

Analysis of an SI epidemic model with nonlinear incidence rate in an environmentally-driven infectious disease

Mimi Yan¹ and Ruiqing Shi^{*1}

¹*School of Mathematics and Computer Science, Shanxi Normal University, Shanxi, Linfen, 041004, P.R. China*

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Abstract

In this paper, a new model for the linking of within- and between-host with nonlinear incidence rate is constructed and investigated. We postulate that the infection process depend on the size of the infective inoculum that susceptible hosts may acquire by interacting with a contaminated environment. Because the dynamical processes associated with the within- and between-host occur on different time scales, the model behaviors can be analyzed by using a singular perturbation argument, which allows us to decouple the full model by separating the fast- and slow-systems. For the fast system, it is shown that the infection-free equilibrium U_0 is locally asymptotically stable if $R_{v,0} < 1$; whereas U_0 is unstable if $R_{v,0} > 1$, and a unique interior equilibrium $U(\bar{E})$ exists and it is locally asymptotically stable. For the slow system, under the condition that the fast system has a stable interior equilibrium $U(\bar{E})$ (i.e., $R_{v,0} > 1$), there exists at least one endemic equilibrium W if $R_h > 1$ and $\beta N < 1$. We also prove that the endemic equilibrium W for the slow system is locally asymptotically stable if $R_{h,0}^* > 1$. Some numerical simulations are provided to verify the theoretical results. In addition, some simple discussion is also presented at the end of the article.

Keywords: Epidemic model; Nonlinear incidence rate; Equilibrium; Basic reproduction number

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1 Introduction

For most infectious diseases there are two key processes in the host-parasite interaction. One is the epidemiological process associated with the disease transmission, and the other is the immunological process at the individual host level. Viral dynamic models will consider the within-host dynamics independent of the interaction at the population level, whereas epidemiological models of population dynamics will consider the interaction between susceptible and infected hosts without an explicit link

^{*}Corresponding author: E-mail: shirq1979@163.com

to the viral dynamics within the hosts. There are, however, questions that can only be studied by using models that explicitly link the two processes. Such questions include: (i) How does the within-host dynamics influence the transmission of a pathogen from individual to individual? (ii) What is the effect of population dynamics of disease transmission on the viral dynamics at the individual level? (iii) Will the model predictions in terms of the virulence and basic reproduction number of the pathogen be altered if the two processes are dynamically linked? Gilchrist and Sasaki have used a nested model to evaluate the direction of natural selection at the within- and between-host levels [Gilchrist M. A. (2002)]. In this nested model, the within-host system is independent of the transmission dynamics at the population level.

In this paper, we propose a framework that explicitly links the epidemiological and immunological dynamics with nonlinear incidence rate. We use this as a conceptual model for the dynamics of *Toxoplasma gondii*. *Toxoplasma gondii* is an obligate intracellular parasite that can infect all warm-blooded vertebrates, including mammals and birds. The parasite reproduces sexually in felines. Once a cat becomes infected, it sheds oocysts, which contaminate the environment. These oocysts can be ingested by mammals and birds which then become infected with the parasite. Eating another organism that is infected can also infect the secondary hosts [Adam et al (2012)].

In modelling communicable disease, the incidence rate (the rate of new infections) is considered to play a vital role in ensuring that the model can give a reasonable qualitative description of the disease dynamics. Epidemiological models with bilinear and standard incidence rate have been frequently used in classical epidemiological models [Zhilan and Jorge (2012); Zhilan et al (2013)]. There are several reasons for using nonlinear incidence rates such as saturating and nearly bilinear. For example, Yorke and London [Yorke and London (1973)] showed that an incidence rate $g(I)S = \beta(1 - cI)IS$ with positive c and time dependent β is consistent with the results of the simulations for measles outbreaks. To incorporate the effect of behavioral changes, Liu and coworkers [Liu and Hethcote (1987); Liu and Levin (1986)] investigated a nonlinear incidence rate given by $\beta I^p S / (1 + aI^q)$ with $\beta, a, p, q > 0$. Mathematical models with nonlinear incidence rates are numerous in the literatures [Alexander and Moghadas (2008); Lijie and Guirong (2007); Li and Guquan (2010); Liming and Jingang (2009); Liming and Xuezhai (2009, 2010); Yu and Liqun (2011); Liming and Shuli (2014)].

The organization of this paper is as follows: In the next section, a mathematical model for coupling within-host and between-host dynamics with nonlinear incidence rate is constructed. The existence and stability of equilibria of the fast system and slow system are investigated in section 3 and 4, respectively. In section 5, some numerical simulations are presented. The paper ends with brief remark in the last section.

