

Original Research Article

A HYBRID OF THE NEW CONJUGATE GRADIENT METHOD AND GALERKIN THEORY FOR OPTIMIZING BEAM DEFLECTION UNDER UNIFORMLY DISTRIBUTED LOAD

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ABSTRACT

A hybrid of the new Conjugate gradient method and Galerkin theory has been used to find the maximum deflection of a beam under uniformly distributed load. Maximum deflection of a beam under a given pressure was found by solving a two-point linear boundary value problem with homogeneous boundary conditions without evaluating the inverse of a matrix. An objective function associated with a given member of this class of boundary value problems was optimized. The numerical results obtained from solving some of these problems are very close to the exact solutions. This method is easy to implement and automate computer-wise.

Keywords: Hybrid of the new conjugate gradient method, Two-point linear boundary value problems, Objective function, Galerkin theory.

1. INTRODUCTION

Burden and Faires [1] have shown how physical problems that are position-dependent rather than time-dependent could be described in terms of differential equations. A differential equation of this type has conditions imposed at more than one point. A common problem in civil engineering is to find the maximum deflection of a beam subject to uniform loading while the ends are supported in order to avoid deflection at the two fixed ends. A mathematical model of this physical phenomenon results in the formulation of a boundary value problem (BVP). A BVP is an ordinary differential equation with specified values at the extreme points or boundaries of a given system [2]. A beam deflection model, formulated from figure (1), seeks the values of a function $y(x)$ from the following two-point linear boundary value problem.

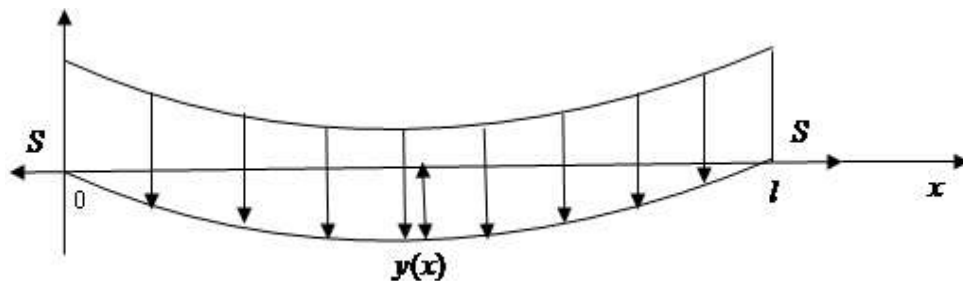


Figure 1: Simply supported beam with a uniformly distributed load.

$$\frac{d^2 y(x)}{dx^2} = \frac{S}{EI} y(x) + \frac{q}{2EI} x(l-x), 0 \leq x \leq l \quad (1)$$

$$y(0) = 0 = y(l),$$

where x is the location along the beam, l is the length of the beam, E is the young's modulus of elasticity of the beam, I is the central moment of inertia, S is the stress at the endpoints and q is the intensity of the uniform load. This paper will concentrate on linear two-point boundary value problem given in equation (1). This type of boundary value problem is assumed to have a unique solution, $y(x)$, since $\frac{q}{2E} x(l-x)$ is continuous in the given interval.

2. LITERATURE REVIEW

Numerical methods have been used to generate an approximate solution of the conventional linear two-point boundary value problem since the analytic solution is very difficult to handle. This problem takes the form

