

On the entire solutions of a nonlinear differential equation of Hayman

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Abstract: Hayman [5] proposed to study the meromorphic solutions of nonlinear differential equations of the form:

$$ff'' - (f')^2 = k_0 + k_1f + k_2f' + k_3f'',$$

where k_j ($j = 0, 1, 2, 3$) are constants. In this note, by using a new method, we give a unified and simplified proof for these known results.

Keywords: entire solutions; differential equation; Nevanlinna theory

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1. Introduction and main results

In this paper, we shall adopt the standard notations in Nevanlinna's value distribution theory of meromorphic functions. For example, the characteristic function $T(r, f)$, the counting function of the poles $N(r, f)$, and the proximity function $m(r, f)$ (see, e.g., [4, 6] and [10]).

The behavior of meromorphic solutions of differential equations has been the subject of much study. Research has concentrated on the value distribution of meromorphic solutions and their rates of growth. The purpose of the present paper is to show that a thorough search will yield a list of all meromorphic solutions of a multi-parameter ordinary differential equation introduced by Hayman. Hayman [5] proposed to study the meromorphic solutions of nonlinear differential equations of the form:

$$ff'' - (f')^2 = k_0 + k_1f + k_2f' + k_3f'', \quad (1.1)$$

where k_j ($j = 0, 1, 2, 3$) are constants. By letting $\omega = f - k_3$, the differential equation (1.1) can be rewritten as

$$\omega\omega'' - (\omega')^2 = \alpha\omega + \beta\omega' + \gamma, \quad (1.2)$$

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where α, β, γ are constants.

The major result concerning the order of growth of meromorphic solutions of first-order differential equations is the following theorem due to Goldberg [3]. A generalization of Goldbergs result to second-order algebraic equations has been conjectured by Bank [1]. Steinmetz [8] proved related results for any second-order polynomial equation which is homogeneous in its dependent variable and its derivatives. Chiang and Halburd [2] studied the Hayman's equation, and they obtain the following results.

Theorem A If not both α and γ are zeros and $\beta \neq 0$, then the meromorphic solutions of (1.2) are

$$\omega(z) = c_1 \exp\left(\frac{\alpha z}{a_{\mp}}\right) - \frac{\gamma}{\alpha}, \text{ if } \alpha \neq 0,$$

and

$$\omega(z) = c_1 + a_{\pm} z, \text{ if } \alpha = 0,$$

where c_1 is a constant, and $a_{\pm} = \frac{-\beta \pm \sqrt{\beta^2 - 4\gamma}}{2}$.

Theorem B If $\beta = 0$, then the general solution of (1.2) is given by

$$\omega(z) = \begin{cases} c_1 \exp(\pm i \frac{\alpha z}{\sqrt{\gamma}}) - \frac{\gamma}{\alpha} & , \text{ if } \alpha \neq 0; \\ c_1 \pm i \sqrt{\gamma} z & , \text{ if } \alpha = 0; \\ \frac{1}{c_1^2} [\alpha + \sqrt{\alpha^2 + \gamma c_1^2} \cosh(c_1 z + c_2)] & , \text{ where } c_1 \neq 0; \\ -\frac{\alpha}{2} z^2 + c_2 \alpha z - \frac{\gamma + c_2^2 \alpha^2}{2\alpha} & , \text{ if } \alpha \neq 0, \end{cases}$$

where c_1, c_2 are constants.

Theorem C If $\alpha = \gamma = 0$, then the general solution of (1.2) is given by

$$\omega(z) = \begin{cases} c_1 e^{c_2 z} + \frac{\beta}{c_2} & , \\ -\beta z + c_1 & , \\ 0 & , \end{cases}$$

where c_1, c_2 are constants.

