.738632<br>1.53 $e^{-1}$ | 0.739299<br>6.3 $e^{-3}$ | 0.73907<br>9.4 $e^{-2}$ | 0.738763<br>1.35 $e^{-1}$ | 0.73974<br>3.7 $e^{-3}$ |
| 20 | 0.370580 | 0.37066<br>2.15 $e^{-2}$ | 0.370008<br>1.54 $e^{-1}$ | 0.37033<br>6.7 $e^{-3}$ | 0.37024<br>9.2 $e^{-2}$ | 0.370082<br>1.34 $e^{-1}$ | 0.37057<br>3.3 $e^{-3}$ |

$T_2$ denotes second modified approximate periods; $T_{W2}$, $T_{M2}$, $T_{R2}$, $T_{H2}$ and $T_{LG2}$ respectively denote second approximate periods obtained by Wu *et al.* Gottlieb, Ma *et al.*, Ramos, Hu and Leung-Guo. Er(%) denotes percentage error.


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