$$p(x)y''(x) + q(x)y'(x) + r(x)y(x) = f(x), [a, b]. \quad (2)$$

A unique solution of equation (2) exists if $p(x)$, $q(x)$, $r(x)$ and $f(x)$ are continuous in $[a, b]$ [1, 3, 4]. Shooting method and finite difference method have been used to solve this class of problems [1, 3, 4, 5]. Fyfe, in 1968 introduced cubic spline interpolation method on solving equation (2). After Fyfe, many researchers, including Burden and Faires, used linear and cubic spline interpolations with Galerkin method and finite element method to solve same problem [6, 7]. Galerkin method is one of the best methods for solving (2) numerically. The numerical solutions of this class of boundary value problems are very good but difficult to implement. The method leads to full matrices that must be inverted in order to obtain the required solutions [8]. Bamigbola and Ejieji [9] showed that the properties of the objective function of (2) could be explored to design an efficient conjugate gradient method for solving (2). The new conjugate gradient method [10] has been used to solve many optimization problems successfully without involving matrix inversion. We, therefore, present a hybrid of the new conjugate gradient method and Galerkin theory for solving problem (1) with ease and high accuracy.

3. GALERKIN METHOD

Galerkin method is a variation technique used in solving equation (2) numerically. In this paper, it is based on the fact that a function $y \in C^2[0, l]$ is the unique solution of (1) if and only if y is the unique function in $C^2[0, l]$ whenever

$$\int_0^l \{y''(x) - r(x)y(x) - f(x)\} \phi_i(x) dx = 0 \quad (3)$$

where

$$r(x) = \frac{S}{EI}, f(x) = \frac{q}{2EI} x(l-x), i = 1, \dots, n \text{ and each } \phi_i \text{ is a weight function. Equation}$$

(3) yields n residual equations in n unknowns. This method approximates the solution of (1) by solving the system of equations derived from equation (3) simultaneously. In order to solve this system of equations, we used the finite element method as a variation tool. Thus, we partition the interval $[0, l]$ into $n+1$ subintervals with n interior points such that

$$0 = x_0 < x_1 < \dots < x_n < x_{n+1} = l; h = x_{j+1} - x_j; x_j = jh; j = 0, 1, 2, \dots, n.$$

75 A small set of functions which consists of linear combinations of some basis functions
 76 $\phi_1(x), \phi_2(x), \dots, \phi_n(x)$ contains the approximate solutions of (1) [11]. These functions are
 77 required to be linearly independent and satisfy the conditions given in equation (4) below.

$$78 \quad \phi_i(0) = \phi_i(1) = 0; \quad i = 1, 2, \dots, n. \quad (4)$$

79 In this paper, we used the conventional basis functions and their derivatives as defined in
 80 equations (5) and (6) below.

$$82 \quad \phi_i(x) = \begin{cases} 0; & 0 \leq x \leq x_{i-1} \\ \frac{x - x_{i-1}}{h}; & x_{i-1} < x \leq x_i \\ \frac{x_{i+1} - x}{h}; & x_i < x \leq x_{i+1} \\ 0; & x_{i+1} < x \leq 1 \end{cases} \quad (5)$$

$$84 \quad \phi'_i(x) = \begin{cases} 0; & 0 < x < x_{i-1} \\ \frac{1}{h}; & x_{i-1} < x < x_i \\ -\frac{1}{h}; & x_i < x < x_{i+1} \\ 0; & x_{i+1} < x < 1 \end{cases} \quad (6)$$

85
 86 $i = 1, \dots, n$. $\phi_i(x_i) = 1, \phi_i(x_{i-1}) = 0$ and $\phi_i(x_{i+1}) = 0$. If we connect (x_i, c_i) by a line
 87 segment we obtain an approximation of the form

$$88 \quad y_n(x) = \sum_{i=1}^n c_i \phi_i(x) \quad (7)$$

89 where c_i approximates the exact solution of (1) at $y(x_i)$. It follows that

$$90 \quad y'_n(x) = \sum_{i=1}^n c_i \phi'_i(x) \quad (8)$$

91 The absolute error of approximation is $|y_n(x) - y(x)|$.

92 Replace y by $y_n(x)$ in (3):

$$93 \quad \int_0^1 \{y''(x) - r(x)y_n(x) - f(x)\} \phi_j(x) dx = 0$$

94 Integrate first term by parts:

$$95$$

$$96 \quad \int_0^1 \phi_j(x) y''(x) dx = y'(x) \phi_j(x) \Big|_0^1 - \int_0^1 y'(x) \phi'_j(x) dx$$

$$97 \quad = \int_0^1 -y'_n(x) \phi'_j(x) dx$$

$$98 \quad \int_0^1 -\{y'_n(x) \phi'_j(x) + r(x)y_n(x) \phi_j(x)\} dx = \int_0^1 f(x) \phi_j(x) dx$$

