

Original Research Article

THE STABILITY AND CONVERGENCE ANALYSIS OF A NEW CONJUGATE GRADIENT METHOD FOR SOLVING NONLINEAR OPTIMIZATION PROBLEMS

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ABSTRACT

The development of any conjugate gradient method could be viewed from the perspective of approximating an objective function $f(x)$ by a functional $F(x)$ noting that the properties of the functional can be used to characterize the method. Using the functional $F(x)$, the new conjugate gradient method was developed and used to solve many nonlinear optimization problems with high efficiency and accuracy. The numeric analysis of its stability and convergence becomes imperative in order to establish the reliability of the method and satisfy the yearnings of its increasing users. In this paper, we present the stability and convergence analysis of the new conjugate gradient method.

Keywords: Conjugate gradient method, convergence, stability, objective function.

1. INTRODUCTION

The new conjugate gradient method (NCGM) [1] is an effective iterative scheme for optimizing non-linear objective functions involved in many optimization problems. It is robust and generally able to achieve rapid convergence to an accurate solution. The traditional criterion for ensuring that a numerical method is stable is called absolute-stability [2]. A conjugate gradient method is said to be absolutely stable for given step lengths α if, for these α , the error bound $|E_j|$ satisfy the criterion $|E_j| < 1$, $j = 0, 1, 2, \dots, n$. An interval (a, b) of the real line is said to be an interval or region of absolute stability if the method is absolutely stable for all $\alpha \in (a, b)$ [3]. Numerical stability analysis of NCGM is carried out using the basic properties of Lanczos algorithm for tridiagonalizing a matrix while its absolute stability proof is derived from the error analysis.

Every conjugate gradient algorithm is known to be closely related to Lanczos algorithm for tridiagonalizing a matrix [4]. Greenbaum [5, 6] have considered the close connection between the Lanczos and conjugate gradient algorithm in the analysis of stability

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of conjugate gradient computations under perturbations in finite arithmetic. Paige [7, 8, 2] stated that every conjugate gradient method (CGM) is a tridiagonalizing procedure. Greenbaum, amongst others, agreed with Paige and added that the symmetric tridiagonal matrix produced by symmetric Lanczos process gives the same pivot recurrence as the tridiagonal matrix produced by the CGM. They concluded that if the original matrix involved in the CGM is symmetric and positive definite, then, the CGM is absolutely stable.

In section 2, we describe the NCGM with its algorithm and how it satisfies the convergence theorem. In section 3, we explain the basic vector updates for the search directions. The natural association between the new conjugate gradient algorithm and the Lanczos process has been established in section 4. This exercise represents the stability proof of NCGM. Some numerical examples were considered in section 5. The examples confirmed that the error bound $|E_j|$, in each case, satisfy the criterion $|E_j| < 1, j = 0, 1, 2, \dots, n$. Section 6 summarizes the findings of this paper with a conclusion.

2. THE NEW CONJUGATE GRADIENT METHOD

The new conjugate gradient method has been designed to optimize a non-linear objective functional, F :

$$\text{Optimize } F(x) \quad (1)$$

where $F(x) = \sum_{j=0}^n \frac{1}{j!} (x^{\frac{j}{2}})^T \nabla^j F(x_0) x^{\frac{j}{2}}$, $n > 2$ is the degree of F . The gradient functional is

$$G(x) = F'(x) = \nabla F(x_0) + \sum_{j=2}^n \frac{1}{(j-1)!} \left(x^{\frac{j-1}{2}} \right)^T \nabla^j F(x_0) x^{\frac{j-1}{2}}. \quad (2)$$

$F(x)$ approximates $f(x)$ at point $x \in \Re^n$ and $g(x) = f'(x)$. The following algorithm characterizes the NCGM.

