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Overview of Graph Colouring and some Ramsey-type Numbers

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Abstract

We introduce the concept of graph colouring and discuss some classical results in this area. In particular, we consider the problem of finding the minimal graphs, complete or not, whose vertex or edge colouring contains or avoids certain subgraphs. This is generally known as Ramsey theory. We give short proofs of some elementary results in this area, and discuss their relationship to colouring integer sequences.

Keywords: Graph Theory; Graph Colouring; Critical Graphs; Extremal Graphs; Ramsey Numbers
2010 Mathematics Subject Classification: 05C15; 05C35; 05C55

1 Graph Colouring

1.1 Definitions

In this paper we use the classical notion of *graph* G as a an ordered pair of its vertex V and edge E sets, symbolically $G = (V, E)$. Unless otherwise specified, we consider undirected graphs, without multiple edges between a pair of vertices, and without self loops. The following basic notions are used in their classical form. The number of edges adjacent to a given vertex v of a graph is called *degree* of that vertex, denoted by $\deg v$. A *hypergraph* is a graph where each edge can connect any number of vertices. A *complete graph* K_n is a graph on n vertices where every vertex is adjacent to all the other vertices. A *cycle graph* C_n is a connected graph on n vertices where every vertex has degree equal to 2. A complete subgraph of a given graph is called *clique*. The size of the largest clique contained in a given graph G is called *clique number* of G and is denoted by $\omega(G)$.

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In graph colouring, we assign colours (or more practically, numbers which are understood to represent different colours) to the vertices, edges, or both of a graph. A colouring is called proper if it does not assign the same colour to adjacent elements.

A *proper vertex colouring* is a colouring such that:

- No two adjacent vertices have the same colour
- Each edge has two endpoints of different colour
- The set of vertices of one colour is an *independent vertex set* (a set of vertices containing none which are adjacent to each other)

Chromatic number, or *vertex colouring number* of a graph G is the smallest number k of colours necessary to properly colour this graph. We denote it by $\chi(G) = k$. Thus we can divide such a graph into k independent vertex sets, G is k-partite. Some properties from [Marcus (2008)]:

- Where α is the size of a maximal independent vertex set, then the maximum number of vertices in a k-colourable graph is $k\alpha$, or equivalently, a graph with n vertices has chromatic number $\chi(G) \geq n/\alpha$
- $\chi(G) \leq \Delta + 1$, where Δ is the maximum degree of the vertices in G. $\chi(G) \leq \Delta$ if G is connected, not complete, and not an odd cycle graph
- If G contains at most k vertices of degree k or greater, then G is k-colourable

A *proper edge colouring* is a colouring such that:

- No two edges with a common endpoint have the same colour
- Each vertex can have no more than one edge of a single colour adjacent to it
- The set of edges of one colour is an *independent edge set* (set of edges which includes none that have an endpoint in common)

Chromatic index, or the edge colouring number $\chi'(G) = k$ of a graph G means that graph is k-edge-colourable. Clearly, $\chi'(G) \geq \Delta$, because each edge occurring at a vertex of maximum degree must have different colours. Some properties from [Marcus (2008)]:

- A complete graph on n vertices has chromatic index equal to n where n is odd, or equal to $(n-1)$ where n is even
- In a graph with e edges where the size of the maximal independent edge set is α', the maximum number of edges in a k-edge-colourable graph of a particular colour is α', so the total number of edges in the graph is at most $k\alpha'$, or equivalently, a graph with e edges has chromatic index $\chi'(G) \geq e/\alpha'$
- $\chi'(G) = \Delta$ (G is of Vizing class 1) or $\chi'(G) = \Delta + 1$ (G is of Vizing class 2)
- Every bipartite graph is of Vizing class 1

A *proper total colouring* is a colouring such that:

- No adjacent edges have the same colour, and
- An edge and its endpoints all have distinct colours

The *total chromatic number*, $\chi''(G)$ of a graph G is the fewest colours needed for a total colouring of it. Clearly, $\chi''(G) \geq \Delta + 1$, because each edge occurring at a vertex of maximum degree must have different colours, and that vertex must itself also have a different colour.

