

Preliminary Test Stochastic Restricted $r-k$ Class Estimator and Preliminary Test Stochastic Restricted $r-d$ Class Estimator in Linear Regression Model

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ABSTRACT

In this study two new preliminary test stochastic restricted estimators, a Preliminary Test stochastic restricted $r-k$ class estimator and a Preliminary Test stochastic restricted $r-d$ class estimator, are proposed. The comparison of one estimator over the other is done in the mean square error matrix sense. Further preliminary test stochastic restricted $r-k$ class estimator is compared with $r-k$ class estimator (Baye and Parker, 1984) and stochastic restricted $r-k$ class estimator (Wu, 2014). Similarly preliminary test stochastic restricted $r-d$ class estimator is compared with $r-d$ class estimator (Kaçiranlar and Sakallıoğlu, 2001) and stochastic restricted $r-d$ class estimator (Wu, 2014). Finally a numerical example and a Monte Carlo simulation study are done to illustrate the theoretical findings of the proposed estimators.

Keywords: Preliminary Test Estimator, $r-d$ class estimator, $r-k$ class estimator, mean square error matrix

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1. INTRODUCTION

Instead of using the Ordinary Least Square Estimator (OLSE), the biased estimators are considered in the regression analysis in the presence of multicollinearity. Some of these are namely the Principal Component Regression Estimator (PCRE) (Massy, 1965), Ridge Estimator (RE) (Hoerl and Kennard, 1970), Liu Estimator (LE) (Liu, 1993), r - k class estimator (Baye and Parker, 1984), Almost Unbiased Ridge Estimator (AURE) (Sing et al., 1986), Almost Unbiased Liu Estimator (AULE) (Akdeniz and Kaçiranlar, 1995) and the r - d class estimator (Kaçiranlar and Sakallıoğlu, 2001). An alternative method to deal with multicollinearity problem is to consider parameter estimation with some restrictions on the unknown parameters, which may be exact or stochastic restrictions. In the presence of stochastic prior information in addition to the sample information, Theil and Goldberger (1961) proposed the Mixed Estimator (ME). By replacing OLSE by ME in the RE, LE, AURE and AULE respectively, the Stochastic Mixed Ridge Estimator (SMRE) (Li and Yang, 2010), Stochastic Restricted Liu Estimator (SRLE) (Hubert and Wijekoon, 2006), Stochastic Restricted Almost Unbiased Ridge Estimator (SRAURE) (Wu and Yang, 2014) and Stochastic Restricted Almost Unbiased Liu Estimator (SRAULE) (Wu and Yang, 2014) are introduced. Recently, Wu (2014) proposed a new stochastic restricted r - k class estimator which is defined by combining the ME and r - k class estimator and a new stochastic restricted r - d class estimator which is defined by combining the ME and r - d class estimator.

When different estimators are available to estimate the unknown parameter, preliminary test estimation procedure is adopted to select a suitable estimator, and it can be also used as another estimator with combining properties of both estimators. The preliminary test approach was first proposed by Bancroft (1944), and then has been studied by many researchers, such as Judge and Bock (1978), Wijekoon (1990) and Arumairajan and Wijekoon (2013). By combining OLSE and ME, the Ordinary Stochastic Preliminary Test Estimator (OSPE) was proposed by Wijekoon (1990). Recently, Arumairajan and Wijekoon (2013) introduced the Preliminary Test

Stochastic Restricted Liu Estimator (PTSRLE) by combining the Stochastic Restricted Liu Estimator and Liu Estimator.

In this study, we propose a new Preliminary Test stochastic restricted $r-k$ class estimator which is defined by combining the $r-k$ class estimator and stochastic restricted $r-k$ class estimator and a new Preliminary Test stochastic restricted $r-d$ class estimator which is defined by combining the $r-d$ class estimator and stochastic restricted $r-d$ class estimator. Further the proposed estimators are compared with some biased estimators in the mean square error matrix sense. Also Preliminary Test stochastic restricted $r-d$ class estimator is compared with Preliminary Test stochastic restricted $r-k$ class estimator. Finally a numerical example and a Monte Carlo simulation study are done to illustrate the theoretical findings of the proposed estimators.

2. MODEL SPECIFICATION AND ESTIMATORS

First we consider the multiple linear regression model

$$Y = X\beta + \varepsilon, \varepsilon \sim N(0, \sigma^2 I), \quad (2.1)$$

where Y is an $n \times 1$ observable random vector, X is an $n \times p$ known design matrix of rank p , β is a $p \times 1$ vector of unknown parameters and ε is an $n \times 1$ vector of disturbances.

In addition to sample model (2.1), let us be given some prior information about β in the form of a set of m independent stochastic linear restrictions as follows;

$$r = R\beta + \delta + v, v \sim N(0, \sigma^2 \Omega) \quad (2.2)$$

where r is an $m \times 1$ stochastic known vector R is a $m \times p$ of full row rank $m \leq p$ with known elements, δ is non zero $m \times 1$ unknown vector and v is an $m \times 1$ random vector of disturbances and Ω is assumed to be known and positive definite. Further it is assumed that v is stochastically independent of ε . i.e. $E(\varepsilon v') = 0$.

The Ordinary Least Squares Estimator (OLSE) for the model (2.1) and the mixed estimator (Theil and Goldberger, 1961) due to a stochastic prior restriction (2.2) are given by

$$\hat{\beta} = S^{-1} X' Y \quad \text{and} \quad \hat{\beta}_m = \hat{\beta} + S^{-1} R' (\Omega + RS^{-1} R')^{-1} (r - R\hat{\beta}) \quad (2.3)$$

respectively. where $S = X' X$.

The expectation vector, and the mean square error matrix of $\hat{\beta}$ are given as $E(\hat{\beta}) = \beta$ and $MSE(\hat{\beta}) = \sigma^2 S^{-1}$ respectively.

The expectation vector, dispersion matrix, and the mean square error matrix of $\hat{\beta}_m$ are given as $E(\hat{\beta}_m) = \beta + H\delta$, $D(\hat{\beta}_m) = \sigma^2 S^{-1} - \sigma^2 G$ and

$MSE(\hat{\beta}_m) = \sigma^2 (S^{-1} - G) + H\delta\delta'H'$ respectively, where,

$$G = S^{-1} R' (\Omega + RS^{-1} R')^{-1} RS^{-1}, \quad H = S^{-1} R' (\Omega + RS^{-1} R')^{-1} \quad \text{and} \quad \delta = E(r) - R\beta.$$

When different estimators are available for the same parameter vector β in the linear regression model one must solve the problem of their comparison. Usually as a simultaneous measure of covariance and bias, the mean square error matrix is used, and is defined by

$$MSE(\hat{\beta}, \beta) = E \left[(\hat{\beta} - \beta)(\hat{\beta} - \beta)' \right] = D(\hat{\beta}) + B(\hat{\beta})B'(\hat{\beta}), \quad (2.4)$$

where $D(\hat{\beta})$ is the dispersion matrix, and $B(\hat{\beta}) = E(\hat{\beta}) - \beta$ denotes the bias vector.

We recall that the Scalar Mean Square Error $SMSE(\hat{\beta}, \beta) = \text{trace} [MSE(\hat{\beta}, \beta)]$.

For two given estimators $\hat{\beta}_1$ and $\hat{\beta}_2$, the estimator $\hat{\beta}_2$ is said to be superior to $\hat{\beta}_1$ under the MSEM criterion if and only if

$$M(\hat{\beta}_1, \hat{\beta}_2) = MSE(\hat{\beta}_1, \beta) - MSE(\hat{\beta}_2, \beta) \geq 0. \quad (2.5)$$

Let us now turn to the question of the statistical evaluation of the compatibility of sample and stochastic information. The classical procedures is to test the hypothesis

$$H_0 : \delta = 0 \quad \text{against} \quad H_1 : \delta \neq 0 \quad (2.6)$$

under linear model (2.1) and stochastic prior information (2.2).