2.1 ALGORITHM (NCGM)

i. Input initial values x_0 and $D_0 = -G_0 = -g_0$.

ii. Repeat:

a. Find step length α_k such that

$$F(x_k + \alpha_k D_k) = \min_{\alpha > 0} F(x_k + \alpha D_k)$$

b. Compute new point:

$$x_{k+1} = x_k + \alpha_k D_k$$

c. Update search direction:

$$D_{k+1} = -G_{k+1} + \beta_k D_k,$$

$$G_{k+1} = \frac{1}{2!} [g(x_k + 2\Delta x_k) + g(x_k)]$$

$$\beta_k = \frac{\|G_{k+1}\|^2}{D_k^T y_k}; \quad \|G_k\| = (G_k^T G_k)^{\frac{1}{2}}.$$

$$y_k = G_{k+1} - G_k$$

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74 d. Check for optimality of g :
 75 Terminate iteration at step m when $\|g_m\|$ is so small that x_m is an acceptable
 76 estimate of the optimal point x^* of F . If not optimal set $k = k + 1$.

77 2.2 CONVERGENCE OF THE NEW CONJUGATE GRADIENT METHOD

78 In order to establish the convergence of the above algorithm, we assume that the
 79 objective function satisfies the following conditions:

- 80 1. F is bounded below in $\mathfrak{R}^N$ and is continuously differentiable in a neighborhood Z of the
 81 level set $L = \{x_1, x \in \mathfrak{R}^N : F(x) \leq F(x_1)\}$
 82 2. The gradient $\nabla F(x)$ is Lipschitz continuous in Z , namely, there exists a constant $L > 0$
 83 such that

$$84 \quad \|\nabla F(x) - \nabla F(y)\| \leq L \|x - y\|, \text{ for any } x, y \in Z \quad (3)$$

85 2.3 LEMMA (EXISTENCE OF A GLOBAL OPTIMUM OF f)

86 Suppose that x_1 is a starting point for which the above assumptions are satisfied.
 87 Consider the conventional conjugate gradient method where d_k is the descent direction and
 88 α_k (the step length of line search) satisfies the standard Wolfe conditions. Then, we have
 89 that

$$90 \quad \sum_{k \geq 1} \frac{(g_k^T d_k)^2}{\|d_k\|^2} < \infty \quad (4)$$

91 **Proof.** (See the proof by Dai and Yuan [10])

92 Dai and Yuan proved this lemma for the algorithm of any conventional conjugate
 93 gradient method (CGM). The proof for our algorithm is same when we put $G(x_k)$ in place of
 94 $g(x_k)$ and D_k in place of d_k . Dai and Yuan made it simple to see that with

$$95 \quad G_{k+1} = \frac{1}{2!} [g(x_k + 2\Delta x_k) + g(x_k)] \text{ we have}$$

$$96 \quad \sum_{k \geq 1} \frac{(G_{k+1}^T D_k)^2}{\|D_k\|^2} = \sum_{k \geq 1} \frac{[(g(x_k + 2\Delta x_k)^T + g(x_k)^T) D_k / 2]^2}{\|D_k\|^2}$$

$$97 \quad = \sum_{k \geq 1} \frac{[g(x_k + 2\Delta x_k)^T D_k / 2]^2}{\|D_k\|^2} + \sum_{k \geq 1} \frac{[g(x_k)^T D_k / 2]^2}{\|D_k\|^2} < \infty$$

98 Hence,

$$99 \quad \sum_{k \geq 1} \frac{(G_{k+1}^T D_k)^2}{\|D_k\|^2} < \infty. \quad (5)$$

100 2.4 THEOREM (CONVERGENCE OF A CONVENTIONAL CGM)

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102 Suppose that x_1 is a starting point for which the above assumptions and the lemma
 103 are satisfied. Let $\{x_k, k = 1, 2, \dots\}$ be generated by an algorithm for a conventional CGM.
 104 Then, the algorithm either terminates at a stationary point or converges in the sense that
 105 $\liminf_{k \rightarrow \infty} \|g(x_k)\| = 0$ (6)

106 Proof: (See the proof by Dai and Yuan [10])

107 Dai and Yuan used proof by contradiction to prove this theorem for a conventional
 108 CGM. The proof of this theorem for the above algorithm is same when we put $G(x_k)$ in
 109 place of $g(x_k)$ and D_k in place of d_k . It is not difficult to see that
 110 $\liminf_{k \rightarrow \infty} \|g(x_k)\| = 0$ when $g(x_k) = 0$ (zero vector) and $\Delta x_k = 0$ (zero vector, no further

111 improvement on x_k). Hence, with $G_{k+1} = \frac{1}{2!} [g(x_k + 2\Delta x_k) + g(x_k)]$ we have

$$112 \liminf_{k \rightarrow \infty} \|G(x_{k+1})\| = \liminf_{k \rightarrow \infty} \|g(x_k + 2\Delta x_k) + g(x_k)\| / 2 = 0$$

113 This implies that $\liminf_{k \rightarrow \infty} \|G(x_k)\| = 0$. We conclude that the new conjugate gradient
 114 method will converge to the global optimum of F and hence f .

