

On The Solution of a Rough Interval Three-level Quadratic Programming Problem

Omar M.Saad*

O. E. Emam**

Marwa M. Sleem***

* Department of Mathematics, Faculty of Science, Helwan University, P.O.Box11795, Cairo, Egypt.

**Department of Information Systems, Faculty Of Computer Science and Information, Helwan University, P.O. Box 11795, Egypt.

***Department of Basic Sciences, Thebes Higher Institute of Engineering, Maadi, Cairo, Egypt.

***E-mail: oemam80@yahoo.com*

ARTICLE INFO

MSC2010:90C20, 90C99,

Keywords:

Quadratic Programming ,
Three-level Programming ,
Rough Intervals Programming.

ABSTRACT

In this paper, a three-level quadratic programming (QP) problem is considered where some or all of its coefficients in the objective function are rough intervals. At the first phase of the solution approach and to avoid the complexity of the problem, two QP problems with interval coefficients will be formulated. One of these problems is a QP where all of its coefficients are upper approximation of rough intervals and the other problem is a QP where all of its coefficients are lower approximations of rough intervals. At the second phase, a membership function is constructed to develop a fuzzy model for obtaining the optimal solution of the three-level quadratic programming problem. Finally, an illustrative numerical example is given to demonstrate the obtained results.

1. Introduction

The rough set expressed by a boundary region of a set which is described by lower and upper approximation where the set is considered as a crisp set if the boundary region is empty. This is exactly the idea of vagueness [6, 9]. The approach for solving rough interval programming problem is to convert the objective function from rough interval to crisp using theorem of crisp evaluation.

Three-level optimization is a kind of multi-level optimization which is a technique developed to solve decentralized problems with multiple decision-makers in hierarchical organization [7]. Three-level programming problem is concerned with minimizing or maximizing some quantity represented by an objective function.

Quadratic Programming (QP) is one of the most popular models used in decision-making and in optimization problems [14]. Quadratic Programming problem aims at minimize (maximize) a quadratic objective function subject to a set of linear constraints. If the coefficients in the objective function are exactly known crisp value, then these models can be solved by classical methods and algorithms.

In some real-world situations, the coefficients of decision-making models are not exactly known. This is due to the fact that some of relevant data are non-existent, scarce, or difficult. Almost all concepts which are used in a natural language are vague. Perhaps some people think that they are subjective probability or fuzzy.

In modern times, scholars are often faced with complex decision-making problems concerning uncertainty. Here uncertainties are stated by interval data, roughness or their hybrid with fuzziness and randomness [1,4,5,11,12].

Emam [2] presented a bi-level integer non-linear programming problem with linear or non-linear constraints, and in which the non-linear objective function at each level were maximized. It proposed a two planner integer model and a solution method for solving this problem. Therefore Emam proposed an interactive approach for solving bi-level integer multi-objective fractional programming problem [3].

Osman et al. in [8] presented rough bi-level programming problems using genetic algorithm (GA) by constructing the fitness function of the upper level programming problems based on the definition of through feasible degree. Hamzehee et al. [5] presented a linear programming (LP) problem which is considered where some or all of its coefficients in the objective function and /or constraints are rough intervals. In order to solve this problem, two LP problems with interval coefficients will be constructed. One of these problems is a LP where all of its coefficients are upper approximations of rough intervals and the other problem is a LP where all of its coefficients are lower approximations of rough intervals. Using these two LPs, two newly solutions are defined.

In [7] Osman et al. provided a solution method for solving multi-level non-linear multi-objective problem under fuzziness. This solution method uses the concepts of tolerance membership functions and multi-objective optimization at every level to develop a fuzzy max-min decision model till generating the optimal solution.

On the other hand, Sultan et al. [14] used the fuzzy approach to study a parametric bi-level quadratic programming problem with random rough coefficient in the objective functions. This approach combines the convert technique of rough coefficient and Stackelberg strategy [14]. In later work for Effati and Pakdaman [1], an interval valued linear fractional programming problem (IVLFP) has been introduced. An IVLFP is a linear fractional programming problem with interval coefficients in the objective function. It is proved that we can convert an IVLFP to an optimization problem with interval valued objective function which bounds are linear fractional functions.

The purpose of the present paper is to find an optimal solution of the model of a three-level quadratic programming problem with rough interval coefficients. The used fuzzy approach is based mainly upon a systematic access to the best results. This paper is organized as follows: In Section 2, the three-level quadratic programming problem with rough interval coefficients is formulated. Section 3 involves the theories used to transform rough interval to crisp variable. The fuzzy approach using membership function to solve the problem under consideration is given in Section 4. Section 5 provides an algorithm of finding the optimal solution of the formulated model. A numerical example which illustrates the theory of the solution algorithm is suggested in Section 6. Finally, the paper is concluded in Section 7 where some points of further research are reported.

2. Problem formulation and solution concept

The three-level quadratic programming problem with rough interval coefficients in the objective functions (TLQPRIC) may be written as follows:

[1stLevel]

$$Max_{x_1} F_1(x) = \sum_{j=1}^n ([\underline{c}_j^L, \underline{c}_j^U], [\bar{c}_j^L, \bar{c}_j^U])x_j + \frac{1}{2}x_j^T([\underline{c}_j^L, \underline{c}_j^U], [\bar{c}_j^L, \bar{c}_j^U])x_j, \quad (1)$$

where x_2 solves

[2ndLevel]

$$Max_{x_2} F_2(x) = \sum_{j=1}^n ([\underline{c}_j^L, \underline{c}_j^U], [\bar{c}_j^L, \bar{c}_j^U])x_j + \frac{1}{2}x_j^T([\underline{c}_j^L, \underline{c}_j^U], [\bar{c}_j^L, \bar{c}_j^U])x_j, \quad (2)$$

where x_3 solves

[3rdLevel]

$$Max_{x_3} F_3(x) = \sum_{j=1}^n ([\underline{c}_j^L, \underline{c}_j^U], [\bar{c}_j^L, \bar{c}_j^U])x_j + \frac{1}{2}x_j^T([\underline{c}_j^L, \underline{c}_j^U], [\bar{c}_j^L, \bar{c}_j^U])x_j, \quad (3)$$

Subject to

$$G = \{x | Ax \leq d, x \geq 0\}. \quad (4)$$

where G is the three-level convex constraint set, F_1, F_2 and F_3 are the objective functions of the first level decision maker (FLDM), second level decision maker (SLDM), and third level decision maker (TLDM), respectively. Also $([\underline{c}_j^L, \underline{c}_j^U], [\bar{c}_j^L, \bar{c}_j^U])$ is rough interval coefficient of the objective function. Let $(j=1,2,\dots,n), x = (x_1, x_2, \dots, x_n)^T$ denote the vector of all decision variables.