The Ordinary Stochastic Pre Test Estimator (OSPE) of β (Wijekoon, 1990) is defined as

$$\hat{\beta}_{OSPE} = \begin{cases} \hat{\beta}_m & \text{if } H_0 : \delta = 0 \\ \hat{\beta} & \text{if } H_1 : \delta \neq 0 \end{cases} \quad (2.7)$$

Further, we can write (2.7) as

$$\hat{\beta}_{OSPE} = \hat{\beta}_m I_{[0, F_{m,n-p}(\alpha)]}(F) + \hat{\beta} I_{[F_{m,n-p}(\alpha), \infty)}(F), \quad (2.8)$$

$$\text{where, } F = \frac{(r - R\hat{\beta})' (\Omega + RS^{-1}R')^{-1} (r - R\hat{\beta})}{m\hat{\sigma}^2} \quad (2.9)$$

which has a non-central $F_{m,n-p,\lambda}$ distribution under $H_1 : \delta \neq 0$, with non-centrality parameter

$$\lambda = \frac{\delta' (\Omega + RS^{-1}R')^{-1} \delta}{2\sigma^2} \quad \text{with} \quad \hat{\sigma}^2 = \frac{(Y - X\hat{\beta})' (Y - X\hat{\beta})}{n-p}, \quad (2.10)$$

and $I_{[0, F_{m,n-p}(\alpha)]}(F)$ and $I_{[F_{m,n-p}(\alpha), \infty)}(F)$ are indicator functions which take the value one if F falls in the subscripted interval, and zero otherwise. $F_{m,n-p}(\alpha)$ is the upper α - level critical value from the central F distribution $F_{m,n-p,0}$.

The expectation vector, dispersion matrix, and the mean square error matrix of $\hat{\beta}_{OSPE}$ are derived by Wijekoon (1990), and given by

$$E(\hat{\beta}_{OSPE}) = \beta + h_\lambda(2)H\delta, \quad (2.11)$$

$$D(\hat{\beta}_{OSPE}) = \sigma^2 S^{-1} - \sigma^2 h_\lambda(2)G + [2h_\lambda(2) - h_\lambda(4) - h_\lambda^2(2)]H\delta\delta'H', \quad (2.12)$$

and

$$MSE(\hat{\beta}_{OSPE}) = \sigma^2 S^{-1} - \sigma^2 h_\lambda(2)G + (2h_\lambda(2) - h_\lambda(4))H\delta\delta'H' \quad (2.13)$$

respectively, where, $h_\lambda(\ell) = \Pr\left(\frac{\chi_{m+\ell, \lambda}^2}{\chi_{n-p}^2} \leq \frac{mF_{m, n-p}(\alpha)}{n-p}\right)$ for $\ell \in \mathbb{N}$.

Now we consider the transformation for model (2.1):

$$y = XTT'\beta + \varepsilon = Z\alpha + \varepsilon \quad (2.14)$$

where $Z = XT$, $\alpha = T'\beta$ and $T = (t_1, t_2, \dots, t_p) = (T_r, T_{p-r})$ is a $p \times p$ orthogonal matrix such that

$$(T_r, T_{p-r})' X'X (T_r, T_{p-r}) = \Lambda = \begin{pmatrix} \Lambda_r & 0 \\ 0 & \Lambda_{p-r} \end{pmatrix}$$

where $0 < k \leq p$, $T_r = (t_1, t_2, \dots, t_r)$, $T_{p-r} = (t_{r+1}, t_{r+2}, \dots, t_p)$, $\Lambda = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_p)$, $\Lambda_r = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_r)$, $\Lambda_{p-r} = \text{diag}(\lambda_{r+1}, \lambda_{r+2}, \dots, \lambda_p)$ and $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_p > 0$ are the eigenvalues of $X'X$. Note that $Z = XT = (z_1, z_2, \dots, z_p) = (Z_r, Z_{p-r})$ is the $n \times p$ matrix of the principal components, where $z_i = Xt_i$ is the i^{th} principal component. When Z_{p-r} contains principal components corresponding to near zero eigenvalues, Z can be separated as Z_r and Z_{p-r} , where Z_{p-r} is to be deleted. Now we can rewrite the model (2.14) as

$$y = XTT'\beta = XT_r T_r' \beta + XT_{p-r} T_{p-r}' \beta + \varepsilon = Z_r \alpha_r + Z_{p-r} \alpha_{p-r} + \varepsilon. \quad (2.15)$$

By omitting Z_{p-r} , the OLSE of α_r is obtained, and $\hat{\alpha}_r = (Z_r' Z_r)^{-1} Z_r' y$. Then PCRE of β is

$$\hat{\beta}_{PCRE} = T_r (T_r' S T_r)^{-1} T_r' X' y. \quad (2.16)$$

Now the r - k class estimator proposed by Baye and Parker (1984) and the r - d class estimator proposed by Kaçiranlar and Sakallıoğlu (2001) are defined as

$$\hat{\beta}_{rk}(r, k) = T_r (T_r' X' X T_r + k I_r)^{-1} T_r' X' y \quad (2.17)$$

and

$$\hat{\beta}_{rd}(r, d) = T_r (T_r' X' X T_r + k I_r)^{-1} (T_r' X' y + d T_r' \hat{\beta}_r) \quad (2.18)$$

respectively.

Followed by Xu and Yang (2011), the r - k class estimator and the r - d class estimator could be rewritten as follows:

$$\hat{\beta}_{rk}(r, k) = T_r T_r' \hat{\beta}(k) = T_r T_r' W_k \hat{\beta} = R_k \hat{\beta} \quad (2.19)$$

and

$$\hat{\beta}_{rd}(r, d) = T_r T_r' \hat{\beta}(d) = T_r T_r' F_d \hat{\beta} = H_d \hat{\beta} \quad (2.20)$$

respectively, where $R_k = T_r T_r' W_k$ and $H_d = T_r T_r' F_d$.

Note that when $r = p$ we may conclude that $T_r T_r' = I_p$. Hence the estimators

$\hat{\beta}_{rk}(r, k) = W_k \hat{\beta}$ called as RE and $\hat{\beta}_{rd}(r, d) = F_d \hat{\beta}$ named as LE.

The mean square error matrices of $\hat{\beta}_{rk}(r, k)$ and $\hat{\beta}_{rd}(r, d)$ can be obtained as

$$MSE[\hat{\beta}_{rk}(r, k)] = \sigma^2 R_k S^{-1} R_k' + (R_k - I) \beta \beta' (R_k - I)' \quad (2.21)$$

and

$$MSE[\hat{\beta}_{rd}(r, d)] = \sigma^2 H_d S^{-1} H_d' + (H_d - I) \beta \beta' (H_d - I)' \quad (2.22)$$

respectively.

Wu (2014) proposed a new stochastic restricted r - k class estimator which is defined by combining the ME and r - k class estimator and a new stochastic restricted r - d class estimator which is defined by combining the ME and r - d class estimator as follows:

$$\hat{\beta}_{SRrk}(r, k) = T_r T_r' \hat{\beta}(k) = T_r T_r' W_k \hat{\beta}_m = R_k \hat{\beta}_m \quad (2.23)$$

and

$$\hat{\beta}_{SRrd}(r, d) = T_r T_r' \hat{\beta}(d) = T_r T_r' F_d \hat{\beta}_m = H_d \hat{\beta}_m \quad (2.24)$$

respectively.

When $r = p$, the estimator $\hat{\beta}_{SRrk}(r, k) = W_k \hat{\beta}_m$ called as SMRE and $\hat{\beta}_{SRrd}(r, d) = F_d \hat{\beta}_m$ named as SRLE.

The mean square error matrices of $\hat{\beta}_{SRrk}(r, k)$ and $\hat{\beta}_{SRrd}(r, d)$ can be derived as

$$MSE \left[\hat{\beta}_{SRrk} (r, k) \right] = \sigma^2 R_k S^{-1} R_k' - \sigma^2 R_k G R_k' + \left[(R_k - I) \beta + R_k H \delta \right] \left[(R_k - I) \beta + R_k H \delta \right]' \quad (2.25)$$

and

$$MSE \left[\hat{\beta}_{SRrd} (r, d) \right] = \sigma^2 H_d S^{-1} H_d' - \sigma^2 H_d G H_d' + \left[(H_d - I) \beta + H_d H \delta \right] \left[(H_d - I) \beta + H_d H \delta \right]' \quad (2.26)$$

respectively.