115 3. BASIC VECTOR UPDATES

116 The two basic vector updates needed to update the search direction and the
 117 computed solutions by NCGM are

$$118 D_{k+1} = r_{k+1} + \beta_k D_k \quad (7)$$

119 and

$$120 r_{k+1} = r_k - \alpha_k W P_k \quad (8)$$

121 where $r_k = -G_k$

122 Equation (4) is an implicit vector update since

$$123 F(x) = \sum_{j=0}^n \frac{1}{j!} (x^{\frac{j}{2}})^T \nabla^j F(x_0) x^{\frac{j}{2}}$$

$$124 F'(x) = \nabla F(x_0) + \sum_{j=2}^n \frac{1}{(j-1)!} \left(x^{\frac{j-1}{2}} \right)^T \nabla^j F(x_0) x^{\frac{j-1}{2}}$$

$$125 F'(x_0) = \nabla F(x_0) + \sum_{j=2}^n \frac{1}{(j-1)!} \left(x_0^{\frac{j-1}{2}} \right)^T \nabla^j F(x_0) x_0^{\frac{j-1}{2}} = 0$$

$$126 \nabla F(x_0) = - \sum_{j=2}^n \frac{1}{(j-1)!} \left(x_0^{\frac{j-1}{2}} \right)^T \nabla^j F(x_0) x_0^{\frac{j-1}{2}};$$

$$127 b = - \sum_{j=2}^n \frac{1}{(j-1)!} \left(x_0^{\frac{j-1}{2}} \right)^T \nabla^j F(x_0) x_0^{\frac{j-1}{2}};$$

$$128 r(x) = -F'(x)$$

$$\begin{aligned}
129 \quad &= -(b + \sum_{j=2}^n \frac{1}{(j-1)!} \left(x^{\frac{j-1}{2}} \right)^T \nabla^j F(x_0) x^{\frac{j-1}{2}}) \\
130 \quad &= -(\sum_{j=2}^n \frac{1}{(j-1)!} \left((x - x_0)^{\frac{j-1}{2}} \right)^T \nabla^j F(x_0) (x - x_0)^{\frac{j-1}{2}}) \\
131 \quad &r(x_k) = -(\sum_{j=2}^n \frac{1}{(j-1)!} \left((x_k - x_0)^{\frac{j-1}{2}} \right)^T \nabla^j F(x_0) (x_k - x_0)^{\frac{j-1}{2}}) \\
132 \quad &r(x_{k+1}) = -F'(x_{k+1}) = -F'(x_k + \alpha_k P_k) \\
133 \quad &= -(\sum_{j=2}^n \frac{1}{(j-1)!} \left((x_k + \alpha_k P_k - x_0)^{\frac{j-1}{2}} \right)^T \nabla^j F(x_0) (x_k + \alpha_k P_k - x_0)^{\frac{j-1}{2}}) \\
134 \quad &= -((\sum_{j=2}^n \frac{1}{(j-1)!} \left((x_k - x_0)^{\frac{j-1}{2}} \right)^T \nabla^j F(x_0) (x_k - x_0)^{\frac{j-1}{2}}) + \\
135 \quad &\quad (\sum_{j=2}^n \frac{1}{(j-1)!} \left((\alpha_k P_k)^{\frac{j-1}{2}} \right)^T \nabla^j F(x_0) (\alpha_k P_k)^{\frac{j-1}{2}})) \\
136 \quad &r(x_{k+1}) = r(x_k) - (\sum_{j=2}^n \frac{1}{(j-1)!} \left((\alpha_k P_k)^{\frac{j-1}{2}} \right)^T \nabla^j F(x_0) (\alpha_k P_k)^{\frac{j-1}{2}})
\end{aligned}$$