Remark 1.[5]

According to rough interval properties we have:

$$[\underline{c}_j^L, \underline{c}_j^U] \subseteq [\bar{c}_j^L, \bar{c}_j^U] \Rightarrow \bar{c}_j^L \leq \underline{c}_j^L \leq \underline{c}_j^U \leq \bar{c}_j^U.$$

Definition 1.[5]

Consider all of the corresponding TLQPRIC problem (1)-(4):

- (a) The interval $[z_*^L, z_*^U]([z^{*L}, z^{*U}])$ is called the surely (possibly) optimal range of problem (1)-(4), if the optimal range of each TLQPRIC Problem is a superset (subset) of $[z_*^L, z_*^U]([z^{*L}, z^{*U}])$.
- (b) Let $[z_*^L, z_*^U]([z^{*L}, z^{*U}])$ be surely optimal (possibly) optimal range of the problem (1)-(4). Then the rough interval $[z_*^L, z_*^U]([z^{*L}, z^{*U}])$ is called the rough optimal range of problem (1)-(4).
- (c) The optimal solution of each corresponding TLQPRIC problem (1)-(4) which its optimal value belongs to $[z_*^L, z_*^U]([z^{*L}, z^{*U}])$ is called a complete (rather) satisfactory solution of problem (1)-(4).

3. The transformation of random rough interval coefficient

To convert the three-level quadratic programming problem with random rough interval coefficient in the objective functions into upper and lower approximations for random rough interval coefficient to crisp equivalent, this process is usually a hard work for many cases in the following manner.

(LI): Lower interval in the objective functions.

[1stLevel]

$$Max_{x_1} \underline{f}_1(x) = \sum_{j=1}^n [\underline{c}_j^L, \underline{c}_j^U] x_j + \frac{1}{2} x_j^T [\underline{c}_j^L, \underline{c}_j^U] x_j, \quad (5)$$

where x_2 solves

[2ndLevel]

$$Max_{x_2} \underline{f}_2(x) = \sum_{j=1}^n [\underline{c}_j^L, \underline{c}_j^U] x_j + \frac{1}{2} x_j^T [\underline{c}_j^L, \underline{c}_j^U] x_j, \quad (6)$$

where x_3 solves

[3rdLevel]

$$Max_{x_3} \underline{f}_3(x) = \sum_{j=1}^n [\underline{c}_j^L, \underline{c}_j^U] x_j + \frac{1}{2} x_j^T [\underline{c}_j^L, \underline{c}_j^U] x_j, \quad (7)$$

Subject to

$$G = \{x | Ax \leq d, x \geq 0\}. \quad (8)$$

(UI): Upper interval in the objective functions.

[1stLevel]

$$Max_{x_1} \bar{f}_1(x) = \sum_{j=1}^n [\bar{c}_j^L, \bar{c}_j^U] x_j + \frac{1}{2} x_j^T [\bar{c}_j^L, \bar{c}_j^U] x_j, \quad (9)$$

where x_2 solves

[2ndLevel]

$$Max_{x_2} \bar{f}_2(x) = \sum_{j=1}^n [\bar{c}_j^L, \bar{c}_j^U] x_j + \frac{1}{2} x_j^T [\bar{c}_j^L, \bar{c}_j^U] x_j, \quad (10)$$

where x_3 solves

[3rdLevel]

$$Max_{x_3} \bar{f}_3(x) = \sum_{j=1}^n [\bar{c}_j^L, \bar{c}_j^U] x_j + \frac{1}{2} x_j^T [\bar{c}_j^L, \bar{c}_j^U] x_j, \quad (11)$$

Subject to

$$G = \{x | Ax \leq d, x \geq 0\}. \quad (12)$$

After the division of random rough interval coefficient in the objective functions into upper and lower interval to build a crisp equivalent model, the following theorems are necessary and useful.

Theorem1. [5]

Suppose that the optimal range of LPIC problem (L) exists. Then, it is equal to the surely optimal range of problem (1)-(4). The optimal range of LPIC Problem (L) can be obtained by solving two classical LPs as follows:

$$P_1: \underline{z}^L := Max \sum_{j=1}^n \underline{c}_j^L x_j,$$

subject to

$$\sum_{j=1}^n \underline{a}_{ij}^U x_j \leq \underline{b}_i^L, i = 1, 2, \dots, m,$$

$$x_j \geq 0, j = 1, 2, \dots, n,$$

$$P_2: \underline{z}^U := Max \sum_{j=1}^n \underline{c}_j^U x_j,$$

subject to

$$\sum_{j=1}^n \underline{a}_{ij}^L x_j \leq \underline{b}_i^U, i = 1, 2, \dots, m,$$

$$x_j \geq 0, j = 1, 2, \dots, n.$$

Theorem2. [5]

Suppose that the optimal range of LPIC Problem (U) exists. Then, it is equal to the surely optimal range of Problem (1)-(4). The optimal range of LPIC Problem (U) can be obtained by solving two classical LPs as follows:

$$\begin{aligned}
 P_3: \bar{z}^L &:= \text{Max} \sum_{j=1}^n \bar{c}_j^L x_j, & P_4: \bar{z}^U &:= \text{Max} \sum_{j=1}^n \bar{c}_j^U x_j, \\
 \text{subject to} & & \text{subject to} & \\
 \sum_{j=1}^n \bar{a}_{ij}^U x_j &\leq \bar{b}_i^L, i = 1, 2, \dots, m, & \sum_{j=1}^n \bar{a}_{ij}^L x_j &\leq \bar{b}_i^U, i = 1, 2, \dots, m, \\
 x_j &\geq 0, j = 1, 2, \dots, n. & x_j &\geq 0, j = 1, 2, \dots, n.
 \end{aligned}$$