3. PROPOSED ESTIMATORS

Arumirajan and Wijekoon (2013) proposed the Preliminary Test Stochastic Restricted Liu Estimator (PTSRLE) by combining the LE and SRLE as follows:

$$\tilde{\beta}_{PTSRLE} (d) = F_d \hat{\beta}_m I_{[0, F_{m,n-p}(\alpha)]} (F) + F_d \hat{\beta} I_{[F_{m,n-p}(\alpha), \infty)} (F) = F_d \hat{\beta}_{OSPE} \quad (3.1)$$

Following Arumirajan and Wijekoon (2013), we may write the Preliminary Test Stochastic Mixed Ridge Estimator (PTSMRE) by combining RE and SMRE as follows:

$$\tilde{\beta}_{PTSMRE} (k) = W_k \hat{\beta}_m I_{[0, F_{m,n-p}(\alpha)]} (F) + W_k \hat{\beta} I_{[F_{m,n-p}(\alpha), \infty)} (F) = W_k \hat{\beta}_{OSPE} \quad (3.2)$$

Following Arumairajan and Wijekoon (2013), we propose a new Preliminary Test stochastic restricted r - k class estimator (PTSRrk) which is defined by combing the r - k class estimator and stochastic restricted r - k class estimator and a new Preliminary Test stochastic restricted r - d class estimator (PTSRrd) which is defined by combing the r - d class estimator and stochastic restricted r - d class estimator as follows:

$$\tilde{\beta}_{PTSRrk} (k) = R_k \left[\hat{\beta}_m I_{[0, F_{m,n-p}(\alpha)]} (F) + \hat{\beta} I_{[F_{m,n-p}(\alpha), \infty)} (F) \right] = R_k \hat{\beta}_{OSPE}, \quad (3.3)$$

and

$$\tilde{\beta}_{PTSRrd} (d) = H_d \left[\hat{\beta}_m I_{[0, F_{m,n-p}(\alpha)]} (F) + \hat{\beta} I_{[F_{m,n-p}(\alpha), \infty)} (F) \right] = H_d \hat{\beta}_{OSPE} \quad (3.4)$$

respectively.

Now we will see some properties of proposed estimators.

1. When $r = p$, we may conclude that $\tilde{\beta}_{PTSRrk}(k) = \tilde{\beta}_{PTSMRE}(k)$ and $\tilde{\beta}_{PTSRrd}(d) = \tilde{\beta}_{PTSRLE}(d)$.
2. When $r = p$, $k = 0$ and $d = 1$, we may conclude that $\tilde{\beta}_{PTSRrk}(k) = \tilde{\beta}_{PTSRrd}(d) = \hat{\beta}_{OSPE}$.
3. When $\alpha = 0$, we may conclude that $\tilde{\beta}_{PTSRrk}(k) = \hat{\beta}_{SRrk}(r, k)$ and $\tilde{\beta}_{PTSRrd}(d) = \hat{\beta}_{SRrd}(r, d)$.
4. When $\alpha = 1$, we may conclude that $\tilde{\beta}_{PTSRrk}(k) = \hat{\beta}_{rk}(r, k)$ and $\tilde{\beta}_{PTSRrd}(d) = \hat{\beta}_{rd}(r, d)$.

By using (2.11), (2.12) and (2.13), the expectation vector, bias vector, dispersion matrix, and the mean square error matrix of $\tilde{\beta}_{PTSRrk}(k)$ can be shown as follows:

$$E[\tilde{\beta}_{PTSRrk}(k)] = R_k E(\hat{\beta}_{OSPE}) = R_k \beta + h_\lambda(2) R_k H \delta, \quad (3.5)$$

$$B[\tilde{\beta}_{PTSRrk}(k)] = E[\tilde{\beta}_{PTSRrk}(k)] - \beta = (R_k - I) \beta + h_\lambda(2) R_k H \delta, \quad (3.6)$$

$$\begin{aligned} D[\tilde{\beta}_{PTSRrk}(k)] &= R_k D(\hat{\beta}_{OSPE}) R_k' \\ &= \sigma^2 R_k S^{-1} R_k' - \sigma^2 h_\lambda(2) R_k G R_k' + [2h_\lambda(2) - h_\lambda(4) - h_\lambda^2(2)] R_k H \delta \delta' H' R_k', \end{aligned} \quad (3.7)$$

and

$$\begin{aligned} MSE[\tilde{\beta}_{PTSRrk}(k)] &= \sigma^2 R_k S^{-1} R_k' - \sigma^2 h_\lambda(2) R_k G R_k' \\ &\quad + [2h_\lambda(2) - h_\lambda(4) - h_\lambda^2(2)] R_k H \delta \delta' H' R_k' \\ &\quad + R_k [(I - R_k^{-1}) \beta + h_\lambda(2) H \delta] [(I - R_k^{-1}) \beta + h_\lambda(2) H \delta]' R_k' \end{aligned} \quad (3.8)$$

respectively.

Similarly the expectation vector, bias vector, dispersion matrix, and the mean square error matrix of $\tilde{\beta}_{PTSRrd}(d)$ can be shown as follows:

$$E[\tilde{\beta}_{PTSRrd}(d)] = H_d E(\hat{\beta}_{OSPE}) = H_d \beta + h_\lambda(2) H_d H \delta, \quad (3.9)$$

$$B[\tilde{\beta}_{PTSRrd}(d)] = (H_d - I) \beta + h_\lambda(2) H_d H \delta, \quad (3.10)$$

$$D[\tilde{\beta}_{PTSRd}(d)] = \sigma^2 H'_d S^{-1} H'_d - \sigma^2 h_\lambda(2) H_d G H'_d + [2h_\lambda(2) - h_\lambda(4) - h_\lambda^2(2)] H_d H \delta \delta' H' H'_d, \quad (3.11)$$

and

$$\begin{aligned} MSE[\tilde{\beta}_{PTSRd}(d)] &= \sigma^2 H'_d S^{-1} H'_d - \sigma^2 h_\lambda(2) H_d G H'_d \\ &\quad + [2h_\lambda(2) - h_\lambda(4) - h_\lambda^2(2)] H_d H \delta \delta' H' H'_d \\ &\quad + H_d \left[(I - H_d^{-1}) \beta + h_\lambda(2) H \delta \right] \left[(I - H_d^{-1}) \beta + h_\lambda(2) H \delta \right]' H'_d \end{aligned} \quad (3.12)$$

respectively.

4. MEAN SQUARE ERROR MATRIX COMPARISONS

In this section the Preliminary Test stochastic restricted r - d class will be compared with r - d class estimator and stochastic restricted r - d class estimator and, the Preliminary Test stochastic restricted r - k class estimator will be compared with r - k class estimator and stochastic restricted r - k class estimator in the mean square error matrix sense. Also Preliminary Test stochastic restricted r - d class estimator will be compared with Preliminary Test stochastic restricted r - k class estimator.

4.1. Comparison between $\hat{\beta}_{rk}(r, k)$ and $\tilde{\beta}_{PTSRk}(k)$

The mean square error matrix difference between $\tilde{\beta}_{PTSRk}(k)$ and $\hat{\beta}_{rk}(r, k)$ is obtained as

$$MSE[\hat{\beta}_{rk}(r, k)] - MSE[\tilde{\beta}_{PTSRk}(k)] = R_k [D + d_1 d'_1 - d_2 d'_2] R'_k \quad (4.1)$$

where $D = \sigma^2 h_\lambda(2) G - \xi H \delta \delta' H'$, $\xi = 2h_\lambda(2) - h_\lambda(4) - h_\lambda^2(2)$, $d_1 = (I - R_k^{-1}) \beta$ and

$$d_2 = (I - R_k^{-1}) \beta + h_\lambda(2) H \delta.$$

Now the following theorem can be stated.