137 But

$$\begin{aligned}
138 \quad &(\sum_{j=2}^n \frac{1}{(j-1)!} \left((\alpha_k P_k)^{\frac{j-1}{2}} \right)^T \nabla^j F(x_0) (\alpha_k P_k)^{\frac{j-1}{2}}) = \\
139 \quad &\alpha_k \left(\nabla^2 F(x_0) + \frac{1}{2!} \alpha_k P_k^T \nabla^3 F(x_0) + \dots + \frac{1}{(j-1)!} (\alpha_k P_k)^{j-2} \nabla^j F(x_0) \right) P_k \\
140 \quad &= \alpha_k W P_k
\end{aligned}$$

141 where

$$142 \quad W = \left(\nabla^2 F(x_0) + \frac{1}{2!} \alpha_k P_k^T \nabla^3 F(x_0) + \dots + \frac{1}{(j-1)!} (\alpha_k P_k)^{j-2} \nabla^j F(x_0) \right)$$

$$143 \quad r(x_{k+1}) = r(x_k) - \alpha_k W P_k \quad (9)$$

144 **4. AN ABSOLUTE-STABILITY PROOF:(A NEW CONJUGATE GRADIENT** 145 **METHOD)**

146 Here, we need to show that the symmetric tridiagonal matrix produced by the
147 symmetric Lanczos process gives the same pivot recurrence as the tridiagonal matrix
148 produced by the NCGM. Also, we need to show that the error bound $|E_j|$ satisfy the
149 criterion $|E_j| < 1, j = 0, 1, 2, \dots, n$.

150 In line with Hageman [9] and Paige [2], the two basic vector updates of the NCGM
151 given in equations (3) and (4) can be summarized as

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$$R(I - J) = WPV, \quad PU = R \quad (10)$$

where R and P are matrices containing the vector sequences $\{r_k\}$ and $\{p_k\}$ as columns.

$J = (\delta_{i,j+1})$, $V = \text{diag}(\alpha_k)$, $c = (\frac{1}{\beta_k})$ and $U = \frac{1}{c} \text{diag}(\|(J^{-1} - cI)\|)$. It follows from equation (10) that

$$I - J = R^{-1}WPV$$

$$(I - J)V^{-1} = R^{-1}WP$$

$$(I - J)V^{-1}U = R^{-1}WPU$$

$$= R^{-1}WR, \text{ a tridiagonal matrix}$$

With $H = (I - J)V^{-1}U$, a tridiagonal matrix, we have that

$$WR = RH$$

since

$$RH = R(I - J)V^{-1}U$$

$$= WPVV^{-1}U$$

$$= WPU$$

$$= WR$$

The Lanczos method for constructing an orthonormal matrix Q that reduces W to symmetric tridiagonal form T by $T = Q^T W Q$ could be constructed from the NCGM by letting $q_i = D_i$ be the diagonal elements of Q. The orthonormal tridiagonalization becomes

$WQ = QT$ with $Q = RC^{-1}$, $T = C^{-1}HC$. Using the fact that H is tridiagonal and that

$H = C^{-1}TC$ with C diagonal, we see that the factorization of T generates the same pivot sequence as W. This is true since

$$H = C^{-1}TC; CH = TC$$

$$WQ = WRC^{-1}$$

$$= RHC^{-1}; WR = RH$$

$$= QCHC^{-1}; R = QC$$

$$= QTCC^{-1}; CH = TC$$

For the region of absolute stability, we recall the step length $\alpha_c = \frac{2(1-\varepsilon)}{M}$, $0 < \varepsilon < 1$ and

error function defined by $E_k = E(x_k) = x_k - x^*$ where $\frac{\|E_{k+1}\|}{\|E_k\|} \leq Z = \|I - \frac{\alpha_c F''_{kc}}{F'_{opt}}\|$

and $Z = \max(|1 - \frac{\alpha_c m}{\|F'_{opt}\|}|, |1 - \frac{M\alpha_c}{\|F'_{opt}\|}|)$ from [1]. M and m are the biggest and

smallest eigenvalues of F''_{kc} respectively. x^* is the exact solution of the optimization problem stated in equation (1). It follows that

$$Z_k = \max(|1 - \frac{2m(1-\varepsilon)}{\|F'_k\| M}|, |1 - \frac{2(1-\varepsilon)}{\|F'_k\|}|), \quad k = 0, 1, 2, \dots, n$$