1.2 Colouring Problems

Generally in graph colouring problems, it is useful to consider colour switches, either interchanging two colours on a set of vertices with those colours (without creating adjacent vertices of the same colour), or on a set of edges with those colours (without creating any edges of the same colour with a common endpoint). To edge colour a graph, consider its adjacency matrix: no colour can occur twice in the same row or column.

Theorem 1.1 (Four Colour Theorem [Marcus (2008)]). *For every planar graph,* $\chi \leq 4$

This theorem was proved with the assistance of computers. Since any map can be represented by a planar graph where each region is a vertex and if two regions border each other, there is an edge between them, then any map is 4-colourable. It is simpler to show that planar graphs have $\chi \leq 6$:

Proof of the Six Colour Theorem. Suppose that a graph G is the smallest planar graph which cannot be coloured with 6 colours. Every planar graph must have a vertex of degree less than or equal to 5, so choose this vertex v in G, and remove it. We now have a graph smaller than G, so it can be coloured with 6 colours. The vertex v was adjacent to at most 5 other vertices, which are coloured with at most 5 distinct colours. So we can return v to the graph G and colour it with the 6th colour that its neighbours do not use. Then G is coloured with 6 colours, a contradiction, so there is no smallest planar graph which cannot be coloured with 6 colours. □

Conjecture 1.2 (Total Colouring Conjecture). $\chi''(G) \leq \Delta + 2$

Two upper bounds on the total chromatic number have been found: $\chi''(G) \leq \Delta + 10^{26}$ [Molloy (1998)], and $\chi''(G) \leq \Delta + 8ln^8\Delta$ [Hind et al. (1998)]

Conjecture 1.3 (Erdős–Faber–Lovász conjecture [Erdős (1981)]). *If k complete graphs, each having exactly k vertices, have the property that every pair of complete graphs has at most one shared vertex, then the union of the graphs can be colored with k colors.*

It is sometimes easier to view this as an edge-colouring problem. In a hypergraph formulation of the conjecture, consider each clique to be a vertex, and the vertices which are shared by multiple cliques to be the hyperedges containing those cliques (and no pair of hyperedges may have more than one clique in common). If every hyperedge is of size 2, and each pair of cliques has such an edge between them, then we are dealing with a complete graph. We know that such a graph can be edge-coloured with k many colours. This is the case with the maximum number of hyperedges, and these edges have minimal size.

On the other end of the spectrum, we could have a single hyperedge of maximal size, containing every clique. This is clearly the easiest case to colour. It remains to be shown that all the intermediate cases can be coloured with k many colours. Below are shown the two cases described, where each G_i is one of the cliques (vertices), and the edges with the vertices in their center are the shared vertices (hyperedges):

To generate these intermediate cases, we would like to create an algorithm which essentially combines smaller hyperedges into a single larger hyperedge, beginning from the worst case (the already coloured complete graph, on the right), with hyperedges of minimal size. The goal is to then maintain the k-colouring of the graph by a series of colour switches as necessary.

This process involves disconnecting and then reconnecting a sequence of cliques from each other. Each time, at the nth clique, this clique will have a number of colours which no hyperedge containing it is coloured in: call these colours *free*. A problem encountered is to prove that there must exist a series of colour switches on hyperedges which contain only cliques up to the nth clique which results in a new, $(n+1)^{st}$ colour becoming free. Proving that some type of order must necessarily exist in an object of some size is the aim of Ramsey theory, which we explore in the next section.

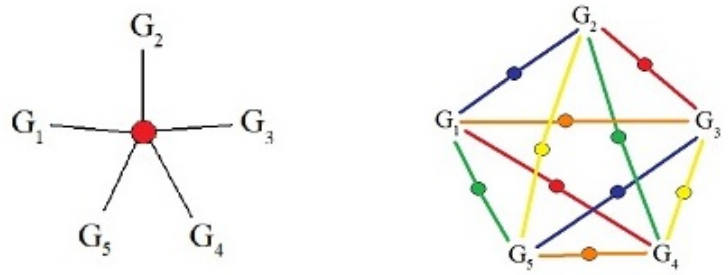


Figure 1: Erdős–Faber–Lovász best and worst cases on K_5

2 Ramsey Theory

2.1 The Basics

Ramsey theory deals with the question of how large some object need be that a property will occur in it: given a set S partitioned into finitely many subsets, must one of those subsets have the property P?

