

On a local equality of distributions and relation of jumps of distributions with

Fourier series

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Abstract. Distribution theory has an important role in applied mathematics. Firstly, in the introduction part of this paper we will give some general notations, definitions and results in distribution theory, as analytic representation of distribution, distributional jump behavior, distributional symmetric jump behavior, tempered distributions, formulas for the jump of distributions in terms of Fourier series and global equality in distributional sense. Then in final part we will state two results, the first one has to do on local equality of two distributions and the second one states if a 2π -periodic distribution has jump behavior at point, then exists the relation of jumps of distributions with Fourier series in a subset of the upper half-plane.

Keywords: Distributions, distributional jump behavior, point value, support, Schwartz space

Introduction

Firstly, let us give some introductory concepts. With P^n we denote the subset of $\mathbb{R}^n$, which elements have nonnegative integers coordinates. For a function $f, f: \Omega \rightarrow \mathbb{C}, \Omega \subseteq \mathbb{R}^n, k = (k_1, k_2, \dots, k_n), k_j \in \mathbb{N} \cup \{0\}, x \in \Omega$, with $f^{(k)}(x)$ we denote the differential operator

$$f^{(k)}(x) = \frac{\partial^{|k|}}{\partial_{x_1}^{k_1} \dots \partial_{x_n}^{k_n}} f(x) = \partial^k f(x), \quad |k| = k_1 + k_2 + \dots + k_n.$$

With $C^\infty(\mathbb{R}^n)$ is denoted the space of all complex valued infinitely differentiable functions on $\mathbb{R}^n$ and $C_0^\infty(\mathbb{R}^n)$ denotes the subspace of $C^\infty(\mathbb{R}^n)$ that consists of those functions of $C^\infty(\mathbb{R}^n)$ which have compact support.

Definition 1. The support of f is the closure of the set $x \in \Omega$, of points for which f is different from zero ($f(x) \neq 0$), and is denoted by $\text{supp } f$.

With D we denote the space of $C_0^\infty(\mathbb{R}^n)$ functions, called the set of test functions in which convergence is defined in the following way : a sequence $\{\phi_\nu\}$ of functions $\phi_\nu \in D$ converges to $\phi \in D$ in D as $\nu \rightarrow \nu_0$ if and only if there is a compact set $K \subset \mathbb{R}^n$ such that $\text{supp}(\phi_\nu) \subseteq K$ for each ν , $\text{supp}(\phi) \subseteq K$ and for every n -tuple k of nonnegative integers the sequence $\{f^{(k)}\phi_\nu(t)\}$ converges to $f^{(k)}\phi(t)$ uniformly on K as $\nu \rightarrow \nu_0$.

Distributions (or generalized functions) are objects that generalize the classical notion of functions in mathematical analysis. Distributions make it possible to differentiate functions whose derivatives do not exist in the classical sense.

Definition 2. A distribution T is continuous linear functional on D . Instead of writing $T(\phi)$, it is conventional to write $\langle T, \phi \rangle$ for the value of T acting on a test function ϕ . The space of all distributions is denoted by D' .

Schwartz space is the vector space

$$S(\mathbb{R}^n) = \left\{ \phi : \mathbb{R}^n \rightarrow \mathbb{C}, \phi \in C^\infty, \sup_{x \in \mathbb{R}^n} |x^\alpha \partial^\beta \phi(x)| < \infty, \alpha, \beta \in \mathbb{N}^n \right\}.$$

S' is the space of all continuous linear functionals on S , called the space of tempered distributions.

Proposition 1. A sequence $\{\phi_\nu\}$ of functions $\phi_\nu \in S$ converges to $\phi \in S$ in S as $\nu \rightarrow \nu_0$ if and only if

$$\lim_{\nu \rightarrow \nu_0} \sup_{x \in \mathbb{R}^n} |x^\alpha f^{(k)}[\phi_\nu(x) - \phi(x)]| = 0.$$

Let ϕ be an element of one of the above function spaces D or S , and f be a function for which

$$\langle T_f, \phi \rangle = \int_{\mathbb{R}^n} f(t)\phi(t)dt, \quad \phi \in D(\text{or } S)$$

exists and is finite. Then T_f is regular distribution on D (or S) generated by f .

Definition 3. Distributions f and g in $D'(\Omega)$ are equal (in global sense) in open set $\Omega_1 \subset \Omega$ if and only if their restrictions are equal in Ω_1 . Distributions f and g are equal (in local sense) in any neighborhood of a point $x \in \Omega$ if and only if are equal in any open neighborhood $U \subset \Omega$ of a point.