2 Formulation of the Model

Zhilan and Jorge [Zhilan et al (2013)] proposed and discussed a mathematical model for coupling within-host and between-host dynamics with a bilinear incidence rate.

The between-host dynamics is governed by the SI system

$$\begin{cases} \frac{dS}{dt} = \mu N - \lambda ES - \mu S, \\ \frac{dI}{dt} = \lambda ES - \mu I, \\ \frac{dN}{dt} = S + I. \end{cases} \quad (2.1)$$

The variables S and I represent the numbers of susceptible and infectious individuals at time t , respectively. The parameter μ is the per-capita host natural mortality rate, which is assumed to be the same as the per-capital birth rate so that the total population size N remains constant for all time t . The parameter λ denotes the per-capita infection rate of hosts in a contaminated environment, and E denotes the level of environment contamination at time t , or the concentration of oocysts per

unit area or volume of a region being considered ($0 \leq E \leq 1$). This level of contamination is dependent of the number of infected hosts (I) and the average parasite load (V) within the host as described by the equation

$$\frac{dE}{dt} = \theta IV(1 - E) - \gamma E, \quad (2.2)$$

where θ is the rate of contamination and γ is the clearance rate.

For the within-host dynamics, the sub-system reads

$$\begin{cases} \frac{dT}{dt} &= \Lambda - kVT - mT, \\ \frac{dT^*}{dt} &= kVT - (m + d)T^*, \\ \frac{dV}{dt} &= g(E) + pT^* - cV. \end{cases} \quad (2.3)$$

The variables T , T^* and V represent the density of healthy cells, infected cells, and parasite load, respectively. The parameter k denotes the per-capita infection rate of cells; m and d are the per-capita background mortality and infection-induced mortality rates of cell, respectively; p is the parasite reproduction rate by an infected cell; and c is the within-host mortality rate of parasites. The function $g(E)$ represents the rate at which an average host is inoculated. In [Zhilan et al (2013)], $g(E) = aE$, where a is a positive constant. Motivated by [Zhilan and Jorge (2012); Zhilan et al (2013)], we will construct the following mathematical model for an environmentally-driven infectious disease with nonlinear incidence rate.

The between-host dynamics is governed by the SI system

$$\begin{cases} \frac{dS}{dt} &= \mu N - \frac{\lambda ES}{1 + \beta S} - \mu S, \\ \frac{dI}{dt} &= \frac{\lambda ES}{1 + \beta S} - \mu I, \\ \frac{dE}{dt} &= \theta IV(1 - E) - \gamma E, \\ N &= S + I. \end{cases} \quad (2.4)$$

For the within-host dynamics, the sub-system reads

$$\begin{cases} \frac{dT}{dt} &= \Lambda - \frac{kVT}{1 + \alpha T} - mT, \\ \frac{dT^*}{dt} &= \frac{kVT}{1 + \alpha T} - (m + d)T^*, \\ \frac{dV}{dt} &= g(E) + pT^* - cV. \end{cases} \quad (2.5)$$

The function $g(E)$ indicates the fact that if the environmental contamination is high, then the inoculum is high as well. For biological considerations, the function $g(E)$ should have the following properties:

$$g(E) \geq 0, \quad g(0) = 0, \quad g'(E) > 0. \quad (2.6)$$

Here we choose the function $g(E)$ as the following nonlinear incidence rate form

$$g(E) = \frac{aE}{1 + E^2}, \quad (2.7)$$

where a is a positive constant.

An important biological feature of this coupled system is that the within-host dynamics occurs on a faster time scale than the dynamics of the between-host and the environment. The multiple time-scale allows us to study the mathematical properties of the model by analyzing the fast- and slow-systems determined by the two time scales.

3 Analysis of the fast system (2.5)

3.1 Existence of the equilibrium point

The subsystem for the within-host dynamics (2.5) can be considered as the fast system in which the variable E is treated as a constant. For the simpler case, when $g(E) = 0$, the system (2.5) always has an infection-free equilibrium $U_0 = (T_0, T_0^*, V_0)$, where $T_0 = \frac{\Lambda}{m}$, $T_0^* = 0$, $V_0 = 0$. The system (2.5) has no infection-free equilibrium when $g(E) > 0$.