The *Dirichlet Principle* sometimes referred to as *Pigeonhole Principle* is often important to finding Ramsey-type numbers. It states:

If a set of more than mn objects is to be partitioned into n subsets, then at least one of those subsets must contain more than m elements.

Proof. Suppose each of the n subsets contains m or fewer elements. Then the union of the subsets contains mn or fewer elements. So one of subsets must contain more than m elements. □

We begin with a classical example of a problem motivating Ramsey theory before introducing Ramsey's theorem itself:

The *Party Problem* [Landman (2004)] considers what is the minimum number of guests which must be invited to a party such that either m guests will know each other (there exists a clique of size m) or n guests will not know any of each other (there is an independent set of size n). This number is denoted $R(m, n)$, and is called a *Ramsey number* (specifically, a 2-colour Ramsey number).

In this example the relation that "two guests know each other" is described by those 2 vertices being adjacent to each other, whereas if they do not know each other, there is no edge between them. Alternatively, we could consider a 2-colouring of the complete graph on the number of guests $R(m, n)$, where one colour represents knowing and the other not knowing a guest.

Example 2.1 ($R(3, 3) = 6$)**.** *Show that in a group of 6 people, either 3 people mutually know each other or do not know each other. If we consider each person to be a vertex in a graph, the relation of knowing another person being that they are connected to them by a red edge, and not knowing each other being connected by a blue edge, then we are dealing with a 2-colouring of the edges of K_6. The problem becomes to show that there must exist a monochromatic triangle. We do this by selecting one person: by the pigeonhole principle, they either know or do not know at least 3 people. Without loss of generality, assume they know 3 people. If any pair among those 3 know each other, then we have a red triangle and are done. If none of them know each other, then we have a blue triangle between the 3 of them.* □

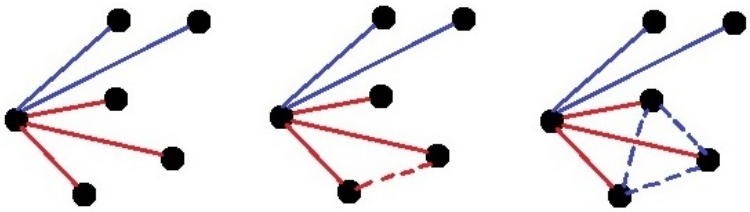


Figure 2: $R(3,3) = 6$

Theorem 2.1 (Ramsey's Theorem for Two Colours [Landman (2004)]). *There exists a least positive integer $R = R(m,n)$, with $m,n \geq 2$, such that any edge colouring of K_R contains a monochromatic K_m of the first colour or K_n of the second colour.*

Proof. Let us assume the first colour is blue and the second colour is red. We proceed with a proof by induction.

Base case: $R(m,2) = m$, $R(2,n) = n$, because an edge colouring of K_{m-1} where all edges are blue does not contain K_m, and has no red edges, so it does not contain a red K_2. Obviously K_m must contain one or the other. So a least positive integer R exists when one of the m,$n = 2$. If $m + n = 5$, one of them must equal 2 and the other 3, so the theorem is true for that case too. So let $m + n \geq 6$, with $m, n \geq 3$.

Our inductive hypothesis is that both $R(m-1,n)$ and $R(m,n-1)$ exist. We want to show that $R(m,n)$ exists, by showing that:

$$R(m,n) \leq R(m-1,n) + R(m,n-1)$$

Consider the complete graph on $R(m-1,n) + R(m,n-1)$ vertices, and edge-colour it with red and blue. Pick any vertex v in this graph. It has $R(m-1,n) + R(m,n-1) - 1$ edges incident on it. So it must have either at least $R(m-1,n)$ blue edges or $R(m,n-1)$ red edges incident on it. Otherwise, the total number of edges incident on it would be $\leq R(m-1,n) + R(m,n-1) - 2$ (an application of the *Pigeonhole Principle*), a contradiction. Without loss of generality, assume there are at least $R(m-1,n)$ blue edges incident on v. Consider the set of $R(m-1,n)$ vertices at the end of these edges. By the inductive hypothesis, there is contained among them either a blue K_{m-1} or a red K_n. If there is a red K_n, then we are done. If there is a blue K_{m-1}, then connect it to v by all of the blue edges to form a blue K_m. So our graph must contain either a blue K_m or a red K_n, so $R(m,n)$ is less than or equal to its order.