99 Substitute for $y_n(x)$ and $y'_n(x)$, from equations (7) and (8):

100

$$\int_0^l - \left\{ \sum_{i=1}^n c_i \phi'_i(x) \right\} \phi'_j(x) + r(x) \left\{ \sum_{i=1}^n c_i \phi_i(x) \right\} \phi_j(x) dx = \int_0^l f(x) \phi_j(x) dx$$

102

$$\sum_{i=1}^n \left[\int_0^l - \left\{ \phi'_i(x) \phi'_j(x) + r(x) \phi_i(x) \phi_j(x) \right\} dx \right] c_i = \int_0^l f(x) \phi_j(x) dx \quad (9)$$

104 The solution of equation (9) will produce $n \times n$ linear system of equations in
105 $c_i, i = 1, 2, \dots, n$, unknown constants. So, we must form and solve a matrix equation. Define

$$a_{i,j} = - \int_0^l \left\{ \phi'_i(x) \phi'_j(x) + r(x) \phi_i(x) \phi_j(x) \right\} dx \quad (10)$$

$$b_j = \int_0^l f(x) \phi_j(x) dx \quad (11)$$

108

109 From equation (10),

$$\begin{aligned} a_{i,i} &= - \int_0^l \left\{ [\phi'_i(x)]^2 + r(x) [\phi_i(x)]^2 \right\} dx \\ &= - \int_{x_{i-1}}^{x_i} \left\{ [\phi'_i(x)]^2 + r(x) [\phi_i(x)]^2 \right\} dx - \int_{x_i}^{x_{i+1}} \left\{ [\phi'_i(x)]^2 + r(x) [\phi_i(x)]^2 \right\} dx \\ &= - \frac{1}{h^2} \int_{x_{i-1}}^{x_i} \left\{ 1 + r(x) [x - x_{i-1}]^2 \right\} dx - \frac{1}{h^2} \int_{x_i}^{x_{i+1}} \left\{ 1 + r(x) [x_{i+1} - x]^2 \right\} dx \end{aligned} \quad (12)$$

114

115 $i = 1, 2, \dots, n$.

$$a_{i,i+1} = - \int_{x_i}^{x_{i+1}} \left\{ \phi'_i(x) \phi'_{i+1}(x) + r(x) \phi_i(x) \phi_{i+1}(x) \right\} dx \quad (13)$$

117

$$= - \frac{1}{h^2} \int_{x_i}^{x_{i+1}} \left\{ 1 + r(x) [x - x_i][x_{i+1} - x] \right\} dx$$

119

120 $i = 1, 2, \dots, n-1$.

$$a_{i,i-1} = - \int_{x_{i-1}}^{x_i} \left\{ \phi'_i(x) \phi'_{i-1}(x) + r(x) \phi_i(x) \phi_{i-1}(x) \right\} dx \quad (14)$$

$$= - \frac{1}{h^2} \int_{x_{i-1}}^{x_i} \left\{ 1 + r(x) [x - x_{i-1}][x_i - x] \right\} dx$$

123 $i = 2, \dots, n$.

124 From equation (11),

$$b_j = \int_{x_{j-1}}^{x_j} \left\{ f(x) \phi_j(x) \right\} dx + \int_{x_j}^{x_{j+1}} \left\{ f(x) \phi_j(x) \right\} dx \quad (15)$$

$$= \frac{1}{h} \int_{x_{j-1}}^{x_j} \left\{ f(x) (x - x_{j-1}) \right\} dx + \frac{1}{h} \int_{x_j}^{x_{j+1}} \left\{ f(x) (x_{j+1} - x) \right\} dx$$

127 $j = 1, 2, \dots, n$.