Let $R_v(E)$ denote the within-host reproduction number [van den Driessche and Watmough (2002)], which is a function of the environmental contamination level, and let $R_{v,0} = R_v(0)$. It is easy to derive that

$$R_{v,0} = \frac{T_0[kp - \alpha c(m + d)]}{c(m + d)}.$$

Consider the case when $g(E) > 0$. Let $\bar{U}(E) = (\bar{T}(E), \bar{T}^*(E), \bar{V}(E))$ denote a nontrivial equilibrium, where $\bar{T}^*(E) > 0$, $\bar{V}(E) > 0$. Then

$$\begin{aligned}\bar{V}(E) &= \frac{1}{c}[g(E) + p\bar{T}^*(E)], \\ \bar{T}^*(E) &= \frac{\Lambda - m\bar{T}(E)}{m + d} = \frac{m}{m + d}[T_0 - \bar{T}(E)].\end{aligned}$$

and $\bar{T}(E)$ is a solution of the following quadratic equation

$$a_1 T^2 + a_2 T + 1 = 0, \quad (3.1)$$

where

$$\begin{aligned}a_1 &= \frac{R_{v,0}}{T_0^2} > 0, \\ a_2 &= -\frac{kg(E)}{cmT_0} - \frac{1 + R_{v,0}}{T_0} < 0.\end{aligned}$$

The discriminant of equation (3.1) is given by

$$\begin{aligned}a_2^2 - 4a_1 &= \left[-\frac{kg(E)}{cmT_0} - \frac{1 + R_{v,0}}{T_0}\right]^2 - 4\frac{R_{v,0}}{T_0^2} \\ &\geq \frac{(1 + R_{v,0})^2}{T_0^2} - 4\frac{R_{v,0}}{T_0^2} \\ &= \frac{(1 - R_{v,0})^2}{T_0^2} \geq 0.\end{aligned}$$

Thus, equation (3.1) has two positive real solutions given by $\bar{T}_{\pm}(E) = \frac{-a_2 \pm \sqrt{a_2^2 - 4a_1}}{2a_1}$. Note that

$$\begin{aligned}a_2'(E) &= -\frac{kg'(E)}{cmT_0} < 0, \\ \bar{T}'_{\pm}(E) &= \frac{a_2'(E)}{2a_1} \left(-1 \pm \frac{a_2}{\sqrt{a_2^2 - 4a_1}}\right).\end{aligned}$$

we obtain that

$$\bar{T}'_+(E) > 0, \quad \bar{T}'_-(E) < 0. \quad (3.2)$$

Note also that

$$\bar{T}_+(0) = \begin{cases} T_0 & R_{v,0} \geq 1, \\ \frac{T_0}{R_{v,0}} & R_{v,0} < 1, \end{cases} \quad (3.3)$$

$$\bar{T}_-(0) = \begin{cases} \frac{T_0}{R_{v,0}} & R_{v,0} > 1, \\ T_0 & R_{v,0} \leq 1. \end{cases} \quad (3.4)$$

From $\bar{T}'_+(0) > 0$ and $\bar{T}_+(0) > T_0$, we know that $\bar{T}_+(E) > T_0$ for all $E > 0$. But $\bar{T}_+(E) \leq T_0$, and thus, $\bar{T}_+(E)$ does not exist.

From equations (3.2) and (3.4) we know that $\bar{U}(E) = (\bar{T}_-(E), \bar{T}^*(E), \bar{V}(E))$ exists when $R_{v,0} > 1$, and it coincides with U_0 when $R_{v,0} \leq 1$.

The existence condition for $\bar{U}(E)$ motivates the following definition for $R_v(E)$ in the case of $E > 0$

$$R_v(E) = \frac{T_0}{\bar{T}_-(E)}. \quad (3.5)$$

Then, we have obtained the following result.

Theorem 3.1. (i) When $g(E) = 0$, the fast system (2.5) always has the infection-free equilibrium $U_0 = (T_0, T_0^*, V_0)$.

(ii) When $g(E) > 0$, the fast system (2.5) has a unique nontrivial equilibrium $\bar{U}(E)$ if $R_{v0} > 1$.

Since R_{v0} does not depend on E , the influence of E on $R_v(E)$ and on $\bar{T}_-(E)$ can be dependent on the magnitude of R_{v0} . Such dependence is illustrated in Fig.1.

3.2 Stability of the equilibrium point

The Jacobian matrix at U_0 is given by

$$J = \begin{bmatrix} -m & 0 & -\frac{kT_0}{1+\alpha T_0} \\ 0 & -(m+d) & \frac{kT_0}{1+\alpha T_0} \\ 0 & p & -c \end{bmatrix}.$$

One of the eigenvalues of the above matrix is $X_1 = -m < 0$, and the other two eigenvalues are determined by the following equation

$$X^2 + (m+d+c)X + c(m+d) - \frac{kp}{1+\alpha T_0}T_0 = 0, \quad (3.6)$$

and

$$\begin{aligned} X_2 + X_3 &= -(m+d+c) < 0, \\ X_2 X_3 &= c(m+d) - \frac{kp}{1+\alpha T_0}T_0. \end{aligned}$$

It is easy to show that $X_2 X_3 > 0$ if $R_{v0} < 1$, then all the roots of the equation (3.6) have negative real parts. Hence, U_0 is locally asymptotically stable. However, we will get $X_2 X_3 < 0$ if $R_{v0} > 1$, then one of the roots of the equation (3.6) has positive real parts, which means that U_0 is unstable.