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185 and
 186 $|E_{k+1}| \leq Z_k |E_k|$, $k = 0, 1, \dots, n$; $|E_0| = |x_0 - x^*|$ or
 187 $|E_k| \leq Z_{k-1} |E_{k-1}|$, $k = 0, 1, \dots, n$; $|E_0| = |x_0 - x^*|$
 188 defines the region of absolute stability of the method. This is true since, with $0 < \varepsilon < 1$ and
 189 $M \geq m > 0$,

$$190 \quad Z_{k-1} = \frac{1}{|F'_{k-1}|} \max(|F'_{k-1}| - \frac{2m(1-\varepsilon)}{M}, |F'_{k-1}| - 2(1-\varepsilon)|), \quad k = 1, 2, \dots, n$$

$$191 \quad |F'_{k-1}| > |F'_{k-1}| - \frac{2m(1-\varepsilon)}{M} \quad \text{and} \quad |F'_{k-1}| > |F'_{k-1}| - 2(1-\varepsilon)|, \quad k = 1, 2, \dots, n$$

192 implies that $Z_{k-1} < 1$. Convergence of the new conjugate gradient method ensures that
 193 $|E_{k-1}| \rightarrow 0$ as $k \rightarrow \infty$. We have shown that the error bound

$$194 \quad |E_k| \leq Z_{k-1} |E_{k-1}| < 1, \quad k = 1, 2, \dots, n$$

195 defines the region of absolute stability of the new conjugate gradient method. We note that
 196 $\nabla^j F(x)$, $j = 3, 4, \dots, n$ are the matrix vector differentials of $\nabla^2 F(x)$. Therefore, if $\nabla^2 F(x)$
 197 is symmetric, then, W must be symmetric too. Also, W must be positive definite since $A =$
 198 $\nabla^2 F(x_0)$ must be positive definite for an optimal point of F to exist. Since W is symmetric
 199 and positive definite, we conclude that the new conjugate gradient method is absolutely
 200 stable.

201 5. NUMERICAL EXAMPLES

202 Numerical results obtained from the optimization of the following problems [11] are
 203 hereby presented.

204 Problem 1(Rastrigin function; $n=2$):

$$205 \quad \text{Minimize } F(x, y) = 20 + x_1^2 + x_2^2 - 10(\cos(2\pi x_1) + \cos(2\pi x_2)),$$

$$206 \quad x_0 = [0.1, 0.1], \quad x^* = [0, 0]$$

207 Problem 2:

$$208 \quad \text{Minimize } F(x_1, x_2) = \exp(x_1)(4x_1^2 + 2x_2^2 + 4x_1x_2 + 2x_2 + 1),$$

$$209 \quad x_0 = [-1, 1], \quad x^* = [-1.5, 1]$$

210 Problem 3:

$$211 \quad \text{Minimize } F(x_1, x_2) = (x_1^2 - x_2)^2 + (x_2 - 1)^2,$$

$$212 \quad x_0 = [1, 0], \quad x^* = [1, 1]$$

213 Problem 4:

$$214 \quad \text{Minimize } f(x_1, x_2) = (x_1 + 1)^3 + x_2$$

215 Subject to

$$216 \quad x_1 - 1 \geq 0, \quad x_2 \geq 0.$$

$$217 \quad x^* = [1, 0]$$

218 Problem 5:

$$219 \quad \text{Minimize } J(x, u) = 0.1 \sum_{i=1}^{10} (x(i)^2 + u(i)^2)$$

220 Subject to

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221 $x_i = x_{i-1} + 0.1u_{i-1}$
 222 $x_0 = 1, u_0 = 0.5; i = 1, 2, \dots, 10.$
 223 Problem 6:

224 Minimize $I(x, u) = \frac{1}{4} \int_0^1 u^4(t) dt$

225 Subject to

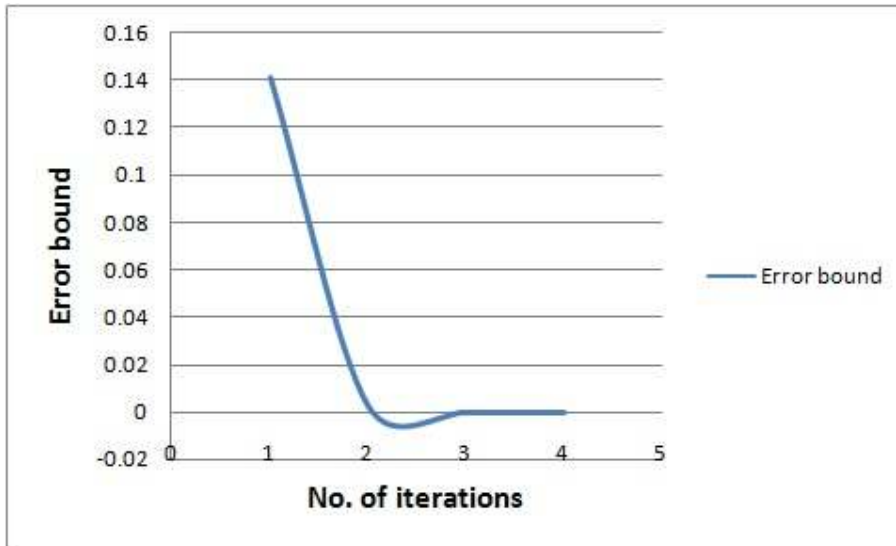
226 $\dot{x} = x(t) + u(t); 0 \leq t \leq 1; x(0) = 1, x(1) = 0; u(0) = -2, \Delta t = 0.1$
 227 $x^* = 0, u^* = -1.29735$
 228

229 The following Tables and Figures confirm the fact that the error bounds generated by
 230 NCGM satisfy the condition $|E_k| \leq Z_{k-1} |E_{k-1}| < 1, k = 1, 2, \dots, n$

231
 232 **Table 1:** (Solution of problem 1 by NCGM)
 233

No. of iterations	x_1	x_2	$F(x_1, x_2)$	Error bound
				$ E_k = x_k - x^* $
1	0.1	0.1	3.824634	0.141421
2	0.001849	0.001849	0.001351	2.615534E-03
3	0.000037	0.000037	0.000001	5.166958E-05
4	0.000001	0.000001	0	1.020735E-06

234
 235 **Figure 1:** (Error bound of problem 1 by NCGM)
 236



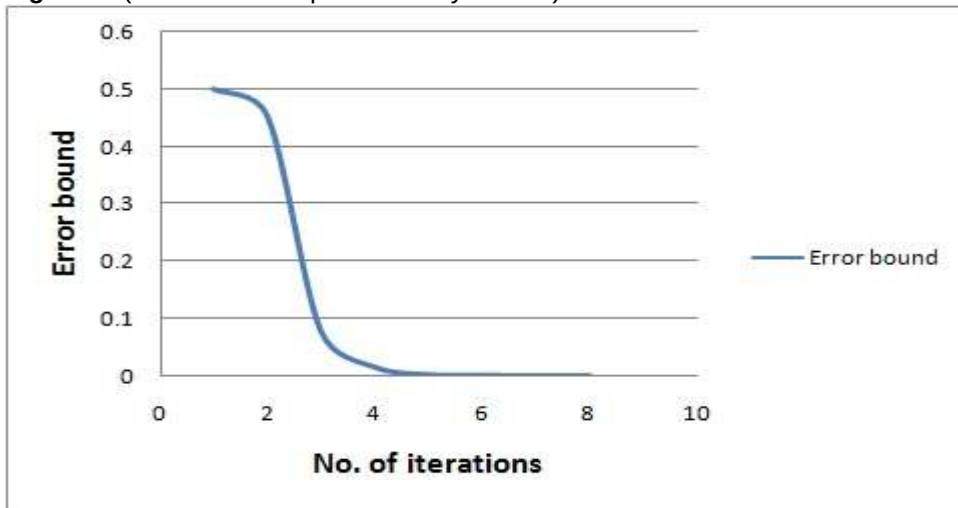
237
 238
 239
 240 **Table 2:** (Solution of problem 2 by NCGM)

No. of iterations	x_1	x_2	$F(x_1, x_2)$	Error bound
				$ E_k = x_k - x^* $

1	-1	1	1.839397	0.5
2	-1.133956	0.732088	1.724263	0.453613
3	-1.547138	0.938846	1.789058	7.721252E-02
4	-1.512419	1.008243	1.78498	1.490518E-02
5	-1.498656	1.001342	1.785044	1.899201E-03
6	-1.499497	0.999665	1.785041	6.045778E-04
7	-1.500055	0.999945	1.785041	7.752504E-05
8	-1.50002	1.000013	1.785041	2.429327E-05