Theorem 4.1: When $\lambda \leq \frac{1}{2[2 - h_\lambda(2)/h_\lambda(4) - h_\lambda(2)]}$, the estimator $\tilde{\beta}_{PTSRk}(k)$ is

superior to $\hat{\beta}_{rk}(r, k)$ if and only if $d'_2 (D + d_1 d'_1)^{-1} d_2 \leq 1$.

Proof:

The mean square error matrix difference between $\tilde{\beta}_{P\text{TSR}rk}(k)$ and $\hat{\beta}_{rk}(r, k)$ is nonnegative definite matrix if and only if $D + d_1 d_1' - d_2 d_2' \geq 0$. To find the conditions that $D + d_1 d_1' - d_2 d_2' \geq 0$ by using lemma 3 (Appendix), we have to show that that D is a nonnegative definite matrix and $d_1, d_2 \in \Re(D)$, where $\Re(\cdot)$ denote the column space of the corresponding matrix.

Note that

$$D = \sigma^2 h_\lambda(2) G - \xi H \delta \delta' H' = \xi \left(\frac{\sigma^2 h_\lambda(2)}{\xi} G - H \delta \delta' H' \right) = \xi D_1$$

$$\text{where } D_1 = \frac{\sigma^2 h_\lambda(2)}{\xi} G - H \delta \delta' H'.$$

$$\text{Set } \gamma = \frac{\sigma^2 h_\lambda(2)}{\xi}, \quad B = G \quad \text{and} \quad a = H \delta.$$

Note that $G \geq 0$, and the generalized inverse of G is $G^- = SR^+ (RS^{-1}R' + \Omega)(R')^+ S$.

Hence $GG^-H\delta = H\delta$. This implies $H\delta \in \Re(G)$. Then according to lemma 1 (Appendix)

$$\delta' H' G^- H \delta \leq \frac{\sigma^2 h_\lambda(2)}{\xi} \quad (4.2)$$

After some straightforward calculations we can easily now show that

$$\delta' H' G^- H \delta = \delta' (RS^{-1}R' + \Omega)^{-1} \delta \quad (4.3)$$

$$\text{By substituting this result to (4.2) we can obtain } \frac{\delta' (RS^{-1}R' + \Omega)^{-1} \delta}{2\sigma^2} \leq \frac{h_\lambda(2)}{2\xi}. \quad (4.4)$$

Using (2.10), this inequality can be rewritten as

$$\lambda \leq \frac{1}{2[2 - h_\lambda(2)/h_\lambda(4) - h_\lambda(2)]}. \quad (4.5)$$

Then according to lemma 1, D_1 is a nonnegative definite matrix, and therefore $D = \xi D_1 \geq 0$ since $\xi \geq 0$.

Now the Moore Penrose inverse of D is obtained by using lemma 2 (Appendix), and is given by

$$D^+ = \frac{1}{\sigma^2 h_\lambda(2)} \times \left[G^+ + \frac{\xi}{\sigma^2 h_\lambda(2) - \xi \delta' H' G^+ H \delta} G^+ H \delta \delta' H' G^+ \right]. \quad (4.6)$$

After some straightforward calculations we can show that

$$\delta' H' G^+ H \delta = 2\sigma^2 \lambda. \quad (4.7)$$

Using (4.6) and (4.7) we can easily prove that $DD^+ = I_p$, where I_p is an identity matrix with order $(p \times p)$. This implies that $DD^+ d_1 = d_1$ and $DD^+ d_2 = d_2$. Then we have $d_1 \in \mathfrak{R}(D)$ and $d_2 \in \mathfrak{R}(D)$. To establish condition (a) in the lemma 3, we find $f_{ij} = d_i' D^- d_j$ for $i = 1, 2, j = 1, 2$ such that

$$\begin{aligned} f_{11} &= \beta' (I - R_k^{-1})' D^+ (I - R_k^{-1}) \beta, \\ f_{22} &= \left[(I - R_k^{-1}) \beta + h_\lambda(2) H \delta \right]' D^+ \left[(I - R_k^{-1}) \beta + h_\lambda(2) H \delta \right] \text{ and} \\ f_{12} &= \beta' (I - R_k^{-1})' D^+ \left[(I - R_k^{-1}) \beta + h_\lambda(2) H \delta \right]. \end{aligned}$$

Note that, instead of D^- , the Moore Penrose inverse D^+ of D is used, since f_{ij} is invariant to the choice of D^- .

Now according to lemma 3 in appendix, $MSE[\hat{\beta}_{rk}(r, k)] - MSE[\tilde{\beta}_{PTSRrk}(k)] \geq 0$ if and only if

$$(d_1' D^+ d_1 + 1)(d_2' D^+ d_2 - 1) \leq (d_1' D_1^+ d_2)^2.$$

4.2. Comparison between $\hat{\beta}_{SRrk}(r, k)$ and $\tilde{\beta}_{PTSRrk}(k)$

The mean square error matrix difference between $\tilde{\beta}_{PTSRrk}(k)$ and $\hat{\beta}_{SRrk}(r, k)$ is obtained as

$$MSE\left[\tilde{\beta}_{PTSRrk}(k)\right]-MSE\left[\hat{\beta}_{SRrk}(k)\right]=D_2+d_3d'_3-d_4d'_4 \quad (4.8)$$

where $D_2 = \sigma^2[1-h_\lambda(2)]R_kGR'_k + \xi R_kH\delta\delta'H'R'_k$, $d_3 = (R_k - I)\beta + h_\lambda(2)R_kH\delta$ and $d_4 = (R_k - I)\beta + R_kH\delta$.

Now we can state the following theorem.

Theorem 4.2: The estimator $\hat{\beta}_{SRrk}(r, k)$ is superior to $\tilde{\beta}_{PTSRrk}(k)$ if and only if $d'_4(D_2 + d_3d'_3)^{-1}d_4 \leq 1$.

Proof: First we consider the matrix

$$D_2 = \sigma^2[1-h_\lambda(2)]R_kGR'_k + \xi R_kH\delta\delta'H'R'_k.$$

The matrix D_2 can be rewritten as $D_2 = R_kM_\ell R'_k$, where $M_\ell = \sigma^2[1-h_\lambda(2)]G + \xi H\delta\delta'H'$. The matrix $\sigma^2[1-h_\lambda(2)]G$ is nonnegative definite matrix since $G \geq 0$ and $0 \leq h_\lambda(2) \leq 1$, and the matrix $\xi H\delta\delta'H'$ is positive definite matrix since $H\delta\delta'H'$ is positive definite matrix and $\xi \geq 0$. Therefore the matrix M_ℓ is positive definite matrix, which leads the matrix D_2 is positive definite matrix since $R_k > 0$. Now according to lemma 4 (Appendix), the matrix $D_2 + d_3d'_3 - d_4d'_4 \geq 0$ if and only if $d'_4(D_2 + d_3d'_3)^{-1}d_4 \leq 1$. This completes the proof.

4.3. Comparison between $\hat{\beta}_{rd}(r, d)$ and $\tilde{\beta}_{PTSRrd}(d)$

The mean square error matrix difference between $\tilde{\beta}_{PTSRrd}(d)$ and $\hat{\beta}_{rd}(r, d)$ is obtained as

$$MSE\left[\hat{\beta}_{rd}(r, d)\right]-MSE\left[\tilde{\beta}_{PTSRrd}(d)\right]=H_d(D+b_1b'_1-b_2b'_2)H'_d \quad (4.9)$$

where, $b_1 = (I - H_d^{-1})\beta$ and $b_2 = (I - H_d^{-1})\beta + h_\lambda(2)H\delta$.

Theorem 4.3: When $\lambda \leq \frac{1}{2[2-h_\lambda(2)/h_\lambda(4)-h_\lambda(2)]}$, the estimator $\tilde{\beta}_{PTSRrd}(d)$ is superior to $\hat{\beta}_{rd}(r, d)$ if and only if $b'_2(D+b_1b'_1)^{-1}b_2 \leq 1$.