The Jacobian matrix at $\bar{U}(E)$ is given by

$$J(E) = \begin{bmatrix} -\frac{\frac{k}{c}[g(E) + \frac{pm}{m+d}(T_0 - \bar{T})]}{(1+\alpha \bar{T})^2} - m & 0 & -\frac{k\bar{T}}{1+\alpha \bar{T}} \\ \frac{\frac{k}{c}[g(E) + \frac{pm}{m+d}(T_0 - \bar{T})]}{(1+\alpha \bar{T})^2} & -(m+d) & \frac{k\bar{T}}{1+\alpha \bar{T}} \\ 0 & p & -c \end{bmatrix}.$$

The characteristic equation is

$$X^3 + b_1 X^2 + b_2 X + b_3 = 0. \quad (3.7)$$

where

$$\begin{aligned} b_1 &= c + m + d + \frac{\frac{k}{c}[g(E) + \frac{pm}{m+d}(T_0 - \bar{T})]}{(1+\alpha \bar{T})^2} + m > 0, \\ b_2 &= \left[\frac{\frac{k}{c}[g(E) + \frac{pm}{m+d}(T_0 - \bar{T})]}{(1+\alpha \bar{T})^2} + m \right] (c + m + d) + c(m + d) - \frac{kp\bar{T}}{1+\alpha \bar{T}} > 0, \\ b_3 &= \frac{(m+d)k[g(E) + \frac{pm}{m+d}(T_0 - \bar{T})]}{(1+\alpha \bar{T})^2} + cm(m+d) - \frac{kp\bar{T}}{1+\alpha \bar{T}}m \\ &> m \left[c(m+d) - \frac{kp\bar{T}}{1+\alpha \bar{T}} \right] \\ &= m \left[\frac{c(m+d) + c\alpha(m+d)\bar{T} - kp\bar{T}}{1+\alpha \bar{T}} \right] \\ &= \frac{m}{1+\alpha \bar{T}} c(m+d) \left(1 - \frac{R_{v0}}{R_v(E)} \right) > 0. \end{aligned}$$

Further, we can verify that $b_1 b_2 - b_3 > 0$. Thus, by use of the Routh-Hurwitz criterion, we obtain the local stability of $\bar{U}(E)$. Then, we will arrive at the following result.

Theorem 3.2. (i) U_0 is locally asymptotically stable if $R_{v,0} < 1$ and it is unstable if $R_{v,0} > 1$.
(ii) For all $E > 0$, $\bar{U}(E)$ is always locally asymptotically stable whenever it exists.

4 Analysis of the slow system (2.4)

4.1 Existence of the equilibrium point

We consider the epidemiological and environmental variables to be slow variables, which consist of S , I and E . Assume that $R_v(E) > 1$, so that the fast system is at the stable nontrivial equilibrium $\bar{U}(E) = (\bar{T}_-(E), \bar{T}^*(E), \bar{V}(E))$ with

$$\begin{aligned}\bar{T}_-(E) &= \frac{T_0}{R_v(E)}, \\ \bar{V}(E) &= \frac{1}{c} \left[g(E) + \frac{p\Lambda}{m+d} \left(1 - \frac{1}{R_v(E)} \right) \right], \\ \bar{T}^*(E) &= \frac{\Lambda}{m+d} \left(1 - \frac{1}{R_v(E)} \right), \\ R_v(E) &= \frac{2T_0 \frac{R_{v,0}}{T_0^2}}{-a_2 - \sqrt{a_2^2 - 4a_1}} = \frac{2R_{v,0}}{T_0(-a_2 - \sqrt{a_2^2 - 4a_1})}.\end{aligned}$$

Since $N = S + I$ remains constant for all time, we only need to consider the following two-dimensional subsystem of system (2.4)

$$\begin{cases} \frac{dI}{dt} = \frac{\lambda E(N - I)}{1 + \beta(N - I)} - \mu I, \\ \frac{dE}{dt} = \theta I \bar{V}(E)(1 - E) - \gamma E. \end{cases} \quad (4.1)$$