Now, the lower interval **LI** and the upper interval **UI** of problems given before by (5)-(12) are the reformulated more explicitly as:

a. Lower Interval:

a.1 Lower Interval coefficient (LIC)

[1stLevel]

$$\text{Max}_{x_1} f_1^L(x) = \sum_{j=1}^n \underline{c}_j^L x_j + \frac{1}{2} x_j^T \underline{c}_j^L x_j, \quad (13)$$

where x_2 solves

[2ndLevel]

$$\text{Max}_{x_2} f_2^L(x) = \sum_{j=1}^n \underline{c}_j^L x_j + \frac{1}{2} x_j^T \underline{c}_j^L x_j, \quad (14)$$

where x_3 solves

[3rdLevel]

$$\text{Max}_{x_3} f_3^L(x) = \sum_{j=1}^n \underline{c}_j^L x_j + \frac{1}{2} x_j^T \underline{c}_j^L x_j, \quad (15)$$

Subject to

$$G = \{x | Ax \leq d, x \geq 0\}. \quad (16)$$

a.2 Upper Interval coefficient (UIC)

[1stLevel]

$$\text{Max}_{x_1} f_1^U(x) = \sum_{j=1}^n \underline{c}_j^U x_j + \frac{1}{2} x_j^T \underline{c}_j^U x_j, \quad (17)$$

where x_2 solves

[2ndLevel]

$$\text{Max}_{x_2} f_2^U(x) = \sum_{j=1}^n \underline{c}_j^U x_j + \frac{1}{2} x_j^T \underline{c}_j^U x_j, \quad (18)$$

where x_3 solves

[3rdLevel]

$$\text{Max}_{x_3} f_3^U(x) = \sum_{j=1}^n \underline{c}_j^U x_j + \frac{1}{2} x_j^T \underline{c}_j^U x_j, \quad (19)$$

Subject to

$$G = \{x | Ax \leq d, x \geq 0\}. \quad (20)$$

b. Upper Interval

b.1 Lower Interval coefficient (LIC)

[1stLevel]

$$\text{Max}_{x_1} \bar{f}_1^L(x) = \sum_{j=1}^n \bar{c}_j^L x_j + \frac{1}{2} x_j^T \bar{c}_j^L x_j, \quad (21)$$

where x_2 solves

[2ndLevel]

$$Max_{x_2} \bar{f}_2^L(x) = \sum_{j=1}^n \bar{c}_j^L x_j + \frac{1}{2} x_j^T \bar{c}_j^L x_j, \quad (22)$$

where x_3 solves

[3rdLevel]

$$Max_{x_2} \bar{f}_2^L(x) = \sum_{j=1}^n \bar{c}_j^L x_j + \frac{1}{2} x_j^T \bar{c}_j^L x_j, \quad (23)$$

Subject to

$$G = \{x | Ax \leq d, x \geq 0\}. \quad (24)$$

b.2 Upper Interval coefficient (UIC)

[1stLevel]

$$Max_{x_1} \bar{f}_1^U(x) = \sum_{j=1}^n \bar{c}_j^U x_j + \frac{1}{2} x_j^T \bar{c}_j^U x_j, \quad (25)$$

where x_2 solves

[2ndLevel]

$$Max_{x_2} \bar{f}_2^U(x) = \sum_{j=1}^n \bar{c}_j^U x_j + \frac{1}{2} x_j^T \bar{c}_j^U x_j, \quad (26)$$

where x_3 solves

[3rdLevel]

$$Max_{x_2} \bar{f}_2^U(x) = \sum_{j=1}^n \bar{c}_j^U x_j + \frac{1}{2} x_j^T \bar{c}_j^U x_j, \quad (27)$$

Subject to

$$G = \{x | Ax \leq d, x \geq 0\}. \quad (28)$$

4. Fuzzy approach of three-level quadratic programming problem[7]

In this section, the three-level quadratic programming problem with rough interval coefficients in the objective functions is solved by using fuzzy approach as described in [7]. At the beginning, we start by stating the first level decision maker problem in the following:

4.1. First level decision maker problem

The FLDM problem may be formulated as follows:

$$Max_{x_1} F_1(x) = \sum_{j=1}^n ([\underline{c}_j^L, \underline{c}_j^U], [\bar{c}_j^L, \bar{c}_j^U]) x_j + \frac{1}{2} x_j^T ([\underline{c}_j^L, \underline{c}_j^U], [\bar{c}_j^L, \bar{c}_j^U]) x_j, \quad (29)$$

Subject to

$$x \in G \quad (30)$$

Find individual optimal solution of problem FLDM by obtaining the best and the worst solutions of his problem after transformation (29) problem into the classical problem by theorem (1,2).

$$Max_{x_1} F_1(x) : \begin{cases} (LIC): \underline{f}_1: \begin{cases} Max_{x_1} \underline{f}_1^L(x) = \sum_{j=1}^n \underline{c}_j^L x_j + \frac{1}{2} x_j^T \underline{c}_j^L x_j \Rightarrow \begin{cases} (\underline{f}_1^L)^* = Max_{x \in G} \underline{f}_1^L, \\ (\underline{f}_1^L)^- = Min_{x \in G} \underline{f}_1^L. \end{cases} \\ Max_{x_1} \underline{f}_1^U(x) = \sum_{j=1}^n \underline{c}_j^U x_j + \frac{1}{2} x_j^T \underline{c}_j^U x_j \Rightarrow \begin{cases} (\underline{f}_1^U)^* = Max_{x \in G} \underline{f}_1^U, \\ (\underline{f}_1^U)^- = Min_{x \in G} \underline{f}_1^U. \end{cases} \end{cases} \\ (UIC): \bar{f}_1: \begin{cases} Max_{x_1} \bar{f}_1^L(x) = \sum_{j=1}^n \bar{c}_j^L x_j + \frac{1}{2} x_j^T \bar{c}_j^L x_j \Rightarrow \begin{cases} (\bar{f}_1^L)^* = Max_{x \in G} \bar{f}_1^L, \\ (\bar{f}_1^L)^- = Min_{x \in G} \bar{f}_1^L. \end{cases} \\ Max_{x_1} \bar{f}_1^U(x) = \sum_{j=1}^n \bar{c}_j^U x_j + \frac{1}{2} x_j^T \bar{c}_j^U x_j \Rightarrow \begin{cases} (\bar{f}_1^U)^* = Max_{x \in G} \bar{f}_1^U, \\ (\bar{f}_1^U)^- = Min_{x \in G} \bar{f}_1^U. \end{cases} \end{cases} \end{cases} \quad (31)$$