However, their proofs are complicated. In this note, by using a new method, we give a unified and simplified proof for these known results. Specifically, our main results can be stated as follows:

Theorem 1.1 If $\gamma \neq 0$, consider the solutions of (1.2), we would have

$$\omega(z) = \begin{cases} c \exp(\frac{\alpha z}{a_{\pm}}) - \frac{\gamma}{\alpha} & , \quad \alpha \neq 0; \\ a_{\pm} z + c & , \quad \alpha = 0, \end{cases}$$

where c is a constant, and $a_{\pm} = \frac{-\beta \pm \sqrt{\beta^2 - 4\gamma}}{2}$.

Theorem 1.2 If $\gamma = 0$, consider the solutions of (1.2), we would have

(1) If $\alpha\beta \neq 0$, then $\omega(z) = ce^{-\frac{\alpha}{\beta}z}$, here c is a constant;

(2) If $\alpha = 0$, then

$$\omega(z) = \begin{cases} c_1 e^{c_2 z} + \frac{\beta}{c_2} & , \\ -\beta z + c_1, & , \\ c_1 & , \end{cases}$$

where c_1, c_2 are constants.

(3) If $\beta = 0$, then

$$\omega(z) = \begin{cases} c_1 e^{\sqrt{A}z} + c_2 e^{-\sqrt{A}z} - \frac{\alpha}{A} & , \quad A \neq 0; \\ -\frac{\alpha}{2}z^2 + c_1 z + c_2 & , \quad A = 0, \end{cases}$$

where c_1, c_2 are constants such that $c_1^2 + 2\alpha c_2 = 0$ if $A = 0$ or $4c_1 c_2 A^2 = \alpha^2$ if $A \neq 0$.

2. Lemmas and Proofs of Theorems

The following lemma is crucial to the proof of our theorems.

Lemma 2.1 ([6]). Let f be a meromorphic solution of an algebraic equation

$$P(z, f, f', \dots, f^{(n)}) = 0, \quad (2.1)$$

where P is a polynomial in $f, f', \dots, f^{(n)}$ with meromorphic coefficients small with respect to f . If a complex constant c does not satisfy equation (2.1), then

$$m(r, \frac{1}{f-c}) = S(r, f).$$

In order to prove the results, we also need the following lemma.

Lemma 2.2 ([10]). Let h be a non-constant entire function, and $f = e^h$, then

$$T(r, h) = o(T(r, f)), \quad T(r, h') = S(r, f).$$

Proof of Theorem 1.1.

Since $\gamma \neq 0$, then (1.2) and Lemma 2.1 imply

$$m(r, \frac{1}{\omega}) = S(r, \omega). \quad (2.2)$$

By the Nevanlinna's first fundamental theorem and (2.2), we get

$$N(r, \frac{1}{\omega}) = T(r, \omega) + S(r, \omega),$$

which, with (1.2), gives

$$N_1(r, \frac{1}{\omega}) = T(r, \omega) + S(r, \omega), \quad (2.3)$$

where $N_1(r, \frac{1}{\omega})$ denotes the counting function corresponding to simple zeros of ω .

Let $\omega(z_0) = 0$, then z_0 is a zero of $(\omega')^2 + \beta\omega' + \gamma$, and thus

$$(\omega'(z_0) - a_+)(\omega'(z_0) - a_-) = 0$$

with $a_+ = \frac{-\beta + \sqrt{\beta^2 - 4\gamma}}{2}$, $a_- = \frac{-\beta - \sqrt{\beta^2 - 4\gamma}}{2}$.

First, we assume that $\omega'(z_0) - a_+ = 0$, and set

$$h_1 = \frac{\omega' - a_+}{\omega}.$$

Next we will show that h_1 is a constant.

To prove this assertion, we first prove h_1 is a small function of ω . In fact, (2.2) and (2.3) give $m(r, h_1) = S(r, \omega)$.

By considering the order of any pole of ω in (1.2), we can check that there exists no possibility for such a pole, as shown in [7] and therefore all solutions are entire, and $N_2(r, \frac{1}{\omega}) = S(r, \omega)$, where $N_2(r, \frac{1}{\omega})$ denotes the counting function corresponding to multiple zeros of ω . Thus, $N(r, h_1) = S(r, \omega)$, and $T(r, h_1) = S(r, \omega)$.