Notice that the dynamic linkage for the between- and within-host systems is through the environmental contamination variable E . The dependence of E on the within-host dynamics is through the term θVI . The two subsystems will be decoupled if V is a constant, in which case we can obtain an isolated reproduction number for the between-host system. Without loss of generality, assume that the constant is $V = 1$. Let $W_0 = (I_0, E_0)$ denote an interior equilibrium of the slow system (4.1), where $I_0 > 0, E_0 > 0$. It is easy to show $E_0 = \frac{\theta I_0}{\theta I_0 + \gamma}$, and I_0 is the positive solution of the following equation

$$I^2 + c_1 I + c_2 = 0, \quad (4.2)$$

where

$$\begin{aligned}c_1 &= \frac{\gamma}{\theta} - \left(\frac{1}{\beta} + N \right) - \frac{\lambda}{\mu\beta}, \\ c_2 &= \frac{\lambda N}{\mu\beta} - \frac{\gamma}{\theta} \left(\frac{1}{\beta} + N \right).\end{aligned}$$

If $c_2 \leq 0$, or equivalently $R_{h,0} = \frac{\gamma\mu(1+N\beta)}{\lambda N\theta} > 1$, we know that equation (4.2) has one unique positive solution.

Thus, we get a unique positive equilibrium for system (4.1) as $W_0 = \left(\frac{c_1 + \sqrt{c_1^2 - 4c_2}}{2}, \frac{1}{1 + \frac{2\gamma}{\theta(c_1 + \sqrt{c_1^2 - 4c_2})}} \right)$.

When the two process are coupled with the fast system being near the equilibrium $\bar{U}(E)$, the variable V in the E equation will be replaced by $\bar{V}(E)$. Let $W = (\bar{I}, \bar{E})$ denote an interior equilibrium of the slow system, where $\bar{I} > 0, \bar{E} > 0$.

Let the right side of the first equation of (4.1) to be zero, we will get

$$\beta\mu I^2 - (\lambda\bar{E} + \mu + \beta\mu N)I + \lambda\bar{E}N = 0.$$

Solving this quadratic equation, we obtain

$$\bar{I}_{\pm} = \frac{\lambda\bar{E} + \mu + \beta\mu N \pm \sqrt{(\lambda\bar{E} + \mu + \beta\mu N)^2 - 4\beta\mu\lambda\bar{E}N}}{2\beta\mu} > 0.$$

Assume that $\frac{1}{\beta} > N$, then we have

$$\bar{I}_{+} = \frac{\lambda\bar{E} + \mu + \beta\mu N + \sqrt{(\lambda\bar{E} + \mu + \beta\mu N)^2 - 4\beta\mu\lambda\bar{E}N}}{2\beta\mu} > \frac{\lambda\bar{E} + \mu}{\beta\mu} \geq \frac{1}{\beta} > N.$$

But $N = S + I$, and thus, $\bar{I}_{+}$ does not exist. Therefore, $\bar{I} = \bar{I}_{-}$ and $\bar{E}$ satisfies the equation

$$\begin{aligned} & \frac{1 - \bar{E}}{c} \left[\frac{a\bar{E}}{1 + \bar{E}^2} + \frac{p\Lambda}{m + d} \left(1 - \frac{1}{R_v(\bar{E})} \right) \right] \\ &= \frac{\gamma}{2\theta\lambda N} (\lambda\bar{E} + \mu + \beta\mu N + \sqrt{(\lambda\bar{E} + \mu + \beta\mu N)^2 - 4\beta\mu\lambda\bar{E}N}). \end{aligned} \quad (4.3)$$

where N is the constant total population size. Let $F(E)$ and $G(E)$ denote the functions on the left and right sides of the equation (4.3), respectively,

$$\begin{aligned} F(E) &= \frac{1 - E}{c} \left[\frac{aE}{1 + E^2} + \frac{p\Lambda}{m + d} \left(1 - \frac{1}{R_v(E)} \right) \right], \\ G(E) &= \frac{\gamma}{2\theta\lambda N} (\lambda E + \mu + \beta\mu N + \sqrt{(\lambda E + \mu + \beta\mu N)^2 - 4\beta\mu\lambda EN}). \end{aligned}$$