This data can then be formulated as the following membership function:

$$(LIC): \mu[f_1(x)] = \begin{cases} \mu[f_1^L(x)] = \begin{cases} 1 & \text{if } f_1(x) > (f_1^L)^* , \\ \frac{f_1(x) - (f_1^L)^-}{(f_1^L)^* - (f_1^L)^-} & \text{if } (f_1^L)^- \leq f_1(x) \leq (f_1^L)^* , \\ 0 & \text{if } (f_1^L)^- \geq f_1(x). \end{cases} \\ \mu[f_1^U(x)] = \begin{cases} 1 & \text{if } f_1(x) > (f_1^U)^* , \\ \frac{f_1(x) - (f_1^U)^-}{(f_1^U)^* - (f_1^U)^-} & \text{if } (f_1^U)^- \leq f_1(x) \leq (f_1^U)^* , \\ 0 & \text{if } (f_1^U)^- \geq f_1(x). \end{cases} \end{cases} \quad (32)$$

$$(UIC): \mu[\bar{f}_1(x)] = \begin{cases} \mu[\bar{f}_1^L(x)] = \begin{cases} 1 & \text{if } f_1(x) > (\bar{f}_1^L)^* , \\ \frac{f_1(x) - (\bar{f}_1^L)^-}{(\bar{f}_1^L)^* - (\bar{f}_1^L)^-} & \text{if } (\bar{f}_1^L)^- \leq f_1(x) < (\bar{f}_1^L)^* , \\ 0 & \text{if } (\bar{f}_1^L)^- > f_1(x). \end{cases} \\ \mu[\bar{f}_1^U(x)] = \begin{cases} 1 & \text{if } f_1(x) > (\bar{f}_1^L)^* , \\ \frac{f_1(x) - (\bar{f}_1^U)^-}{(\bar{f}_1^U)^* - (\bar{f}_1^U)^-} & \text{if } (\bar{f}_1^L)^- \leq f_1(x) < (\bar{f}_1^L)^* , \\ 0 & \text{if } (\bar{f}_1^L)^- > f_1(x). \end{cases} \end{cases} \quad (33)$$

Now, the solution of the FLDM problem can be obtained by solving the following Tchebycheff problem:

$$(LIC): \begin{cases} \text{Max } \lambda, \\ \text{subject to} \\ x \in G, \\ \mu[f_1^L(x)] \geq \lambda, \\ \lambda \in [0,1]. \end{cases} \quad (34)$$

$$(UIC): \begin{cases} \text{Max } \lambda, \\ \text{subject to} \\ x \in G, \\ \mu[\bar{f}_1^L(x)] \geq \lambda, \\ \lambda \in [0,1]. \end{cases} \quad (35)$$

whose solution are assumed to be $(\underline{x}_1^L, \underline{x}_2^L, \underline{x}_3^L)^F, (\underline{x}_1^U, \underline{x}_2^U, \underline{x}_3^U)^F, (\bar{x}_1^L, \bar{x}_2^L, \bar{x}_3^L)^F, (\bar{x}_1^U, \bar{x}_2^U, \bar{x}_3^U)^F, \lambda^F$, and $\left([f_1^L, f_1^U], [\bar{f}_1^L, \bar{f}_1^U]\right)^F$, (satisfactory level).

4.2. Second level decision maker problem

The SLDM does the same action like the FLDM till he/she obtains his/her solution $(\underline{x}_1^L, \underline{x}_2^L, \underline{x}_3^L)^S, (\underline{x}_1^U, \underline{x}_2^U, \underline{x}_3^U)^S, (\bar{x}_1^L, \bar{x}_2^L, \bar{x}_3^L)^S, (\bar{x}_1^U, \bar{x}_2^U, \bar{x}_3^U)^S, \beta^S$, and $\left([f_2^L, f_2^U], [\bar{f}_2^L, \bar{f}_2^U]\right)^S$, (satisfactory level).

4.3. Third level decision maker problem

The TLDM does the same action like the SLDM till he/she obtains his/her solution $(\underline{x}_1^L, \underline{x}_2^L, \underline{x}_3^L)^T, (\underline{x}_1^U, \underline{x}_2^U, \underline{x}_3^U)^T, (\bar{x}_1^L, \bar{x}_2^L, \bar{x}_3^L)^T, (\bar{x}_1^U, \bar{x}_2^U, \bar{x}_3^U)^T, \delta^T$, and $([\underline{f}_3^L, \underline{f}_3^U], [\bar{f}_3^L, \bar{f}_3^U])^T$.

Now, the solution of the FLDM, SLDM, and TLDM are disclosed. However, three solutions are usually different because of the nature between three levels objective functions.

The FLDM knows that using the optimal decisions $x_1^F: \{(x_1^L)^F, (x_1^U)^F, (\bar{x}_1^L)^F, (\bar{x}_1^U)^F\}$, as control factors for the SLDM, are not practical. And also the SLDM knows that using the optimal decisions $x_2^S: \{(x_2^L)^S, (x_2^U)^S, (\bar{x}_2^L)^S, (\bar{x}_2^U)^S\}$ as control factors for the TLDM is not practical. It is more reasonable to have some tolerance that gives the SLDM and TLDM an extent feasible region to search for the optimal solution, and also reduce searching time or interactions.