128 From the above equations, we obtain the matrix equation for the system:

$$AC = B \quad (16)$$

130 where

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$$A = a_{ij}, B = b_j, C = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}, j = 1, 2, \dots, n \text{ and } i = 1, 2, \dots, n.$$

By Galerkin method, the unknown constants, $c_i, i = 1, 2, \dots, n$ will be found from equation (16):

$$C = A^{-1}B \quad (17)$$

The absolute error is: $\text{Error} = |y_n(x) - y(x)|$; $y(x)$ is the exact solution of (1).

4. NEW CONJUGATE GRADIENT METHOD

The new conjugate gradient method (NCGM) [10] seeks to optimize a multivariable function f whose gradient vector is g . At a point x_k , the objective function F is represented by

$$F(x) = f(x_k) + w f(x_k) + \frac{1}{2} w^2 f(x_k) + \dots + \frac{1}{m!} w^m f(x_k)$$

where

$$w^m F(x_k) = \sum_{i_1=1}^N \sum_{i_2=1}^N \dots \sum_{i_N=1}^N h_{i_1} h_{i_2} \dots h_{i_N} \frac{\partial^m f(x_k)}{\partial x_{i_1} \partial x_{i_2} \dots \partial x_{i_N}}, x, h \in \Re^m, h_j = x_j - x_{j-i}; n \geq 2. \text{ The}$$

gradient function of F is represented by G . The algorithm for the scheme is given below.

4.1 ALGORITHM I (NCGM)

Input initial values: x_0 and $D_0 = -G_0$.

Repeat:

Find step length α_k such that $F(x_k + \alpha_k D_k) = \min F(x_k + \alpha D_k)$.

Compute new point: $x_{k+1} = x_k + \alpha_k D_k$. Update search direction:

$$D_{k+1} = -G_{k+1} + \beta_k D_k, G_{k+1} = \frac{1}{2} [g(x_k + 2\Delta x_k) + g(x_k)], \beta_k = \frac{\|G_{k+1}\|^2}{D_k^T u_k}$$

$u_k = G_{k+1} - G_k$. $\|\cdot\|$ denotes Euclidean norm.

Check for optimality of g : Terminate iteration at step k if $\|g(x_k)\|$ is so small that x_k is acceptable.

5. NEW CONJUGATE GRADIENT METHOD WITH GALERKIN THEORY

The hybrid of the new Conjugate Gradient Method and Galerkin theory seeks to

Maximize $y(x)$
 $0 \leq x \leq l$

Subject to

$$y'' - r(x)y - f(x) = 0$$

$$y(0) = 0 = y(l).$$

161 where $r(x) = \frac{S}{EI}$ and $f(x) = \frac{q}{2EI} x(l-x)$. By our method, we require the matrix A, in
 162 equation (16), to be symmetric otherwise we replace A with an equivalent symmetric matrix
 163 M. That is,

$$164 \quad A^T A C = A^T B$$

165 or

$$166 \quad MC = Z \quad (18)$$

167 where $M = A^T A$ and $Z = A^T B$. By our assumption of uniqueness of solution of (1), we
 168 state that the matrix M is also positive definite. It follows that we are guaranteed to form an
 169 objective function from equation (18) as given below.

$$170 \quad F(C) = \frac{1}{2} C^T MC - C^T Z \quad (19)$$

171 The gradient function is

$$172 \quad G(C) = MC - Z \quad (20)$$

173 Next, we seek to solve the following optimization problem.

$$174 \quad \text{Minimize } (-F) \text{ over } C \quad (21)$$

175 Since our objective function F in equation (19) is a quadratic functional, we used the new
 176 Conjugate Gradient Method to solve (20). This technique solves the optimization method
 177 through an iterative procedure

$$178 \quad C_{k+1} = C_k + \alpha_k d_k \quad (22)$$

179 $k = 0, 1, 2, \dots$, d_k is a direction vector and α_k is the line search step length at iteration k .