Then, $\bar{E}$ is a solution of the equation $F(E) = G(E)$ within $(0, 1)$. Note that

$$\begin{aligned} G'(E) &= \frac{\gamma}{2\theta N} \left(1 + \frac{\lambda E + \mu(1 - \beta N)}{\sqrt{(\lambda E + \mu + \beta\mu N)^2 - 4\beta\mu\lambda EN}} \right) > 0, \\ G(0) &= \frac{\gamma\mu(1 + \beta N)}{\theta\lambda N} > 0, \\ G(1) &= \frac{\gamma}{2\theta\lambda N} (\lambda + \mu + \beta\mu N + \sqrt{(\lambda + \mu + \beta\mu N)^2 - 4\beta\mu\lambda N}) > 0, \\ F(0) &= \frac{p\Lambda}{c(m + d)} \left(1 - \frac{1}{R_{v0}} \right) > 0, \\ F(1) &= 0. \end{aligned}$$

it is easy to see that there exists at least one solution $\bar{E} \in (0, 1)$ if $F(0) > G(0)$, which is equivalent to

$$\frac{pmT_0}{c(m + d)} \left(1 - \frac{1}{R_{v0}} \right) \frac{1}{R_{h0}} > 1. \quad (4.4)$$

It follows from $F(0) > G(0)$ and $F(1) = 0$, that $F'(\bar{E}) < 0$.

From the condition (4.4), we can get the basic reproduction number for the slow system, when it is coupled with the fast system, as

$$R_h = \frac{pmT_0}{c(m + d)} \left(1 - \frac{1}{R_{v0}} \right) \frac{1}{R_{h0}}.$$

From the above discussion, we can summarize the following conclusion.

Theorem 4.1. (i) When $V = 1$, the system (4.1) has a unique nontrivial equilibrium W_0 if $R_{h0} \geq 1$.
(ii) If $V = \bar{V}(E)$, and $R_{v0} > 1$, then the system (4.1) has at least one endemic equilibrium $W = (\bar{I}, \bar{E})$ if $R_h > 1$ and $\beta N < 1$.

Although we cannot exclude the possibility of multiple endemic equilibria for the slow system, our numerical exploration suggests that for a wide range of parameter values the endemic equilibrium is unique as the curves of $F(E)$ and $G(E)$ have only one intersection for $E \in (0, 1)$ (see Fig.2).

4.2 Stability of the equilibrium point

The Jacobian matrix of the slow system at $W = (\bar{I}, \bar{E})$ is given by

$$J = \begin{bmatrix} -\frac{\lambda \bar{E}}{[1+\beta(N-\bar{I})]^2} - \mu & \frac{\lambda(N-\bar{I})}{1+\beta(N-\bar{I})} \\ \theta \bar{V}(\bar{E})(1-\bar{E}) & \theta \bar{I} \bar{V}'(\bar{E})(1-\bar{E}) - \theta \bar{I} \bar{V}(\bar{E}) - \gamma \end{bmatrix}$$

The characteristic equation is

$$X^2 + e_1 X + e_2 = 0, \quad (4.5)$$

where

$$e_1 = \frac{\lambda \bar{E}}{[1+\beta(N-\bar{I})]^2} + \mu - \theta \bar{I} \bar{F}'(\bar{E}) + \gamma > 0,$$

$$e_2 = \left[\frac{\lambda \bar{E}}{(1+\beta(N-\bar{I}))^2} + \mu \right] (\gamma - \theta \bar{I} \bar{F}'(\bar{E})) - \frac{\theta \lambda \bar{F}(\bar{E})(N-\bar{I})}{1+\beta(N-\bar{I})}.$$

Note that

$$\begin{aligned} X_1 + X_2 &= -e_1 < 0, \\ X_1 X_2 &= e_2. \end{aligned}$$

Therefore, both X_1 and X_2 have negative real parts if $e_2 > 0$, which is equivalent to

$$\frac{\left[\frac{\lambda \bar{E}}{1+\beta(N-\bar{I})} + \mu(1+\beta(N-\bar{I})) \right] (\gamma - \theta \bar{I} \bar{F}'(\bar{E}))}{\theta \lambda \bar{F}(\bar{E})(N-\bar{I})} > 1.$$

Let $R_{h0}^* = \frac{\left[\frac{\lambda \bar{E}}{1+\beta(N-\bar{I})} + \mu(1+\beta(N-\bar{I})) \right] (\gamma - \theta \bar{I} \bar{F}'(\bar{E}))}{\theta \lambda \bar{F}(\bar{E})(N-\bar{I})}$. Then the following result holds.

Theorem 4.2. *The endemic equilibrium $W = (\bar{I}, \bar{E})$ of the system (4.1) is locally asymptotically stable if $R_{h0}^* > 1$.*

5 Numerical results

In this section, some numerical simulation are provided to illustrate the above theoretical results.