Moreover, it follows by the definition of h_1 that

$$\omega' = h_1\omega + a_+, \quad \omega'' = (h_1' + h_1^2)\omega + h_1a_+,$$

which, with (1.2), gives $h_1' \equiv 0$, and $h_1 = \frac{\alpha}{a_-}$. Thus, $\omega' - a_+ = \frac{\alpha}{a_-}\omega$, and we have $\omega(z) = c \exp(\frac{\alpha z}{a_-}) - \frac{\gamma}{\alpha}$, if $\alpha \neq 0$ or $\omega = a_+z + c$, if $\alpha = 0$, where c is a constant.

If $\omega'(z_0) - a_- = 0$, we set

$$h_2 = \frac{\omega' - a_-}{\omega}.$$

In the same way, we get $\omega = c \exp(\frac{\alpha z}{a_+}) - \frac{\gamma}{\alpha}$, if $\alpha \neq 0$ or $\omega = a_-z + c$, if $\alpha = 0$, where c is a constant.

This completes the proof of Theorem 1.1.

Proof of Theorem 1.2.

To prove Theorem 1.2, now we distinguish three cases to discuss.

Case 1. $\alpha\beta \neq 0$. Since $\gamma = 0$, it follows by (1.2) that

$$\omega\omega''' - \omega'\omega'' = \alpha\omega' + \beta\omega'',$$

this gives

$$(\omega'' + \alpha)(\omega')^2 = \omega\omega'\omega''' - \beta\omega'\omega''. \quad (2.4)$$

From (1.2) and (2.4), we get

$$\omega[(\omega'')^2 - \omega'\omega''' - \alpha^2] = \alpha\beta\omega'. \quad (2.5)$$

Note that all the solutions of (1.2) are entire functions, by (2.5), we see that $\omega \equiv 0$ or $\omega = e^h$, in which h is an entire function.

Substituting $\omega = e^h$ into (2.5), we have

$$\{[(h')^2 + h'']^2 - h'[3h'h'' + (h')^3 + h''']\}e^{2h} = \alpha^2 + \alpha\beta h'. \quad (2.6)$$

By the standard Valiron-Mohon'ko lemma (see, e.g., [9]), Lemma 2.2 and (2.6), we obtain $h' = -\frac{\alpha}{\beta}$ and so

$$\omega = ce^{-\frac{\alpha}{\beta}z},$$

where c is a constant.

Case 2. $\alpha = 0$. In this case, (1.2) gives $\omega\omega'' = \omega'(\omega' + \beta)$, thus $\omega = c_1$, or $\omega = -\beta z + c_1$, or

$$\frac{\omega''}{\omega' + \beta} = \frac{\omega'}{\omega},$$

and so

$$\omega = c_1 e^{c_2 z} + \frac{\beta}{c_2},$$

where c_1, c_2 are constants.

Case 3. $\beta = 0$. From (1.2), we conclude

$$[(\frac{\omega'}{\omega})^2]' = -2\alpha(\frac{1}{\omega})',$$

this leads to

$$(\frac{\omega'}{\omega})^2 = -\frac{2\alpha}{\omega} + A, \quad (2.7)$$

where A is a constant.

Again, by (1.2) and (2.7) we find

$$\omega'' - A\omega + \alpha = 0,$$

which gives

$$\omega = \begin{cases} c_1 e^{\sqrt{A}z} + c_2 e^{-\sqrt{A}z} - \frac{\alpha}{A} & , \quad A \neq 0; \\ -\frac{\alpha}{2}z^2 + c_1 z + c_2 & , \quad A = 0, \end{cases}$$

where c_1, c_2 are constants such that $c_1^2 + 2\alpha c_2 = 0$ if $A = 0$ or $4c_1 c_2 A^2 = \alpha^2$ if $A \neq 0$.

This completes the proof of Theorem 1.2.

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