55 **Note 1.** Fourier transform $\hat{f}$ (see [4]) is continuous linear function from $S(\mathbb{R}^n)$ to $S(\mathbb{R}^n)$.

56 The (complex) Fourier series for the distribution T is the series

57 $T \sim \sum_{n=-\infty}^{\infty} c_n e^{inx}$ with $c_n \stackrel{\text{def}}{=} \frac{1}{2\pi} \langle T, e^{-inx} \rangle, \forall n \in \mathbb{Z}$.

58 **Proposition 2.** A trigonometric series

59
$$\sum_{n=-\infty}^{\infty} c_n e^{inx}$$

60 converges in D' , that is, $\lim_{k \rightarrow \infty} \sum_{n=-k}^k c_n e^{inx}$ exists as a distribution, if and only if there are constants

61 B and β such that

62 $|c_n| \leq B |n|^\beta, \forall n \neq 0.$

63 For every distribution T the Fourier series converges in S' to T .

64 **Definition 4.** The value of distribution f at point x_0 is defined as the limit

65
$$f(x_0) = \lim_{h \rightarrow 0} f(x_0 + hx),$$

66 if the limit exists in D' , that is, if

67
$$\lim_{h \rightarrow 0} \langle f(x_0 + hx), \phi(x) \rangle = f(x_0) \int_{-\infty}^{\infty} \phi(x) dx$$

68 for each $\phi \in D(\mathbb{R})$.

69 The Heaviside function is defined by

70
$$H(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}.$$

71 A distribution $f \in D'(\mathbb{R})$ is said to have a distributional jump behavior (or jump behavior) at

72 $x = x_0 \in \mathbb{R}$ if it satisfies the distributional asymptotic relation

73
$$f(x_0 + hx) = c_- H(-x) + c_+ H(x) + o(1),$$

74 as $h \rightarrow 0^+$ in D' , $c_{\pm}$ are constants and H is the Heaviside function.

75 The jump of f at $x = x_0$ is defined as the number $[f]_{x=x_0} = c_+ - c_-$.

76 A distribution $f \in D'(\square)$ is said to have a distributional symmetric jump behavior at
 77 $x = x_0 \in \square$ if the jump distribution $\psi_{x_0}(x) = f(x_0 + x) - f(x_0 - x)$ has a jump behavior at $x = 0$.

78 In such a case, we define the jump of f at $x = x_0$ as $[f]_{x=x_0} = [\psi_{x_0}]_{x=0} / 2$.

79 **Note 2.** Jump behavior implies symmetric jump behavior, but the converse is not true as shown
 80 by *Dirac delta function* (see [4]).

81 Let us define the subset of upper half-plane $\Delta_\theta^+(x_0)$ as the set of z such that
 82 $\theta \leq \arg(z - x_0) \leq \pi - \theta$, where $0 < \theta \leq \pi/2$. Similarly we define the lower half-plane $\Delta_\theta^-(x_0)$.

83 We may see f as hyperfunction, that is $f(x) = G(x + i0) - G(x - i0)$, where G is analytic for
 84 $\text{Im} z \neq 0$ or in the sense of distributions it means $f(x) = \lim_{y \rightarrow 0^+} (G(x + iy) - G(x - iy))$, then we say
 85 G is analytic representation of f .

86 We say that $U(z)$, harmonic on $\text{Im} z > 0$, is a harmonic representation of $f \in D'(\square)$ if

87
$$\lim_{y \rightarrow 0^+} U(x + iy) = f(x)$$

88 in $D'(\square)$.

89 **Theorem 1.** If K is compact subset of Ω , exists $\psi \in C_0^\infty(\Omega)$ with the property $0 \leq \psi(x) \leq 1$ and
 90 $\psi(x) = 1$ in any neighborhood of K .

91

92

93 Main results

94 **Theorem 2.** If $f \in D'(\Omega)$ is equal to zero in neighborhood of each point in Ω , then $f = 0$ in
 95 Ω .