Fig.1 plots the curves $R_v(E)$ and $\bar{T}_-(E)$ for two different R_{v0} values. We observe that while $\bar{T}_-(E)$ is more sensitive to E in the case when $R_{v0} < 1$ than in the case when $R_{v0} > 1$, $R_v(E)$ is more sensitive to changes in E when $R_{v0} > 1$. The parameter values used are $\Lambda = 5000$, $m = 0.31$, $d = 0.1$, $p = 10^4$, $c = 20$, $a = 10^8$, $\alpha = 10^{-3}$. The two R_{v0} values used in the two plots are determined by using different k values $k = 9 \times 10^{-7}$ and $k = 8.5 \times 10^{-7}$.

Fig.2 plots the curves determined by $F(E)$ and $G(E)$, whose intersection determines a nontrivial equilibrium of the system (4.1). The parameter values used are $\mu = 0.1$, $\lambda = 0.1$, $\gamma = 0.05$, $a = 10^4$, $\theta = 2 \times 10^{-6}$, $\Lambda = 5000$, $m = 0.3$, $d = 0.1$, $k = 0.5 \times 10^{-5}$, $p = 1000$, $c = 20$, $N = 10000$, $\beta = 0.5 \times 10^{-5}$, $\alpha = 0.5 \times 10^{-4}$.

Fig.3 are computed by use of the full system. We observe that the convergence to the interior equilibrium of the within-host variables V and T^* are much faster than the between-host variable I and the environment variable E . The parameter values used are $\mu = 0.1$, $\lambda = 0.1$, $\gamma = 0.05$, $a = 10^4$, $\theta = 2 \times 10^{-6}$, $\Lambda = 5000$, $m = 0.3$, $d = 0.1$, $k = 0.5 \times 10^{-5}$, $p = 1000$, $c = 20$, $N = 10000$, $\beta = 0.5 \times 10^{-5}$, $\alpha = 0.5 \times 10^{-4}$.

6 CONCLUSIONS

In this work, we developed and analyzed a model that couples explicitly the between- and within-host systems with nonlinear incidence rate through their connection to a contaminated environment. This coupled system provided new insights into the effect of each of the processes on the other.

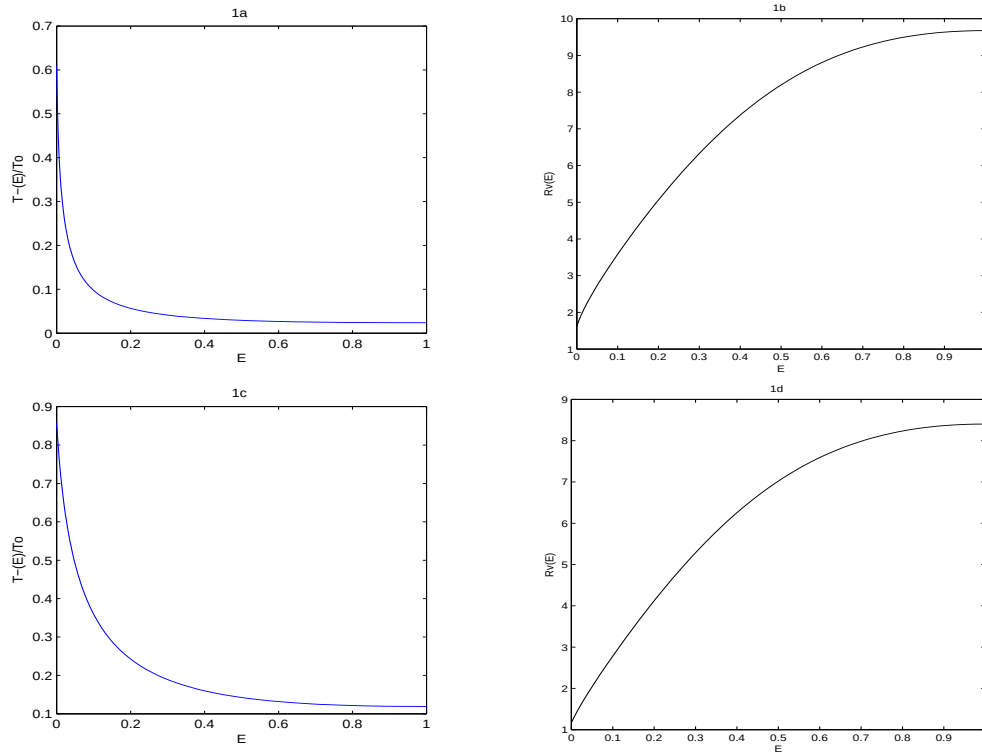


Figure 1: Fig.1 plots the curves $R_v(E)$ and $\bar{T}_-(E)$ for two different $R_{v,0}$ values. (1a) and (1b) for the case $R_{v,0} = 1.57 > 1$, in which case $\lim_{E \rightarrow 0} R_v(E) = R_{v,0}$; whereas figures (1c) and (1d) is for the case of $R_{v,0} = 0.6 < 1$, in which case $\lim_{E \rightarrow 0} R_v(E) = 1$.