Proof: The mean square error difference between $\tilde{\beta}_{PTSRrd}(d)$ and $\hat{\beta}_{rd}(r, d)$ is nonnegative definite matrix if and only if $D + b_1 b'_1 - b_2 b'_2 \geq 0$. We have already proved that $D \geq 0$ and $DD^+ = I_p$. Therefore $DD^+ b_1 = b_1$ and $DD^+ b_2 = b_2$, which implies that $b_1, b_2 \in \Re(D)$. Now according to lemma 3, $D + b_1 b'_1 - b_2 b'_2 \geq 0$ if and only if $(b'_1 D^+ b_1 + 1)(b'_2 D^+ b_2 - 1) \leq (b'_1 D^+ b_2)^2$. This completes the proof.

4.4. Comparison between $\hat{\beta}_{SRrd}(r, d)$ and $\tilde{\beta}_{PTSRrd}(d)$

The mean square error matrix difference between $\tilde{\beta}_{PTSRrd}(d)$ and $\hat{\beta}_{SRrd}(r, d)$ is obtained as

$$MSE[\tilde{\beta}_{PTSRrd}(d)] - MSE[\hat{\beta}_{SRrd}(d)] = B_1 + b_3 b'_3 - b_4 b'_4 \quad (4.10)$$

where $B_1 = \sigma^2 [1 - h_\lambda(2)] H_d G H'_d + \xi H_d H \delta \delta' H' H'_d$, $b_3 = (H_d - I) \beta + h_\lambda(2) H_d H \delta$ and

$$b_4 = (H_d - I) \beta + H_d H \delta.$$

Now the following theorem can be stated.

Theorem 4.4: The estimator $\hat{\beta}_{SRrd}(r, d)$ is superior to $\tilde{\beta}_{PTSRrd}(d)$ if and only if $b'_4 (B_1 + b_3 b'_3)^{-1} b_4 \leq 1$.

Proof: The matrix B_1 can be rewritten as $B_1 = H_d M_\ell H'_d$. We have already proved that M_ℓ is positive definite matrix. Therefore the matrix B_1 is positive definite matrix since $H_d > 0$. Now according to lemma 1, the matrix $B_1 + b_3 b'_3 - b_4 b'_4 \geq 0$ if and only if $b'_4 (B_1 + b_3 b'_3)^{-1} b_4 \leq 1$. This completes the proof.

4.5. Comparison between $\tilde{\beta}_{PTSRrd}(d)$ and $\tilde{\beta}_{PTSRrk}(k)$

The mean square error matrix between $\tilde{\beta}_{PTSRrd}(d)$ and $\tilde{\beta}_{PTSRrk}(k)$ is given as

$$MSE[\tilde{\beta}_{PTSRrk}(k)] - MSE[\tilde{\beta}_{PTSRrd}(d)] = \vartheta + d_3 d'_3 - b_3 b'_3 \quad (4.11)$$

where $\vartheta = R_k M_x R'_k - H_d M_x H'_d$ and $M_x = \sigma^2 S^{-1} - \sigma^2 h_\lambda(2) G + \xi H \delta \delta' H'$.

Theorem 4.5: When maximum eigenvalues of $(H_d M_x H'_d)(R_k M_x R'_k)^{-1}$ is less than one, the estimator $\tilde{\beta}_{PTSRrd}(d)$ is superior to $\tilde{\beta}_{PTSRrk}(k)$ if and only if $b'_3(\vartheta + d_3 d'_3)^{-1} b_3 \leq 1$.

Proof: First we consider $M_x = \sigma^2 S^{-1} - \sigma^2 h_\lambda(2)G + \xi H \delta \delta' H'$. Note that

$$\sigma^2 S^{-1} - \sigma^2 h_\lambda(2)G \geq \sigma^2 (S^{-1} - G) \text{ since } 0 \leq h_\lambda(2) \leq 1. \text{ But}$$

$$S^{-1} - G = (S + R\Omega^{-1}R')^{-1} \geq 0.$$

Therefore $\sigma^2 S^{-1} - \sigma^2 h_\lambda(2)G \geq 0$. Since $\sigma^2 S^{-1} - \sigma^2 h_\lambda(2)G \geq 0$ and $\xi H \delta \delta' H' > 0$, the matrix $M_x > 0$. Therefore $H_d M_x H'_d$ and $R_k M_x R'_k$ are positive definite matrices since $H_d > 0$ and $R_k > 0$ respectively. So that, according to lemma 5, $\vartheta = R_k M_x R'_k - H_d M_x H'_d > 0$ if and only if maximum eigenvalues of $(H_d M_x H'_d)(R_k M_x R'_k)^{-1}$ is less than one. Applying lemma 4, we can easily prove that the estimator $\tilde{\beta}_{PTSRrd}(d)$ is superior to $\tilde{\beta}_{PTSRrk}(k)$ if and only if $b'_3(\vartheta + d_3 d'_3)^{-1} b_3 \leq 1$. This completes the proof.

Note

In the above theorem, when $\alpha = 0$ and $\alpha = 1$, we can obtain the superiority conditions of the Stochastic Restricted r-k class estimator over the Stochastic Restricted r-d class estimator, and the superiority conditions of the r-k class estimator over the r-d class estimator respectively.

5. ILLUSTRATION OF THEORETICAL RESULTS

5.1. Numerical Example

To illustrate our theoretical results in this section we consider the data set which was discussed in Gruber (1998) and later considered by Akdeniz and Erol (2003), Li and Yang (2011), Alheety and Kibria (2013), Toker and Kaçiranlar (2013), Wu (2013) and, Wu and Yang (2013). Table 5.1 gives Total National Research and Development Expenditures-as a Percent of Gross National Product by Country:

1972-1986. It represents the relationship between the dependent variable y the percentage spent by the United States and the dependent variables x_1, x_2, x_3 and x_4 . The variable x_1 represents the percent spent by France, x_2 that spent by West Germany, x_3 that spent by Japan, and x_4 that spent by the former Soviet Union.

Table 5.1: Total National Research and Development Expenditures-as a Percent of Gross National Product by Country: 1972-1986.

Year	y	x_1	x_2	x_3	x_4
1972	2.3	1.9	2.2	1.9	3.7
1975	2.2	1.8	2.2	2.0	3.8
1979	2.2	1.8	2.4	2.1	3.6
1980	2.3	1.8	2.4	2.2	3.8
1981	2.4	2.0	2.5	2.3	3.8
1982	2.5	2.1	2.6	2.4	3.7
1983	2.6	2.1	2.6	2.6	3.8
1984	2.6	2.2	2.6	2.6	4.0
1985	2.7	2.3	2.8	2.8	3.7
1986	2.7	2.3	2.7	2.8	3.8

We assemble the data in the matrix form as follows:

$$X = \begin{pmatrix} 1.9 & 2.2 & 1.9 & 3.7 \\ 1.8 & 2.2 & 2.0 & 3.8 \\ 1.8 & 2.4 & 2.1 & 3.6 \\ 1.8 & 2.4 & 2.2 & 3.8 \\ 2.0 & 2.5 & 2.3 & 3.8 \\ 2.1 & 2.6 & 2.4 & 3.7 \\ 2.1 & 2.6 & 2.6 & 3.8 \\ 2.2 & 2.6 & 2.6 & 4.0 \\ 2.3 & 2.8 & 2.8 & 3.7 \\ 2.3 & 2.7 & 2.8 & 3.8 \end{pmatrix}, y = \begin{pmatrix} 2.3 \\ 2.2 \\ 2.2 \\ 2.3 \\ 2.4 \\ 2.5 \\ 2.6 \\ 2.6 \\ 2.7 \\ 2.7 \end{pmatrix}$$

The four column of the 10×4 matrix X comprise the data on x_1, x_2, x_3 and x_4 respectively, and y is the response variable. Note that the eigen values of S are

$\lambda_1 = 302.9626, \lambda_2 = 0.7283, \lambda_3 = 0.0447$ and $\lambda_4 = 0.0345$ and the condition number of X is approximately 8781.53. This implies the existence of multicollinearity in the data set. The OLSE is given by

$$\hat{\beta}_{OLSE} = S^{-1}X'y = (0.6455, 0.0896, 0.1436, 0.1526)'$$

with $MSE(\hat{\beta}_{OLSE}, \beta) = 0.0808$ and $\hat{\sigma}^2 = 0.0015$.