In this way, the range of decision variable $x_1: \{\underline{x}_1^L, \underline{x}_1^U, \bar{x}_1^L, \bar{x}_1^U\}$, $x_2: \{\underline{x}_2^L, \underline{x}_2^U, \bar{x}_2^L, \bar{x}_2^U\}$ should be around $x_1^F: \{(x_1^L)^F, (x_1^U)^F, (\bar{x}_1^L)^F, (\bar{x}_1^U)^F\}$, $x_2^S: \{(x_2^L)^S, (x_2^U)^S, (\bar{x}_2^L)^S, (\bar{x}_2^U)^S\}$ with maximum tolerance $(t_1)^F, (t_2)^S$ and the following membership function specify $\{(x_1^L)^F, (x_1^U)^F, (\bar{x}_1^L)^F, (\bar{x}_1^U)^F\}, \{(x_2^L)^S, (x_2^U)^S, (\bar{x}_2^L)^S, (\bar{x}_2^U)^S\}$ as follows:

$$\mu(x_1) = \begin{cases} \mu(\underline{x}_1) = \begin{cases} \frac{\underline{x}_1 - ((x_1^L)^F - (t_1)^F)}{(t_1)^F} & \text{if } (x_1^L)^F - (t_1)^F \leq \underline{x}_1 \leq (x_1^L)^F \\ \frac{-\underline{x}_1 + ((x_1^L)^F + (t_1)^F)}{(t_1)^F} & \text{if } (x_1^L)^F \leq \underline{x}_1 \leq (x_1^L)^F - (t_1)^F \end{cases} \\ \mu(\underline{x}_1^U) = \begin{cases} \frac{\underline{x}_1 - ((x_1^U)^F - (t_1)^F)}{(t_1)^F} & \text{if } (x_1^U)^F - (t_1)^F \leq \underline{x}_1 \leq (x_1^U)^F \\ \frac{-\underline{x}_1 + ((x_1^U)^F + (t_1)^F)}{(t_1)^F} & \text{if } (x_1^U)^F \leq \underline{x}_1 \leq (x_1^U)^F - (t_1)^F \end{cases} \\ \mu(\bar{x}_1) = \begin{cases} \frac{\bar{x}_1 - ((\bar{x}_1^L)^F - (t_1)^F)}{(t_1)^F} & \text{if } (\bar{x}_1^L)^F - (t_1)^F \leq \bar{x}_1 \leq (\bar{x}_1^L)^F \\ \frac{-\bar{x}_1 + ((\bar{x}_1^L)^F + (t_1)^F)}{(t_1)^F} & \text{if } (\bar{x}_1^L)^F \leq \bar{x}_1 \leq (\bar{x}_1^L)^F - (t_1)^F \end{cases} \\ \mu(\bar{x}_1^U) = \begin{cases} \frac{\bar{x}_1 - ((\bar{x}_1^U)^F - (t_1)^F)}{(t_1)^F} & \text{if } (\bar{x}_1^U)^F - (t_1)^F \leq \bar{x}_1 \leq (\bar{x}_1^U)^F \\ \frac{-\bar{x}_1 + ((\bar{x}_1^U)^F + (t_1)^F)}{(t_1)^F} & \text{if } (\bar{x}_1^U)^F \leq \bar{x}_1 \leq (\bar{x}_1^U)^F - (t_1)^F \end{cases} \end{cases} \quad (36)$$

The $\mu(x_2)$ does the same action like the $\mu(x_1)$.

The FLDM goals may reasonably consider $F_1(\underline{f}_1^L, \underline{f}_1^U, \bar{f}_1^L, \bar{f}_1^U) \geq F_1^F((f_1^L)^F, (f_1^U)^F, (\bar{f}_1^L)^F, (\bar{f}_1^U)^F)$ is absolutely acceptable and $F_1 \leq F_1^F(x_1^S, x_2^S, x_3^S): (\{(x_1^L)^S, (x_1^U)^S, (\bar{x}_1^L)^S, (\bar{x}_1^U)^S\}, \{(x_2^L)^S, (x_2^U)^S, (\bar{x}_2^L)^S, (\bar{x}_2^U)^S\}, \{(x_3^L)^S, (x_3^U)^S, (\bar{x}_3^L)^S, (\bar{x}_3^U)^S\})$, is absolutely unacceptable, and that the preference with $\hat{F}_1, F_1^F$ is linearly increasing. This is due to the fact that the SLDM obtained the optimum at (x_1^S, x_2^S, x_3^S) , which in turn provides the FLDM the objective function values $\hat{F}_1$, makes any $F_1 \geq \hat{F}_1 = F_1(x_1^S, x_2^S, x_3^S)$ unattractive in practice.

The following membership functions of the FLDM can be stated as:

$$\mu[F_1(x)] = \begin{cases} 1 & \text{if } F_1(x) > F_1^F, \\ \frac{F_1(x) - \hat{F}_1}{F_1^F - \hat{F}_1} & \text{if } \hat{F}_1 \leq F_1(x) \leq F_1^F, \\ 0 & \text{if } \hat{F}_1 \geq F_1(x). \end{cases} \quad (37)$$

Second, The SLDM and TLDM do the same action like the FLDM.

Finally, in order to generate the satisfactory solution, which is also a Pareto optimal solution with overall satisfaction for all decision-makers, the following Tchebycheff problem will be solved:

$$\begin{aligned}
& \text{Max } \omega, \\
& \text{Subject to} \\
& \quad \mu[F_1(x)] \geq \omega, \\
& \quad \mu[F_2(x)] \geq \omega, \\
& \quad \mu[F_3(x)] \geq \omega, \\
& \quad \frac{[x_1 - (x_1^F - t_1)]}{t_1} \geq \omega, \\
& \quad \frac{-x_1 + (x_1^F + t_1)}{t_1} \geq \omega, \\
& \quad \frac{[x_2 - (x_2^S - t_2)]}{t_2} \geq \omega, \\
& \quad \frac{-x_2 + (x_2^S + t_2)}{t_2} \geq \omega, \\
& \quad x \in G, \\
& \quad t_i > 0, \\
& \quad \omega \in [0,1].
\end{aligned} \tag{38}$$

5. An algorithm for solving problem (TLQPRIC)

A solution algorithm to solve (TLQPRIC) problems (1)-(4) is described in a series of steps as follows:

Step1: Determine the surly random rough interval coefficient rang (lower (L) interval problem) in FLDM, SLDM, and TLDM problem, respectively.

Step2: Determine the possible random rough interval coefficient rang (upper (U) interval problem) in FLDM, SLDM, and TLDM problem, go to step3.

Step3: Formulate the corresponding equivalent problem (TLQPI).