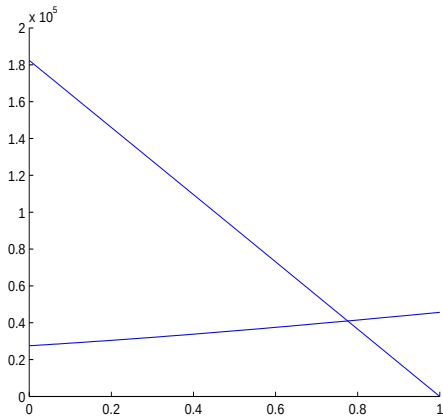


Figure 2: Plots of the curves determined by $F(E)$ and $G(E)$ whose intersection determines a nontrivial equilibrium of the fast system. The parameter values used are $\mu = 0.1, \lambda = 0.1, \gamma = 0.05, a = 10^4, \theta = 2 \times 10^{-6}, \Lambda = 5000, m = 0.3, d = 0.1, k = 0.5 \times 10^{-5}, p = 1000, c = 20, N = 10000, \beta = 0.5 \times 10^{-5}, \alpha = 0.5 \times 10^{-4}$.

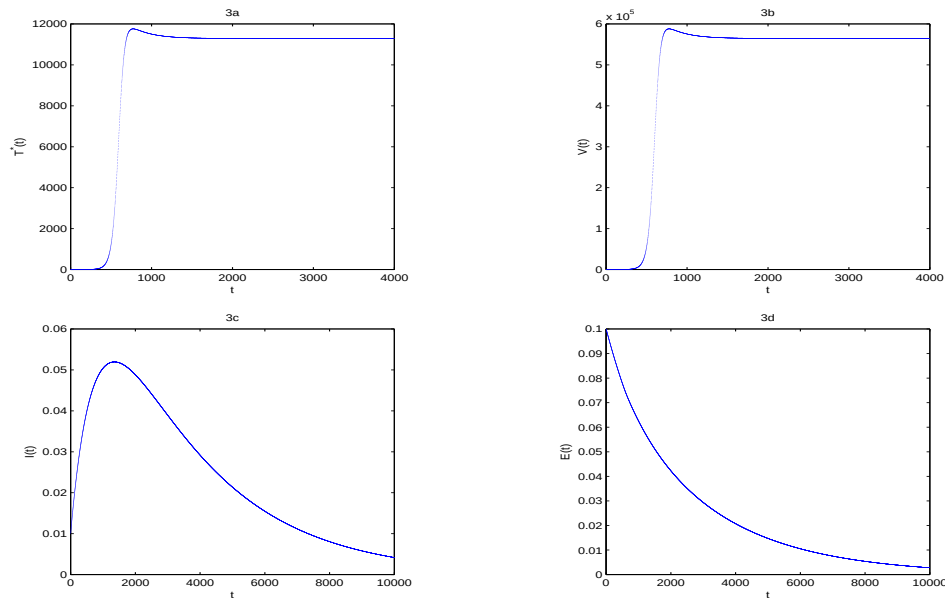


Figure 3: Fig.3 illustrates the dynamic behaviors of the slow system. The slow variables I and E are plotted as a function of time and are shown to tend to an endemic equilibrium as t increases.

Results 3.1-4.2 suggest that the existence and stability of equilibrium points of the fast and slow systems are connected to three reproduction numbers: R_{v0} , R_h and R_{h0}^* , respectively. These results can be summarized as follows:

- A** For the fast system, the infection-free equilibrium U_0 is locally asymptotically stable if $R_{v0} < 1$. This means that the disease will distinct in the host as time is larger enough. When $R_{v0} > 1$, U_0 is unstable and a unique interior equilibrium $U(E)$ exists and is locally asymptotically stable. For this case, the disease will persist in the host for all time.
- B** For the slow system, under the condition that the fast system has the stable interior equilibrium $U(E)$ (when $R_{v0} > 1$), there exists at least one endemic equilibrium W if $R_h > 1$ and $\beta N < 1$. That is, we get the sufficient condition for the possible disease persistence in the population level.
- C** The endemic equilibrium W of the slow system is locally asymptotically stable if $R_{h0}^* > 1$. This means that if the parameters satisfy the threshold, the disease will persist in population level.

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