96 **Proof.** Let $\phi \in C_0^\infty(\Omega)$. We take $K = \text{supp } \phi$. Each point of K has neighborhood in which $f = 0$.
 97 Since K is compact set, exists finite number of open sets Ω_i , $i = 1, 2, \dots, k$, which cover K in
 98 which $f = 0$. Now we will show that exists compact sets K_i , $i = 1, 2, \dots, k$, such that

99

$$K_i \subset \Omega_i \text{ and } K \subset \bigcup_{i=1}^k K_i .$$

100

$C(\bigcup_{i=2}^k \Omega_i)$ is closed (as complement of open) and bounded, so it is compact. The set

101

$K \cap C(\bigcup_{i=2}^k \Omega_i)$ is a compact (as intersection of finite number of compact sets) and is subset of Ω_1 ,

102

since $K \subset \bigcup_{i=1}^k \Omega_i \Rightarrow C\left(\bigcup_{i=1}^k \Omega_i\right) \subset C(K)$ we obtain

103

$$\begin{aligned} K \cap C\left(\bigcup_{i=2}^k \Omega_i\right) &= K \cap C\left(\bigcup_{i=1}^k \Omega_i \setminus \Omega_1\right) = K \cap C\left(\bigcup_{i=1}^k \Omega_i \cap C(\Omega_1)\right) = \\ &= \left(K \cap \left(\bigcap_{i=1}^k C(\Omega_i)\right)\right) \cup (K \cap \Omega_1) \subset (K \cap C(K)) \cup (K \cap \Omega_1) = \emptyset \cup (K \cap \Omega_1) \subset \Omega_1. \end{aligned}$$

104

Exists bounded open set V_1 such that

105

$$\bar{V}_1 \subset \Omega \text{ and } K \cap C\left(\bigcup_{i=2}^k \Omega_i\right) \subset V_1.$$

106

The sets $V_1, \Omega_2, \dots, \Omega_k$ cover K so exists bounded open set V_2 such that $\bar{V}_2 \subset \Omega$ and

107

$$K \cap C\left(V_1 \bigcup_{i=3}^k \Omega_i\right) \subset V_2.$$

108

In same way we construct sets $V_3, \dots, V_k$. If put that $K_i = \bar{V}_i$, $i = 1, \dots, k$, obtain compact sets for

109

which is valid

110

$$K_i \subset \Omega_i \text{ and } K \subset \bigcup_{i=1}^k K_i .$$

111

From theorem 1 , exists a function $\psi_i \in C_0^\infty(\Omega)$ such that $0 \leq \psi_i \leq 1$ and $\psi_i = 1$ in

112

neighborhood of ψ_i . We put that $\phi_1 = \psi_1$; $\phi_i = \psi_i(1 - \psi_1) \dots (1 - \psi_{i-1})$; $i = 2, \dots, k$. While

113

$\psi_1 + \psi_2(1 - \psi_1) = 1 - (1 - \psi_2)(1 - \psi_1) \dots$, is obtained

114

$$\sum_{i=1}^k \phi_i = 1 - (1 - \psi_1)(1 - \psi_2) \dots (1 - \psi_k) ,$$

115 respectively $\phi_i \geq 0$; $\sum_{i=1}^k \phi_i \leq 1$ and $\sum_{i=1}^k \phi_i = 1$ so exists $\phi \in D(\Omega)$

116
$$\bar{\phi} = \sum_{i=1}^k \bar{\phi} \phi_i, \quad \bar{\phi} \text{ is conjugate function of } \phi.$$

117 Next, from the conditions stated in the theorem we have

118
$$(f, \phi) = \sum_{i=1}^k (f, \phi_i \phi) = \sum_{i=1}^k (0, \phi_i \phi) = (0, \phi)$$

119 in the sense of distribution or $f = 0$ in Ω .

120

121 **Theorem 3.** If f is a 2π – periodic distribution and has a jump behavior at $x = x_0$, with jump
 122 $[f]_{x=x_0}$, then for each positive integer k we have that for $0 < \theta \leq \pi/2$,

123
$$\lim_{z \rightarrow x_0, z \in \Delta_{\theta}^+(x_0)} (z - x_0)^k \sum_{n=0}^{\infty} n^k c_n e^{inz} = -\frac{(k-1)!}{2\pi(-i)^{k+1}} [f]_{x=x_0}.$$

124

Proof. Firstly we will show that if the condition

125
$$\lim_{z \rightarrow z_0, z \in \Delta_{\theta}^{\pm}(x_0)} (z - x_0)^k G^{(k)}(z) = (-1)^k \frac{(k-1)!}{2\pi i} [f]_{x=x_0}$$

126

127 holds for one analytic representation, then it holds for any analytic representation of f . From
 128 *edge of the wedge theorem*, any two such analytic representation differ by an entire function,
 but for entire function the relation

129
$$\lim_{z \rightarrow z_0, z \in \Delta_{\theta}^{\pm}(x_0)} (z - x_0)^k G^{(k)}(z) = (-1)^k \frac{(k-1)!}{2\pi i} [f]_{x=x_0}$$

130 gives 0. We prove that we may assume that $f \in S'$.