Consider the following stochastic restrictions

$$r = R\beta + \delta + \nu \quad \text{where} \quad R = (1, -2, -2, -2)', \quad r = 1 \quad \text{and}$$

$\nu \sim N(0, \hat{\sigma}^2 = 0.0015)$. We select the significance level $\alpha = 0.05$. Following Chang and Yang (2012) we choose the number of the principal components $r = 3$. Fig.1, Fig.2 and Fig.3 are obtained by using the equations (2.21), (2.22), (2.25), (2.26), (3.8) and (3.12) for different shrinkage parameters d and k values selected from the interval $(0, 1)$.

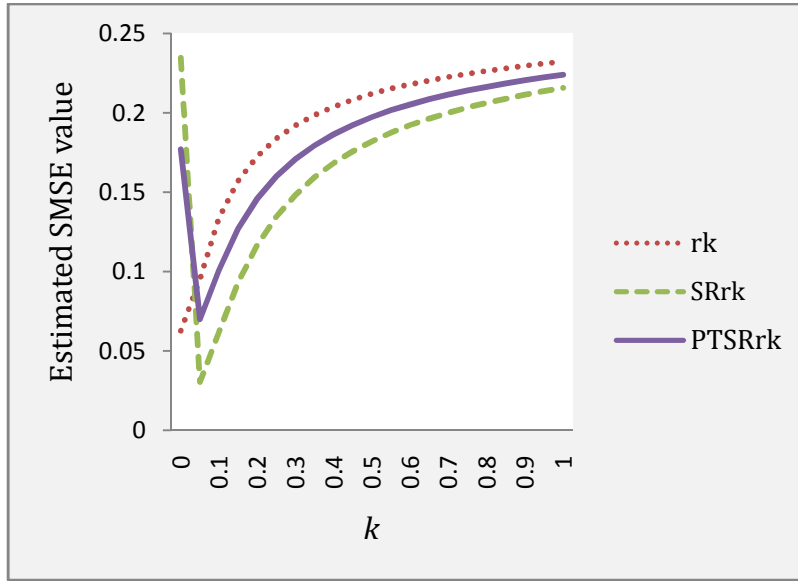


Fig.1: Estimated SMSE value of rk, SRrk and PTSRrk.

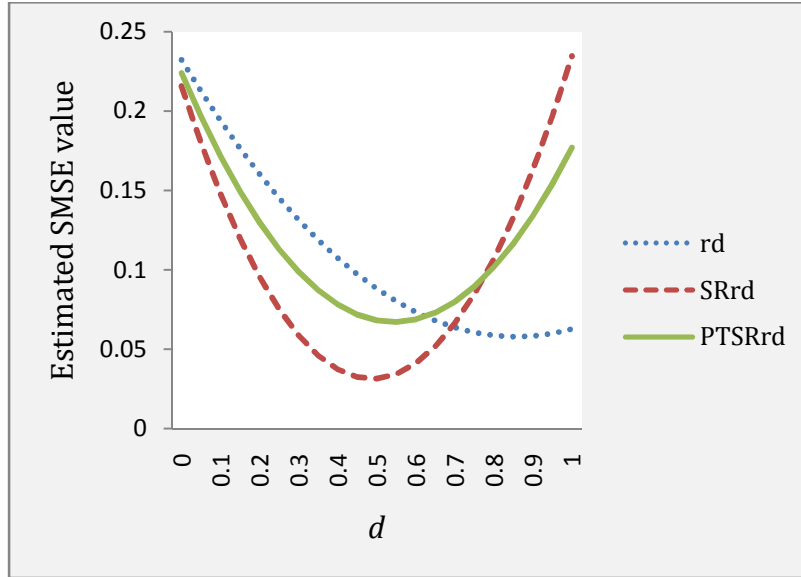


Fig.2: Estimated SMSE value of rd, SRrd and PTSRrd.

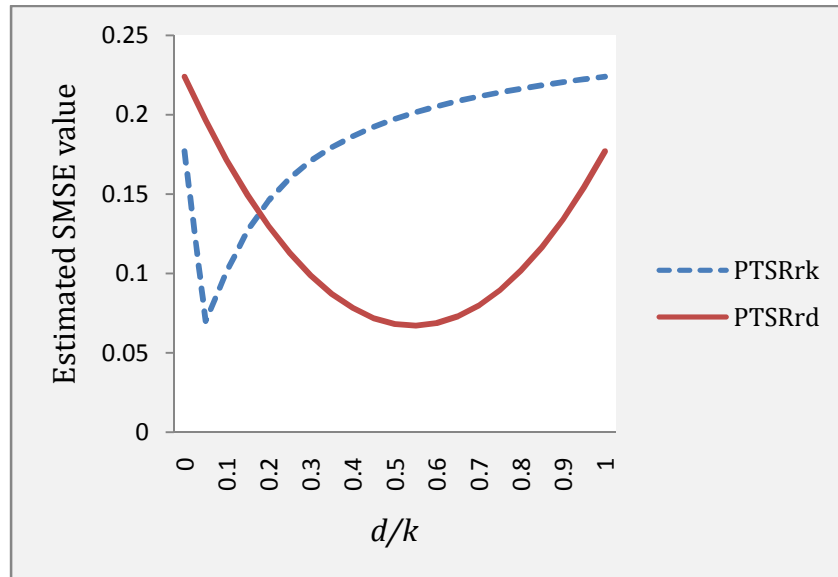


Fig.3: Estimated SMSE value of PTSRrk and PTSRrd

From Fig.1, when k is small the PTSRrk has the smallest SMSE than stochastic restricted r - k class estimator. When k is large, the PTSRrk has the smallest SMSE than r - k class estimator. From Fig.2, when d is small the PTSRrd has the smallest SMSE than r - d class estimator. When d is large, the PTSRrd has the smallest SMSE

than stochastic restricted r-d class estimator. From Fig. 3, when d/k is small, the PTSRrk has the smallest SMSE than PTSRrd, the situation is reversed when d/k is large. From Fig.1, Fig.2 and Fig.3, we can say that no estimator is always superior to other estimators.

5.2. Simulation Study

To illustrate the statistical behavior of our proposed estimators, we perform a Monte Carlo Simulation study by considering different levels of multicollinearity. Following McDonald and Galarneau (1975) we generate explanatory variables as follows:

$$x_{ij} = (1 - \rho^2)^{1/2} z_{ij} + \rho z_{i,p+1}, \quad i = 1, 2, \dots, n, \quad j = 1, 2, \dots, p,$$

where z_{ij} is an independent standard normal pseudo random number, and ρ is specified so that the theoretical correlation between any two explanatory variables is given by ρ^2 . A dependent variable is generated by using the equation.

$$y_i = \beta_1 x_{i1} + \beta_2 x_{i2} + \beta_3 x_{i3} + \beta_4 x_{i4} + \varepsilon_i, \quad i = 1, 2, \dots, n,$$

where ε_i is a normal pseudo random number with mean zero and variance σ_i^2 . Newhouse and Oman (1971) have noted that if the MSE is a function of σ^2 and β , and if the explanatory variables are fixed, then subject to the constraint $\beta' \beta = 1$, the MSE is minimized when β is the normalized eigenvector corresponding to the largest eigenvalue of the $X'X$ matrix. In this study we choose the normalized eigenvector corresponding to the largest eigenvalue of $X'X$ as the coefficient vector β , $n = 50$, $p = 4$ and $\sigma^2 = 1$. Two different sets of correlations are considered by selecting the values as $\rho = 0.8$ and 0.9 , and the significance level is taken as $\alpha = 0.05$. Fig.4, Fig.5, Fig.6, Fig.7, Fig.8 and Fig.9 are obtained by using the equations (2.21), (2.22), (2.25), (2.26), (3.8) and (3.12) for different shrinkage parameters d and k values selected from the interval $(0, 1)$.