Step4 Convert the lower and upper random interval coefficient in FLDM problem in to equivalent crisp models can be solved by classical methods.

Step5: Convert the lower and upper random interval coefficient SLDM, and TLDM problem in to equivalent crisp models, go to step 6.

Step6: Using the fuzzy approach as described in [7] to solve the resulting multi-level decision-making problems in Step 5.

Step7: Build membership functions of the FLDM, SLDM, and TLDM after determine the best and the worst solution of all (LIC) and (UIC) problems.

Step8: Solve a Tchebycheff problem for all decision makers level problem.

Step9: Control assumed the FLDM his/her decision by tolerance t_1 .

Step10: Control assumed the SLDM his/her decision by tolerance t_2 .

Step11: If $\beta < 0$, increase t_1, t_2 then go to step7, otherwise go to step 12.

Step12: The FLDM, SLDM, and TLDM calculating membership function μ .

Step13: Compute tolerance functions for x_1, x_2 using t_1, t_2 .

Step14: Solve the Tchebycheff problem defined by (38), then go to step15.

Step15: If the FLDM not satisfied with solution then go to step 9 with modifying $\omega(\underline{\omega}^L, \underline{\omega}^U, \bar{\omega}^L, \bar{\omega}^U)$.

Step16: Stop.

6. Numerical example:

To demonstrate the solution method for three-level interval quadratic programming problem under random rough coefficient in objective functions, we consider the following problem:

[1stLevel]

$$\text{Max}_{x_1} F_1(x) = 2([2,3], [1,5])x_1 + 3([0,3], [0,4])x_2 + 8x_3^2,$$

where x_2 solves

[2ndLevel]

$$\text{Max}_{x_2} F_2(x) = 6x_1 + 4([2,3], [0,4])x_2^2 + 2([3,4], [1,5])x_3,$$

where x_3 solves

[3rdLevel]

$$\text{Max}_{x_3} F_3(x) = 2x_1^2 + 12x_2 + 4([1,2], [0,3])x_2^2 + 5([1,3], [0,4])x_3^2,$$

Subject to

$$3x_1 + 5x_2 + x_3 \leq 35, \quad 2x_1 - x_2 + 12x_3 \leq 20, \quad 5x_2 + 6x_3 \leq 16, \quad x_i \geq 0, i = 1,2,3.$$

Now by using Theorem (1,2), the equivalent crisp problems which are equivalent to three-level interval quadratic programming problem under rough parameters in objective functions can be written as:

1stLevel

Lower	Upper
QP1: $\underline{f}_1^L := \text{Max } 4x_1 + 8x_3^2,$ Subject to $3x_1 + 5x_2 + x_3 \leq 35,$ $2x_1 - x_2 + 12x_3 \leq 20,$ $5x_2 + 6x_3 \leq 16,$ $x_i \geq 0, i = 1,2,3.$	QP3: $\bar{f}_1^L := \text{Max } 2x_1 + 8x_3^2,$ Subject to $3x_1 + 5x_2 + x_3 \leq 35,$ $2x_1 - x_2 + 12x_3 \leq 20,$ $5x_2 + 6x_3 \leq 16,$ $x_i \geq 0, i = 1,2,3.$
QP2: $\underline{f}_1^U := \text{Max } 6x_1 + 9x_2 + 8x_3^2,$ Subject to $3x_1 + 5x_2 + x_3 \leq 35,$ $2x_1 - x_2 + 12x_3 \leq 20,$ $5x_2 + 6x_3 \leq 16,$ $x_i \geq 0, i = 1,2,3.$	QP4: $\bar{f}_1^U := \text{Max } 10x_1 + 12x_2 + 8x_3^2,$ Subject to $3x_1 + 5x_2 + x_3 \leq 35,$ $2x_1 - x_2 + 12x_3 \leq 20,$ $5x_2 + 6x_3 \leq 16,$ $x_i \geq 0, i = 1,2,3.$

Table (1.a)

2ndLevel

Lower	Upper
QP1: $\underline{f}_2^L := \text{Max } 6x_1 + 8x_2^2 + 6x_3,$ Subject to $3x_1 + 5x_2 + x_3 \leq 35,$ $2x_1 - x_2 + 12x_3 \leq 20,$ $5x_2 + 6x_3 \leq 16,$ $x_i \geq 0, i = 1,2,3.$	QP3: $\bar{f}_2^L := \text{Max } 6x_1 + 2x_3,$ Subject to $3x_1 + 5x_2 + x_3 \leq 35,$ $2x_1 - x_2 + 12x_3 \leq 20,$ $5x_2 + 6x_3 \leq 16,$ $x_i \geq 0, i = 1,2,3.$
QP2: $\underline{f}_2^U := \text{Max } 6x_1 + 12x_2^2 + 8x_3,$ Subject to $3x_1 + 5x_2 + x_3 \leq 35,$ $2x_1 - x_2 + 12x_3 \leq 20,$ $5x_2 + 6x_3 \leq 16,$ $x_i \geq 0, i = 1,2,3.$	QP4: $\bar{f}_2^U := \text{Max } 6x_1 + 16x_2^2 + 10x_3,$ Subject to $3x_1 + 5x_2 + x_3 \leq 35,$ $2x_1 - x_2 + 12x_3 \leq 20,$ $5x_2 + 6x_3 \leq 16,$ $x_i \geq 0, i = 1,2,3.$

Table (2.b)

3rd Level

Lower	Upper
QP1: $\underline{f}_3^L := \text{Max } 2x_1^2 + 12x_2 + 4x_2^2 + 5x_3^2,$ Subject to $3x_1 + 5x_2 + x_3 \leq 35,$ $2x_1 - x_2 + 12x_3 \leq 20,$ $5x_2 + 6x_3 \leq 16,$ $x_i \geq 0, i = 1,2,3.$	QP3: $\bar{f}_3^L := \text{Max } 2x_1^2 + 12x_2,$ Subject to $3x_1 + 5x_2 + x_3 \leq 35,$ $2x_1 - x_2 + 12x_3 \leq 20,$ $5x_2 + 6x_3 \leq 16,$ $x_i \geq 0, i = 1,2,3.$
QP2: $\underline{f}_3^U := \text{Max } 2x_1^2 + 12x_2 + 8x_2^2 + 15x_3^2$ Subject to $3x_1 + 5x_2 + x_3 \leq 35,$ $2x_1 - x_2 + 12x_3 \leq 20,$ $5x_2 + 6x_3 \leq 16,$ $x_i \geq 0, i = 1,2,3.$	QP4: $\bar{f}_3^U := \text{Max } 2x_1^2 + 12x_2 + 12x_2^2 + 20x_3^2$ Subject to $3x_1 + 5x_2 + x_3 \leq 35,$ $2x_1 - x_2 + 12x_3 \leq 20,$ $5x_2 + 6x_3 \leq 16,$ $x_i \geq 0, i = 1,2,3.$