131 Indeed we can decompose $f = f_1 + f_2$ where f_2 is zero in a neighborhood of x_0 and $f_1 \in S'$.
 132 Let G_1 and G_2 be analytic representations of f_1 and f_2 , respectively, then G_2 can be
 133 continued across a neighborhood of x_0 (once again edge of the wedge theorem), then
 134 $G_2(z) = G_2(x_0) + O(|z - x_0|) = O(1)$

135 as $z \rightarrow x_0$. Additionally, f_1 has the same jump behavior as f . Hence we may assume that
 136 $f \in S^{\sim}$.

137 Let consider the analytic representation

$$138 \quad G(z) = \begin{cases} \frac{1}{2\pi} \langle \hat{f}_+(t), e^{izt} \rangle, & \text{Im } z > 0 \\ -\frac{1}{2\pi} \langle \hat{f}_-(t), e^{izt} \rangle, & \text{Im } z < 0 \end{cases}$$

139 where $f = \hat{f}_- + \hat{f}_+$ is decomposition with $\text{supp } \hat{f}_- \subseteq (-\infty, 0]$ and $\hat{f}_+ \subseteq [0, \infty)$.

140 For the number z on a compact subset of $\Delta_{\theta}^{\pm}(x_0)$, when we use the relation

$$141 \quad x^k e^{i\lambda x_0 x} \hat{f}_{\pm}(\lambda x) = (\pm 1)^{k-1} \frac{1}{i\lambda} [f]_{x=x_0} x_{\pm}^{k-1} + o\left(\frac{1}{\lambda}\right)$$

142 as $\lambda \rightarrow \infty$, we obtain

$$\begin{aligned} G^{(k)}(x_0 + \frac{z}{\lambda}) &= \pm \frac{i^k}{2\pi} \lambda^{k+1} \langle t^k e^{i\lambda x_0 t} \hat{f}_{\pm}(\lambda t), e^{izt} \rangle \\ 143 \quad &= \pm \frac{(\pm i)^k}{2\pi} [f]_{x=x_0} \lambda^k \int_0^{\infty} t^{k-1} e^{\pm izt} dt + o(\lambda^k) \\ &= (-1)^k \frac{(k-1)!}{2\pi i} [f]_{x=x_0} \left(\frac{\lambda}{z}\right)^k + o(\lambda^k) \end{aligned}$$

144 as $\lambda \rightarrow \infty$.

145 Next, we substitute $z = \lambda z - \lambda x_0$, obtain

$$\begin{aligned} 146 \quad G^{(k)}(x_0 + \frac{\lambda z - \lambda x_0}{\lambda}) &= (-1)^k \frac{(k-1)!}{2\pi i} [f]_{x=x_0} \left(\frac{\lambda}{\lambda(z - x_0)}\right)^k + o(\lambda^k) \\ (z - x_0)^k G^{(k)}(z) &= (-1)^k \frac{(k-1)!}{2\pi i} [f]_{x=x_0} + o(\lambda^k) \end{aligned}$$

147 equals to

$$148 \quad \lim_{z \rightarrow z_0, z \in \Delta_{\theta}^{\pm}(x_0)} (z - x_0)^k G^{(k)}(z) = (-1)^k \frac{(k-1)!}{2\pi i} [f]_{x=x_0}.$$

While f have jump behavior it implies symmetric jump behavior and every distributionally convergent series may be differentiated term by term(see [5]) , for any positive integer k and $0 < \theta \leq \pi / 2$ is valid

$$\lim_{z \rightarrow x_0, z \in \Delta_{\theta}^{+}(x_0)} \sum_{n=0}^{\infty} n^k c_n e^{inz} = (-1)^k \frac{(k-1)!}{\pi i^{k+1} (z-x_0)^k} [f]_{x=x_0}$$

we obtain the result

$$\lim_{z \rightarrow x_0, z \in \Delta_{\theta}^{+}(x_0)} (z-x_0)^k \sum_{n=0}^{\infty} n^k c_n e^{inz} = -\frac{(k-1)!}{2\pi(-i)^{k+1}} [f]_{x=x_0} .$$

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