241
242
243

Figure 2: (Error bound of problem 2 by NCGM)



244

245
246

Table 3: (Solution of problem 3 by NCGM)

No. of iterations	x_1	x_2	$F(x_1, x_2)$	Error bound
				$ E_k = x_k - x^* $
1	1	0	2	1
2	0.5002	0.4998	0.3125	0.707107
3	0.99997	1.001372	0.000004	1.371892E-03
4	1.00054	1.000814	0.000001	9.764222E-04
5	0.999993	1.000252	0	2.523227E-04
6	1.000099	1.000149	0	1.789777E-04
7	0.999999	1.000046	0	4.554963E-05
8	1.000018	1.000027	0	3.219506E-05
9	1	1.000008	0	8.070761E-06
10	1.000003	1.000005	0	5.683448E-06
11	1	1.000001	0	1.402926E-06
12	1.000001	1.000001	0	9.841288E-07

247
248

Figure 3: (Error bound of problem 3 by NCGM)

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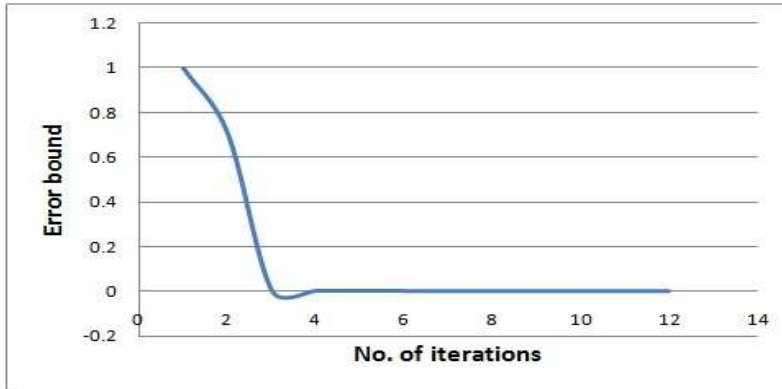


Table 4: (Solution of problem 4 by NCGM)

No. of iterations	x_1	x_2	$F(x_1, x_2)$	Error bound
				$ E_k = x_k - x^* $
1	1.2	0.2	3.749334	0.2
2	1.139556	0.2604439	3.525193	0.139556
30	1.000002	0.03880354	2.705478	1.788139E-06
31	1.000002	0.0388035	2.705478	1.788139E-06
32	1.000002	0.0388035	2.705478	1.788139E-06
33	1.000002	0.0388035	2.705478	2.026558E-06
34	1.000002	5.871513E-06	2.66668	2.026558E-06
35	1.000002	3.903678E-06	2.66668	2.026558E-06
36	1.000002	3.928141E-06	2.66668	2.026558E-06
37	1.000002	3.928141E-06	2.66668	2.026558E-06

Figure 4: (Error bound of problem 4 by NCGM)

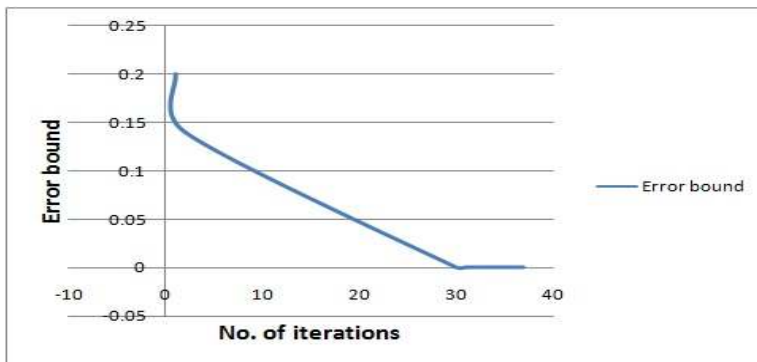


Table 5: (Solution of problem 5 by NCGM)

No. of iterations	x	u	$F(x, u)$	Error bound
				$ E_k $
1	1.011796	4.084E-04	0.102373	1.346319E-02