Table (3.c)

By using (32, 33), the FLDM builds the membership functions $\mu(\underline{f}_1^L, \underline{f}_1^U, \bar{f}_1^L, \bar{f}_1^U)(x)$ and from table (1.a) then one solves problem as follows:

QP1: Max λ , subject to $4x_1 + 8x_3^2 \leq 41.53846\lambda,$ $x \in G$, $\lambda \in [0,1].$ Whose solution is $(\underline{x}_1^L, \underline{x}_2^L, \underline{x}_3^L)^F = (1.234568, 1.234568, 1.234568),$ $(\underline{f}_1^L)^F = 17.13153717$ $\lambda^F = 0.9.$	QP3: Max λ , subject to $2x_1 + 8x_3^2 \leq 24.71258\lambda,$ $x \in G$, $\lambda \in [0,1].$ Whose solution is $(\bar{x}_1^L, \bar{x}_2^L, \bar{x}_3^L)^F = (1.234568, 1.234568, 1.234568),$ $(\bar{f}_1^L)^F = 14.66240117,$ $\lambda^F = 0.9.$
---	--

QP2: Max λ , subject to $6x_1 + 9x_2 + 8x_3^2 \leq 69.53746\lambda$, $x \in G$, $\lambda \in [0,1]$. Whose solution is $(\underline{x}_1^U, \underline{x}_2^U, \underline{x}_3^U)^F = (1.234568, 1.234568, 1.234568)$, $(f_1^U)^F = 30.71178517$ $\lambda^F = 0.9$.	QP4: Max λ , subject to $10x_1 + 12x_2 + 8x_3^2 \leq 113.0769\lambda$, $x \in G$, $\lambda \in [0,1]$. Whose solution is $(\bar{x}_1^U, \bar{x}_2^U, \bar{x}_3^U)^F = (1.234568, 1.234568, 1.234568)$, $(f_1^U)^F = 39.35376117$. $\lambda^F = 0.9$.
--	--

Table (1.2.a)

Secondly, the SLDM defines his/her problem in view of the FLDM as follows:

$$QP1: (\underline{x}_1^L, \underline{x}_2^L, \underline{x}_3^L)^S = (1.234568, 1.234568, 1.234568), QP2: (\underline{x}_1^U, \underline{x}_2^U, \underline{x}_3^U)^S = (1.234568, 1.234568, 1.234568).$$

$$QP3: (\bar{x}_1^L, \bar{x}_2^L, \bar{x}_3^L)^S = (0, 3.2000, 0), QP4: (\bar{x}_1^U, \bar{x}_2^U, \bar{x}_3^U)^S = (1.234568, 1.234568, 1.234568).$$

Third, the TLDM defines his/her problem in view of the SLDM as follows:

$$QP1: (\underline{x}_1^L, \underline{x}_2^L, \underline{x}_3^L)^T = (1.234568, 1.234568, 1.234568), QP2: (\underline{x}_1^U, \underline{x}_2^U, \underline{x}_3^U)^T = (1.234568, 1.234568, 1.234568).$$

$$QP3: (\bar{x}_1^L, \bar{x}_2^L, \bar{x}_3^L)^T = (1.234568, 1.234568, 0), QP4: (\bar{x}_1^U, \bar{x}_2^U, \bar{x}_3^U)^T = (1.234568, 1.234568, 1.234568).$$

Finally, in order to generate the satisfactory solution, which is also a Pareto optimal solution with overall satisfaction for all decision-makers, by (38), calculating the tolerance function also.

- 1- We assume the FLDM'S control decision $\underline{x}_1^F$ with the tolerance 1, and assume the SLDM'S control decision $\underline{x}_2^S$ with the tolerance 1.2.
- 2- By using (36)–(37) calculating membership functions μ , then solves the Tchebycheff problem as follows :

Max ω^L , Subject to $4x_1 + 8x_3^2 - 17.13153717 \geq 0$, $6x_1 + 8x_2^2 + 6x_3 - 27.00808117 \geq 0$, $2x_1^2 + 12x_2 + 4x_2^2 + 5x_3^2 - 31.58055561 \geq 0$, $x_1 - 0.234568 \geq \underline{\omega}^L$, $-x_1 + 2.234568 \geq \underline{\omega}^L$, $x_2 - 0.234568 \geq 1.2\underline{\omega}^L$, $-x_2 + 2.234568 \geq 1.2\underline{\omega}^L$, $x \in G$, $t_i > 0, (i = 1, 2)$, $\underline{\omega}^L \in [0, 1]$. $(\underline{x}_1^L, \underline{x}_2^L, \underline{x}_3^L) = (1.234567, 1.234568, 1.234568)$.	Max $\bar{\omega}^L$, Subject to $2x_1 + 8x_3^2 \geq 14.66240117\bar{\omega}^L$, $6x_1 + 2x_3 - 7.47408 \geq -7.47408\bar{\omega}^L$, $2x_1^2 + 12x_2 - 38.4 \geq -20.5368677\bar{\omega}^L$, $x_1 - 0.234568 \geq \bar{\omega}^L$, $-x_1 + 2.234568 \geq \bar{\omega}^L$, $x_2 - 2.20000 \geq 1.2\bar{\omega}^L$, $-x_2 + 4.2000 \geq 1.2\bar{\omega}^L$, $x \in G$, $t_i > 0, (i = 1, 2)$, $\bar{\omega}^L \in [0, 1]$. $(\bar{x}_1^L, \bar{x}_2^L, \bar{x}_3^L) = (1.863676, 2.645070, 0.4624413)$.
Max ω^U , Subject to $6x_1 + 9x_2 + 8x_3^2 - 30.71178517 \geq 0$, $6x_1 + 12x_2^2 + 8x_3 - 35.57384976 \geq 0$, $2x_1^2 + 12x_2 + 8x_2^2 + 15x_3^2 - 52.91876966 \geq 0$, $x_1 - 0.234568 \geq \underline{\omega}^U$, $-x_1 + 2.234568 \geq \underline{\omega}^U$, $x_2 - 0.234568 \geq 1.2\underline{\omega}^U$, $-x_2 + 2.234568 \geq 1.2\underline{\omega}^U$, $x \in G$, $t_i > 0, (i = 1, 2)$, $\underline{\omega}^U \in [0, 1]$. $(\underline{x}_1^U, \underline{x}_2^U, \underline{x}_3^U) = (1.340082, 1.234568, 1.219810)$.	Max $\bar{\omega}^U$, Subject to $10x_1 + 12x_2 + 8x_3^2 - 39.35376117 \geq 0$, $6x_1 + 16x_2^2 + 10x_3 - 44.13961835 \geq 0$, $2x_1^2 + 12x_2 + 12x_2^2 + 20x_3^2 - 66.63619299 \geq 0$, $x_1 - 0.234568 \geq \bar{\omega}^U$, $-x_1 + 2.234568 \geq \bar{\omega}^U$, $x_2 - 0.234568 \geq 1.2\bar{\omega}^U$, $-x_2 + 2.234568 \geq 1.2\bar{\omega}^U$, $x \in G$, $t_i > 0, (i = 1, 2)$, $\bar{\omega}^U \in [0, 1]$. $(\bar{x}_1^U, \bar{x}_2^U, \bar{x}_3^U) = (1.860786, 1.786029, 0.3895630)$.