180 Usually, we find α_k such that $\alpha_k = \min_{\alpha > 0} (-F(C_k + \alpha d_k))$.

181 The algorithm is given below.

182

183 **5.1 ALGORITHM II** (Hybrid of the new Conjugate gradient method and Galerkin theory)

184 **Let** $G_k = G(C_k)$ and $\|G_k\| = (G_k^T G_k)^{\frac{1}{2}}$. **Choose a whole number** n and $h = \frac{1}{n+1}$. Use

185 Galerkin method to obtain equations (16) and (19) and $G(C_k) = MC_k - Z$ from equation
 186 (20). Use the following initial values:

$$187 \quad C_0 = (0, 0, \dots, 0)^T, \quad k = 0, \quad D_0 = -G_0.$$

188 Repeat the following steps of the new Conjugate gradient method.

189 a. Find step length α_k such that

$$190 \quad F(C_k + \alpha_k D_k) = \min_{\alpha > 0} F(C_k + \alpha D_k)$$

191 b. Compute new point:

$$192 \quad C_{k+1} = C_k + \alpha_k D_k$$

193 c. Update search direction:

$$194 \quad D_{k+1} = -G_{k+1} + \beta_k D_k,$$

$$195 \quad G_{k+1} = \frac{1}{2} [G(C_k + 2\Delta C_k) + G(C_k)]$$

$$196 \quad \beta_k = \frac{\|G_{k+1}\|^2}{D_k^T y_k}$$

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197 $y_k = G_{k+1} - G_k$
 198 d. Check for optimality of G : Terminate iteration at step k when $\|G_k\|$ is so small that C_k is
 199 an acceptable estimate of the optimal point of F . If not optimal set $k = k + 1$.
 200

201 **6. NUMERICAL EXAMPLES**

202 We used the hybrid of the new Conjugate Gradient Method and Galerkin theory to find the
 203 maximal deflection of a simply supported beam under uniformly distributed load from the
 204 following boundary value problems. Each boundary value problem governs the deflection of
 205 a structured beam as described in exercise 11.3, page 666, of [1]. Nine interior points were
 206 used in each case. The driving program (MatLab) terminates when an optimal point is found.
 207

208 **Problem 1:**

$$209 \frac{d^2 y(x)}{dx^2} = \frac{S}{EI} y(x) + \frac{q}{2EI} x(120 - x), \quad 0 \leq x \leq 120$$

$$210 y(0) = 0 = y(120),$$

$$211 S = 1000N, \quad l = 120m, \quad q = 8\frac{1}{3}N/m, \quad E = 3 \times 10^7 N/m^2 \text{ and } I = 625m^4.$$

212 **Exact solution:**

$$213 y(x) = C_1 e^{x\sqrt{a}} + C_2 e^{-x\sqrt{a}} + \frac{b}{a} x^2 - \frac{120b}{a} x + \frac{2b}{a^2}$$

$$214 C_1 = -77042.53752198143, \quad C_2 = -79207.46247801857, \quad a = 5.33333 \times 10^{-8}$$

$$215 b = 2.22222222 \times 10^{-10}$$

218 **Problem 2:**

$$219 \frac{d^2 y(x)}{dx^2} = \frac{q}{2EI} x(20 - x), \quad 0 \leq x \leq 20$$

$$220 y(0) = 0 = y(20),$$

$$221 l = 20m, \quad q = 5000N/m, \quad EI = 133.333 \times 10^6 Nm^2.$$

222 **Exact solution:**

$$223 y(x) = \frac{b}{12} (40x^3 - x^4 - 8000x)$$

$$224 b = \frac{q}{2EI}$$

227 **Problem 3:**

$$228 \frac{d^2 y(x)}{dx^2} = \frac{S}{D} y(x) + \frac{q}{2D} x(50 - x), \quad 0 \leq x \leq 50$$

$$229 y(0) = 0 = y(50),$$

$$230 S = 100N, \quad l = 50m, \quad q = 200N/m, \quad D = 8.8 \times 10^7 Nm^2.$$