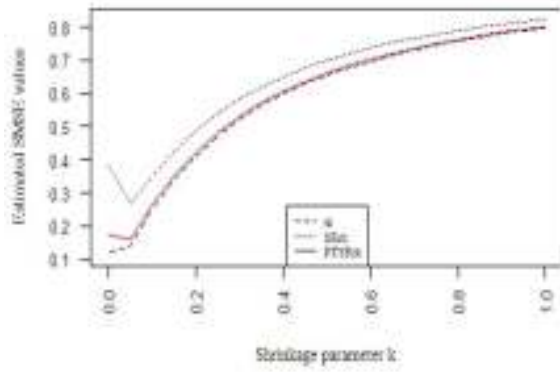


Fig.4: Estimated SMSE value of rk, SRrk and PTRrk for $\rho = 0.8$.

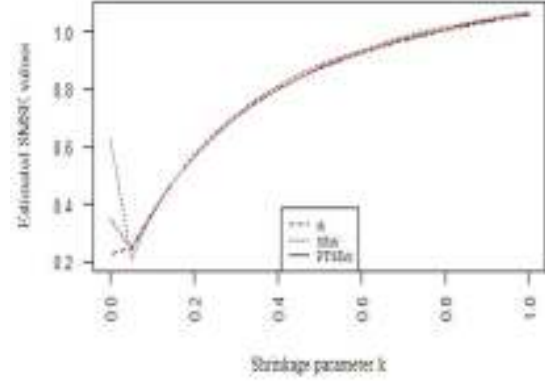


Fig.5: Estimated SMSE value of rk, SRrk and PTRrk for $\rho = 0.9$.

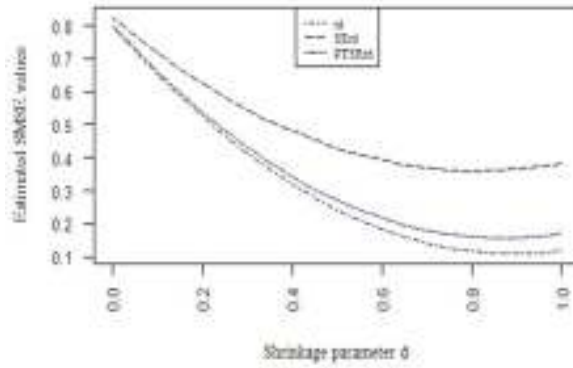


Fig.6: Estimated SMSE value of rd, SRrd and PTRrd for $\rho = 0.8$.

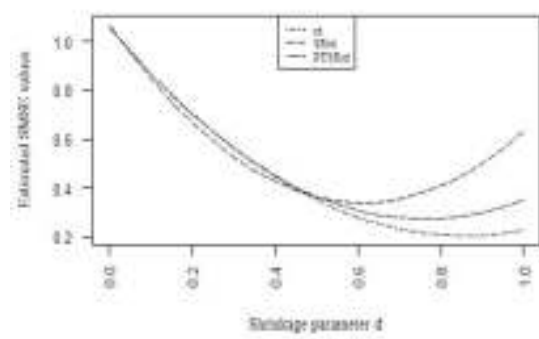


Fig.7: Estimated SMSE value of rd, SRrd and PTRrd for $\rho = 0.9$.

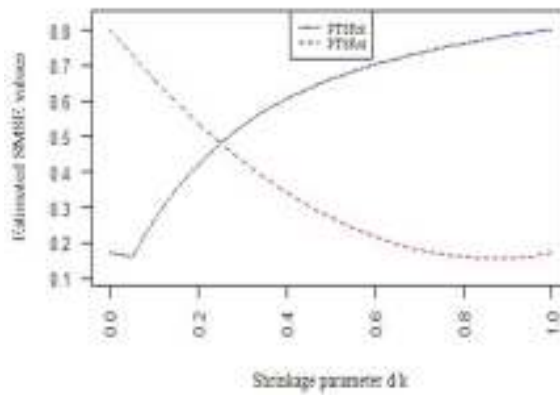


Fig.8: Estimated SMSE value of PTRrk and PTRrd for $\rho = 0.8$.

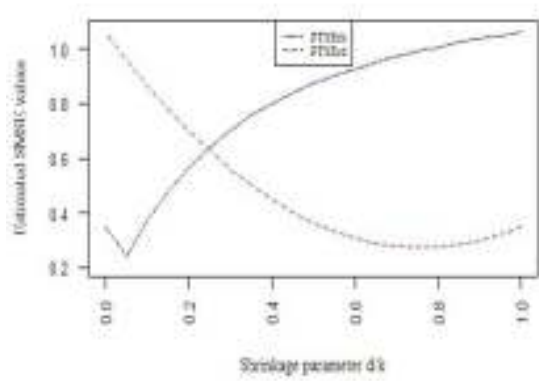


Fig.9: Estimated SMSE value of PTRrk and PTRrd for $\rho = 0.9$.

Based on Fig.4, r - k class estimator and PTSRrk have smallest SMSE than stochastic restricted r - k class estimator. According to Fig.5, there is no big difference in the SMSE except $k=0$. From Fig.6 and Fig.7, we can say that, when d is large the estimator r - d class estimator has the smallest SMSE than PTSRrd. From Fig.8 and Fig.9 we can notice that when d or k is small, the PTSRrk has the smallest SMSE than PTSRrd, the situation is reversed when d or k is large.

Remark:

Arumairajan and Wijekoon (2014) proposed the Generalized Preliminary Test Stochastic Restricted Estimator (GPTSRE) as follows:

$$\tilde{\beta}_{GPTSRE} = A_{(j)} \hat{\beta}_m I_{[0, F_{m,n-p}(\alpha)]}(F) + A_{(j)} \hat{\beta} I_{[F_{m,n-p}(\alpha), \infty)}(F) = A_{(j)} \hat{\beta}_{OSPE}$$

where $A_{(j)}$ is a positive definite matrix.

Note that when $A_{(j)} = R_k$, $\tilde{\beta}_{GPTSRE}$ gives the Preliminary Test stochastic restricted r - k class estimator (PTSRrk), and when $A_{(j)} = H_d$, it gives the Preliminary Test stochastic restricted r - d class estimator (PTSRrd).

6. CONCLUSION

In this study, we propose a new Preliminary Test stochastic restricted r - k class estimator which is defined by combining the r - k class estimator and stochastic restricted r - k class estimator and a new Preliminary Test stochastic restricted r - d class estimator which is defined by combining the r - d class estimator and stochastic restricted r - d class estimator. Further the proposed estimators are compared with some biased estimators in the mean square error matrix sense. Also Preliminary Test stochastic restricted r - d class estimator is compared with Preliminary Test stochastic restricted r - k class estimator. Finally a numerical example and a Monte Carlo simulation study are done to illustrate the theoretical findings of the proposed estimators. Based on the theoretical findings and numerical illustration, we can conclude that no estimator is always superior to other estimators.

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REFERENCE

1. Akdeniz. F and Kaçiranlar. S. On the almost unbiased generalized Liu estimator and unbiased estimation of the bias and MSE. *Communications in Statistics - Theory and Methods*, 1995; 24(07):1789 -1797.
2. Akdeniz F, Erol H. Mean squared error matrix comparisons of some biased estimators in linear regression. *Communication in Statistics - Theory and Methods*. 2003; 32(12): 2389–2413.
3. Alheety. M. I and Kibria. B. M. G. Modified Liu-Type Estimator Based on $(r-k)$ Class Estimator. *Communications in Statistics-Theory and Methods*. 2013;42(02): 304-319.
4. Arumairajan. S and Wijekoon. P. Improvement of the Preliminary Test Estimator When Stochastic Restrictions are Available in Linear Regression Model. *Open Journal of Statistics*. 2013;3(04): 283-292.
5. Arumairajan. S and Wijekoon. P. Generalized preliminary test stochastic restricted estimator in the linear regression model. *Communication in Statistics - Theory and Methods* (Accepted for publication). 2014.
6. Baksalary J.K and Kala. R. Partial orderings between matrices one of which is of rank one. *Bulletin of the Polish Academy of Science, Mathematics*. 1983;31: 5-7.
7. Baksalary. J.K and Trenkler. G. Nonnegative and Positive Definiteness of Matrices Modified by Two Matrices of Rank One. *Linear Algebra and its Applications*.1991;151: 169-184.
8. Bancroft. A. On biases in estimation due to use of preliminary tests of significance. *Annals of Mathematical Statistics*. 1944;15(02):190-204.