Overall satisfaction for both decisions-makers are:

$$([\underline{f}_1^L, \underline{f}_1^U], [\bar{f}_1^L, \bar{f}_1^U]) = ([17.13153317, 31.05509549], [5.438167648, 41.25428265]).$$

$$([\underline{f}_2^L, \underline{f}_2^U], [\bar{f}_2^L, \bar{f}_2^U]) = ([27.00807517, 36.08886976], [12.1069386, 66.09873942]).$$

$$([\underline{f}_3^L, \underline{f}_3^U], [\bar{f}_3^L, \bar{f}_3^U]) = ([38.68741647, 52.91876725], [31.58055067, 69.67137876]).$$

7. Summary and concluding remarks

A three-level quadratic programming (QP) problem was considered where some or all of its coefficients in the objective function are rough intervals. At the first phase of the solution approach and to avoid the complexity of the problem, two QP problems with interval coefficients will be constructed. One of these problems was a QP where all of its coefficients are upper approximation of and the other problem was a QP where all of its coefficients are lower approximations of rough intervals. At the second phase, a membership function was constructed to develop a fuzzy model for obtaining the optimal solution of the three-level quadratic programming problem. In addition, the author put forward the satisfactoriness concept as the first-level decision-maker preference.

However, there are many open points for discussion in future, which should be explored and studied in the area of multi-level rough interval optimization such as:

1. Interactive algorithm is required for treating multi-level integer quadratic multi-objective decision-making problems with rough parameters in the objective functions; in the constraints and in both.
2. Interactive algorithm is needed for dealing with multi-level mixed integer quadratic multi-objective decision-making problems with rough parameters in the objective functions; in the constraints and in both.
3. Interactive algorithm is necessary for solving multi-level integer fractional multi-objective decision-making problems with rough parameters in the objective functions; in the constraints and in both.
4. Interactive algorithm must be investigated for treating multi-level mixed integer fractional multi-objective decision-making problems with rough parameters in the objective functions; in the constraints and in both.

References

- [1]S. Effati, M. Pakdaman, Solving the interval-valued linear fractional programming problem, American Journal of Computational Mathematics 2 (2012) 51-55.
- [2]O. Emam, Interactive Bi-level multi-objective integer non-linear programming problem, Applied Mathematics Sciences 5 (65) (2011) 3221-3232.
- [3]O. Emam, Interactive approach to bi-level integer multi-objective fractional programming problem, Applied Mathematics and Computation 223 (2013) 17-24.
- [4]O. Emam, Fuzzy approach for bi-level integer non-linear programming problem, Applied Mathematics and Computation 172 (2006) 62-71.
- [5]A. Hamzehee, M. A. Yaghoobi, M. Mashinchi, Linear programming with rough interval coefficients, Intelligent and Fuzzy Systems 26 (2014)1179-1189.
- [6]J. Komorowski, L. Polkowskia, A. Skowron, Rough sets: a tutorial, Polish-Japanese Institute of Information Technology Warszawa, Poland.
- [7]M. Osman, M. Abo-Sinna, A. Amer and O. Emam, A multi-level non-linear multi-objective decision-making under fuzziness, Applied Mathematics and Computation 153 (2004) 239-252.
- [8]M. Osman, W.A. El-Wahed, M. El-Shafei, H.A. El-Wahab, A proposed approach for solving rough bi-level programming problems by genetic algorithm, International Journal of Mathematical 6 (2011) 1453-1465.
- [9]Z. Pawlak, Rough sets, Kluwer academic publishers, (1991).
- [10]O. M.Saad, E. F.Elsayed, T. B.Farag, A comparative study on the solution of stochastic and fuzzy integer nonlinear programming problem, International Journal of Mathematical Archive 5(2) (2014) 1-8.
- [11]O. M.Saad, T.B. Farag, On the solution of chance-constrained multi-objective integer quadratic programming problem with some stability notions, General Mathematics Notes, 20(2) (2014) 111-124.
- [12]O. M.Saad, T. B.Farag, Multi objective integer linear programming problem with fuzzy discrete right-hand side parameterization, International Journal Of Innovation In Science and Mathematics, 2(1) (2014) 2347-9051.
- [13]M. Sakawa, Fuzzy sets and interactive multi-objective optimization, Plenum press publishers.
- [14]T.I. Sultan, O.E. Emam, S.A. Nasr, On the solution of a parametric bi-level quadratic programming problem, General Mathematics Notes,19(1) (2013) 43-52.