231 **Exact solution:**

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$$y(x) = C_1 e^{x\sqrt{a}} + C_2 e^{-x\sqrt{a}} + x^2 - 50x + \frac{2}{a}$$

$$C_1 = -856553.471726025, C_2 = -903446.5282739757,$$

$$a = 1.1363636363636 \times 10^{-6}$$

7. NUMERICAL SOLUTIONS

We used MatLab to solve the given boundary value problems. The exact solution is $y(x_i)$ at each x_i .

Table 1: Solution of Problem 1 (Maximal deflection is 0.12cm downwards)

$n = 9, h = \frac{1}{10}, x_i \in [0, 120]$				
i	x_i	$y_9(x_i) = c_i$	$y(x_i)$	ERROR = $ y_9(x_i) - y(x_i) $
1	12	-0.000376675239629	-0.000376675016014	2.23615×10^{-10}
2	24	-0.000712649320008	-0.000712648849003	4.71005×10^{-10}
3	36	-0.000975668780120	-0.000975668110186	6.69934×10^{-10}
4	48	-0.001142695610452	-0.001142694818554	7.91898×10^{-10}
5	60	-0.001199907076070	-0.001199906197144	8.78927×10^{-10}

Table 2: Solution of Problem 2 (Maximal deflection is 7.81cm downwards)

$n = 9, h = \frac{1}{10}, x_i \in [0, 20]$				
i	x_i	$y_9(x_i) = c_i$	$y(x_i)$	ERROR = $ y_9(x_i) - y(x_i) $
1	2	-0.024525000061357	-0.024525000061312	4.4×10^{-14}
2	4	-0.046400000116089	-0.046400000116000	8.9×10^{-14}
3	6	-0.063525000158946	-0.063525000158812	1.33×10^{-13}
4	8	-0.074400000186200	-0.074400000186000	1.78×10^{-13}
5	10	-0.078125000195535	-0.078125000195312	2.22×10^{-13}

Table 3: Solution of Problem 3 (Maximal deflection is 18.49cm downwards).

$n = 9, h = \frac{1}{10}, x_i \in [0, 50]$				
i	x_i	$y_9(x_i) = c_i$	$y(x_i)$	ERROR = $ y_9(x_i) - y(x_i) $
1	5	-0.058044685436589	-0.058044549310580	$1.36126009 \times 10^{-7}$
2	10	-0.109817315930269	-0.109817057382315	$2.58547953 \times 10^{-7}$
3	15	-0.150347747852438	-0.150347393937409	$3.53915029 \times 10^{-7}$
4	20	-0.176085979463466	-0.176085565239191	$4.14224275 \times 10^{-7}$
5	25	-0.18490205005989	-0.184901615371928	$4.34687963 \times 10^{-7}$

8. DISCUSSION OF NUMERICAL RESULTS

The data shown in tables 1 to 3 reveal that results obtained by the hybrid of the new Conjugate Gradient Method and Galerkin theory are very close to the exact solutions of the given boundary value problems. The **symmetric feature of the Hessian matrix obtained from** each objective function, accelerated the convergence of this method. We, also, observed that errors on numerical solutions decrease as interior points increase.

9. CONCLUSION

We have used this hybrid of a new Conjugate Gradient Method and Galerkin theory to find the maximal deflection value of a simply supported beam under uniformly distributed load. The characterized two point linear boundary value problems with homogeneous boundary conditions were solved, numerically. Simple basis functions were used as trial functions in our approximations. Results from tables 1 to 3 confirm that our method is accurate and reliable.

Herein, we present the hybrid of a new Conjugate Gradient Method and Galerkin theory to engineers and scientists who wish to solve real life problems in this class of boundary value problems.

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