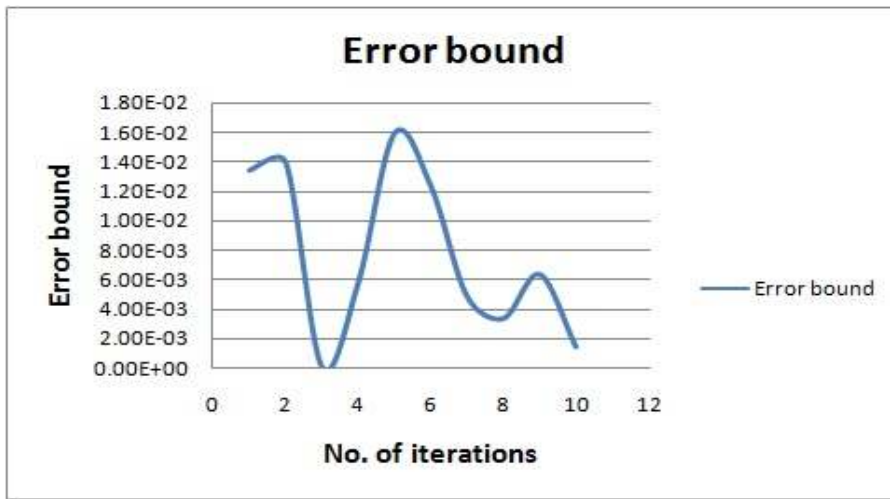
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2	0.974485	4.27E-7	0.197335	1.3985E-2
3	0.969326	9.77E-12	0.291295	1.13E-4
4	0.947307	3.927E-15	0.381034	5.838E-3
5	0.890482	1.068E-17	0.460329	0.015985
6	0.878034	1.652E-21	0.537424	1.239899E-2
7	0.825365	4.942E-24	0.605546	0.004888
8	0.775855	1.221E-26	0.665741	3.398E-03
9	0.729315	3.32E-29	0.718931	6.397E-03
10	0.658226	2.346E-31	0.762258	1.4801E-03

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Figure 5: (Error bound of problem 5 by NCGM)



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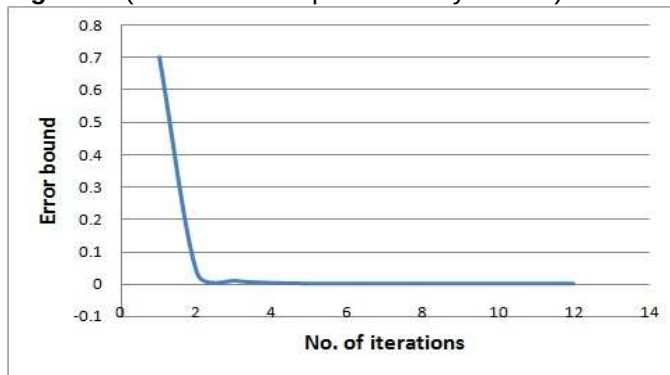
Table 6: (Solution of problem 6 by NCGM)

No. of iterations	x	u	$F(x, u)$	Error bound
				$ E_k $
1	1	-2	4	0.702651
2	0.000000001	-1.330793	0.784119	3.344284E-02
3	0.000000001	-1.306902	0.729310	9.552149E-03
4	0.000000001	-1.300079	0.714199	2.728966E-03
5	0.000000001	-1.298130	0.709926	7.802653E-04
6	0.000000001	-1.297573	0.708709	2.237162E-04
7	0.000000001	-1.297415	0.708362	6.476587E-05
8	0.000000001	-1.297369	0.708263	1.936964E-05
9	0.000000001	-1.297356	0.708235	6.404475E-06
10	0.000000001	-1.297353	0.708227	2.701624E-06
11	0.000000001	-1.297353	0.708227	2.689312E-06
12	0.000000001	-1.297352	0.708227	2.676999E-06

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268 **Figure 6:** (Error bound of problem 6 by NCGM)



270 **6. CONCLUSION**

271 The new conjugate gradient method has a natural connection with the Lanczos process for
 272 solving systems of equations. Also, it satisfies the traditional criterion for ensuring that a
 273 numerical method is stable. Therefore, the NCGM is efficient, accurate, reliable and
 274 absolutely stable.

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