It suffices that we have established an upper bound for $R(m,n)$ to prove that it exists, since there are obviously numbers R such that K_R will be too small to contain a K_m or K_n. So the proof is complete by induction. □

By now we are starting to recognize a general strategy for working with Ramsey-type numbers. We have to keep the Pigeonhole Principle in mind, and try to establish upper and lower bounds on its value. To find a Ramsey-type number $R(F,k;r)$ generally (where F is the property, k is the size of the desired substructure, and r is the number of colours), we want to establish upper and lower bounds for that number which are equal to each other. To find a lower bound u, we find an example of an r-colouring of K_{u-1}, or $[1, u-1]$ which does not contain a monochromatic element of size k with the property F. To find an upper bound v, we show that every r-colouring of K_v, or $[1, v]$ contains a monochromatic element of size k with the property F. We conclude the value of a Ramsey number by finding upper and lower bounds for it which are equal to each other.

As mentioned above, Ramsey numbers exist for more than just 2 colours. Consider the following example, how large must a complete graph be for a 3-colouring of it to contain a monochromatic triangle:

Example 2.2 ($R(3,3,3) = 17$ [Greenwood and Gleason (1955)]). *First we want to establish an upper bound, by showing that any colouring of K_{17} must contain a monochromatic triangle. Choose a vertex v. If it has edges of a single colour with 6 or more other vertices, then either a pair of those vertices also has that colour, forming a triangle, and we are done, or all those 6 have the other 2 colours as edges between them (but $R(3,3) = 6$, so there is a triangle in that K_6). So v can have at most 5 edges of each colour incident at it to prevent a monochromatic triangle from appearing. Since we have 3 colours, there can be at most 15 edges at v, and if there were 16 edges (as in K_{17}) a triangle would appear, so $R(3,3,3) \leq 17$.*

To find a lower bound, we want to find an example of a 3-colouring of K_{16} which does not contain a monochromatic triangle. Such an example is the union of edges of 3 overlapping copies of the 5-regular Clebsch graph, shown below. Since the Clebsch graph itself is triangle free, we can colour each of these graphs in a single colour. Then the union forms K_{16}, so $R(3,3,3) \geq 17$.

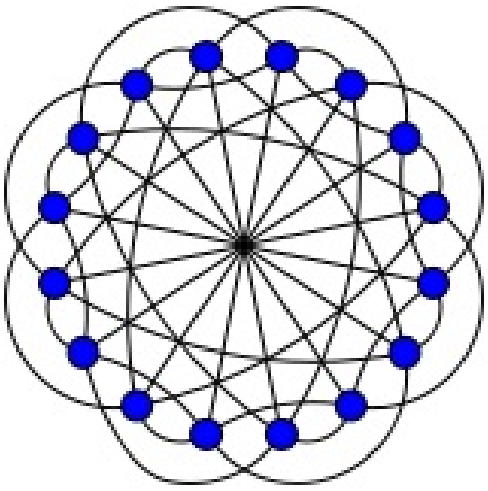


Figure 3: The 5-regular Clebsch graph

Since $R(3,3,3) \leq 17$ and $R(3,3,3) \geq 17$, $R(3,3,3) = 17$. □

Now we can state Ramsey's Theorem for an arbitrary number of colours:

Theorem 2.2 (Ramsey's Theorem [Keisler (1996)]). *For any given number of colours c, and any given integers $n_1, ..., n_c$, there exists a smallest number, $R = R(n_1, ..., n_c)$, such that if the edges of K_R are coloured with c different colours, then for some i between 1 and c, it must contain a complete subgraph of order n_i whose edges are all colour i.*

Proof. The proof of this is by induction on the number of colours, since Ramsey's Theorem has already been proven for the base case $c = 2$ above.

Our inductive hypothesis is that Ramsey's Theorem is true for $(c-1)$ colours. We want to show that it is true for c colours.