9. Baye. R. M and Parker. F. D. Combining ridge and principal component regression: a money demand illustration, *Communications in Statistics-Theory and Methods*. 1984;13(02): 197-205.
10. Chang. X and Yang. H. Combining two-parameter and principal component regression estimators, *Statistical Papers*. 2012; 53(03): 549-562.
11. Gruber. M.H.J. Improving efficiency by shrinkage: the James-Stein and ridge regression estimators, *Dekker, Inc., New York*. 1998.
12. Hoerl. A.E and Kennard. R.W. Ridge Regression: Biased Estimation for Nonorthogonal Problems, *Technometrics*.1970;12(01):55-67.
13. Hubert M.H and Wijekoon. P. Improvement of the Liu Estimator in Linear Regression Model, *Statistical Papers*.2006;47(03):471-479.
14. Judge. G and Bock. E. *The Statistical Implications of Pre-Test and Stein-Rule Estimators in Econometrics*, North Holland, New York. 1978.
15. Kaçiranlar. S and Sakallıoğlu. S. Combining the Liu estimator and the principal component regression estimator, *Communications in Statistics-Theory and Methods*.2001; 30(12): 2699-2705.
16. Li. Y and Yang. H. A new stochastic mixed ridge estimator in linear regression, *Statistical Papers*: 2010;51(02): 315-323.
17. Liu. K. A New Class of Biased Estimate in Linear Regression, *Communications in Statistics-Theory and Methods*.1993;22(02): 393-402.
18. Massy .F. Principal components regression in exploratory statistical research. *Journal of the American Statistical Association*. 1965;60(309): 234-266.
19. McDonald. C and Galarneau. A. A Monte Carlo Evaluation of some Ridge-Type Estimators. *Journal of American Statistical Association*.1975: 70(350):407-416.
20. Newhouse JP and Oman SD. An evaluation of ridge estimators. Rand Report, 1971, No. R-716-Pr: 1-28.
21. Singh. B., Chaubey. Y. P., Dwivedi. T. D. An almost unbiased ridge estimator. *Sankhya: The Indian Journal of Statistics B*.1986;48:342-346.

22. Theil. H and Goldberger. A.S. On pure and Mixed estimation in Economics, *International Economic review*. 1961;2(01): 65-77.
23. Toker. S and Kaçiranlar. S. On the performance of two parameter ridge estimator under the mean square error criterion. *Applied Mathematics and Computation*. 2013;219(09): 4718-4728.
24. Trenkler. G. Mean Square error matrix comparisons of estimators in linear regression. *Communication in Statistics A*. 1985;14: 2495-2509.
25. Trenkler. G and Toutenburg. H (1990). Mean square error matrix comparisons between biased estimators-an overview of recent results, *Statistical papers*.1990: 31(01);165-179.
26. Wang. S. G et al.. Matrix inequalities, 2nd Edition, *Chinese Science Press, Beijing*. 2006.
27. Wijekoon. P. Mixed estimation and Preliminary test estimation in the Linear Regression model. Ph.D. Thesis, *University of Dortmund*. 1990.
28. Wu J. B. On the performance of principal component Liu-type estimator under the mean square error criterion. *Journal of Applied Mathematics*. 2013; Article ID 858794, 7 pages.
29. Wu. J.B and Yang. H. Two Stochastic Restricted Principal Components Regression Estimator in Linear Regression. *Communications in Statistics-Theory and Methods*.2013;42(20): 3793-3804.
30. Wu. J. B. On the Stochastic Restricted r - k Class Estimator and Stochastic Restricted r - d Class Estimator in Linear Regression Model. *Journal of Applied Mathematics*.2014; Article ID 173836, 6 pages.
31. Wu. J.B and Yang. H. On the stochastic restricted almost unbiased estimators in linear regression model. *Communication in Statistics- Simulation and Computation*.2014;43(02): 428-440.
32. Xu. W. J. and Yang. H. On the restricted r - k class estimator and the restricted r - d class estimator in linear regression, *Journal of Statistical Computation and Simulation*.2011;81(06): 679-691.

APPENDIX

Lemma 1: (Baksalary and Kala, 1983)

Suppose B is a symmetric real $(n \times n)$ matrix, a is an $(n \times 1)$ real vector and γ is a positive real number. Then the following two properties are equivalent

- a) $\gamma B - aa'$ is nonnegative definite (n.n.d)
- b) B is n.n.d, $a \in \mathfrak{R}(B)$ and $a'B^-a \leq \gamma$.

Lemma 2: (Trenkler, 1985)

Let A be a symmetric $(n \times n)$ matrix, and let a, a_1 , and a_2 be $(n \times 1)$ vectors. Suppose that

- a) $a \in \mathfrak{R}(A)$, and the real numbers ϕ and ψ satisfy $\phi \neq 0$ and $\phi + \psi a'A^+a \neq 0$.

Then we have the identity

$$[\phi A + \psi aa']^+ = \frac{1}{\phi} \left[A^+ - \frac{\psi}{\phi + \psi a'A^+a} A^+ aa' A^+ \right]$$

- b) $a_j \in \mathfrak{R}(A)$, $j = 1, 2$, and the real number ρ satisfies $1 + \rho a_1'A^+a_1 \neq 0$. Then we have $a_2 \in \mathfrak{R}(A + \rho a_1 a_1')$.

Lemma 3: (Baksalary and Trenkler, 1991)

Let C be a nonnegative definite matrix and c_1, c_2 be linearly independent vectors. Furthermore for some generalized inverse C^- of C , let $f_{ij} = c_i' C^- c_j$; $i = 1, 2, j = 1, 2$ and let

$$s = \frac{c_2'(I - CC^-)'(I - CC^-)c_2}{c_1'(I - CC^-)(I - CC^-)c_1}$$

where $c_1 \in \mathfrak{R}(C)$ and $\mathfrak{R}(\cdot)$ denote the column space of the corresponding matrix.

Then we have $C + c_1 c_1' - c_2 c_2' \geq 0$ if and only if

- a) $c_1 \in \mathfrak{R}(C), c_2 \in \mathfrak{R}(C)$ and $(f_{11} + 1)(f_{22} - 1) \leq f_{12}^2$ or
- b) $c_1 \notin \mathfrak{R}(C), c_2 \in \mathfrak{R}(C, c_1)$ and $(c_2 - s c_1)' C^- (c_2 - s c_1) \leq 1 - s^2$

and all expressions in (a) and (b) are independent of the choice of C^- .

Lemma 4: (Trenkler and Toutenburg, 1990)

Let $\hat{\beta}_1$ and $\hat{\beta}_2$ be two linear estimator of β . Suppose that $D = D(\hat{\beta}_1) - D(\hat{\beta}_2)$ is positive definite then $\Delta = MSE(\hat{\beta}_1) - MSE(\hat{\beta}_2)$ is nonnegative definite if and only if $b_2'(D + b_1 b_1')^{-1} b_2 \leq 1$, where b_j denotes the bias vector of $\hat{\beta}_j$, $j = 1, 2$.

Lemma 5: (Wang et al., 2006)

Let $n \times n$ matrices $M > 0$, $N > 0$ (or $N \geq 0$), then $M > N$ if and only if $\lambda_1(NM^{-1}) < 1$. where $\lambda_1(NM^{-1})$ is the largest eigenvalue of the matrix NM^{-1} .