We do this by establishing an upper bound for such a Ramsey number in terms of Ramsey numbers which our inductive hypothesis claims exist (as we did in the proof for two colours). Namely, we want to show that:

$$R(n_1, ..., n_c) \leq R(n_1, ..., n_{c-2}, R(n_{c-1}, n_c))$$

The right hand side (call it v) of the inequality consists of a Ramsey number with $(c-1)$ colours (which exists by our hypothesis), and a nested Ramsey number of 2 colours, which we also know to exist. So to show that v is an upper bound for $R(n_1, ..., n_c)$, we must show that K_v must necessarily contain a complete subgraph of the desired size and colour.

Colour K_v with c many colours. The first $(c-2)$ colours correspond to the $n_1, ..., n_{c-2}$ part of $R(n_1, ..., n_{c-2}, R(n_{c-1}, n_c))$. The last 2 colours correspond to the final element, $R(n_{c-1}, n_c)$.

Since Ramsey's Theorem is assumed true for $(c-1)$ many colours, by treating the last 2 colours as if they were a single colour, we can consider K_v as if it had been coloured with $(c-1)$ many colours and then conclude that there either exist monochromatic $K_{n_1}, ..., K_{n_{c-2}}$, or there exists a $K_{R(n_{c-1}, n_c)}$ which is coloured in only the last 2 colours (being $c-1$ and c).

But then by its definition, $K_{R(n_{c-1}, n_c)}$ being coloured with $(c-1)$ and c must contain either $K_{n_{c-1}}$ or K_{n_c}. So in any case, K_v must contain one of $K_{n_1}, ..., K_{n_c}$, and we have established the finite number v as an upper bound for $R(n_1, ..., n_c)$, which must therefore exist, proving the theorem for an arbitrary number of colours. □

Both of the preceding proofs of Ramsey's Theorem established upper bounds for Ramsey numbers, which were defined in terms of smaller Ramsey numbers, and so we can compute upper bounds on all Ramsey numbers using this definition, though they grow quickly. As such, exact values for Ramsey numbers are difficult to compute, and known only for the smaller ones: we have only weak upper and lower bounds for the rest. This is summarized in the table below.

Table 1: Some small Ramsey numbers

$R(r, s)$	3	4	5	6
3	6			
4	9	18		
5	14	25	43–49	
6	18	36–41	58–87	102–165

Clearly Ramsey numbers are equal across the diagonal, and we recall from an earlier proof that $R(2, s) = R(s, 2) = s$.

There are numerous interesting results on general bounds for Ramsey numbers. In addition to the upper bounds described in the proofs above, it is known that

$$R(k, k) \leq 4R(k, k-2) + 2$$

due to Walker [Walker (1968)].
The best known general lower bound is due to Erdős [Erdős (1947)]

$$R(k, k) \geq ck2^{k/2}$$

with further refinement by Spencer [Spencer (1975)], who specified that

$$c = \frac{\sqrt{2}}{e} .$$

Both results are based on the use of probabilistic methods.
In the case of $R(3,k)$ a much tighter bounds are proved by Kim [Kim (1995)]

$$c_1 \cdot \frac{k^2}{\log k} \leq R(3,k) \leq c_2 \cdot \frac{k^2}{\log k} \ .$$

The lower bound is tight, $c_1 = \frac{1}{4}$ as proven in [Bohman and Keevash (2013)]. The best upper bound so far is $c_2 = 1$ by [Shearer (1983)]. The most complete and up to date table of Ramsey Numbers and list of best known bounds on them is due to Radziszowski, see [Radziszowski (2014)], page 4.

2.2 Folkman Numbers

Next we consider some other Ramsey-type numbers: Folkman numbers. There are a number of definitions to be addressed first:

Definition 2.1 (Arrowing [Wood (2014)]). We say $G \to (a_1, ..., a_r; q)^e$ if G does not contain K_q and any edge colouring of G contains a monochromatic K_{a_i} in the colour i.

Note that if we remove the parameter q, then this arrowing corresponds to standard Ramsey numbers.

Definition 2.2. The edge *Folkman number* [Wood (2014)] $F_e(a_1, ..., a_r; q)$ is equal to the smallest possible order of a graph G which arrows $(a_1, ..., a_r; q)^e$

The vertex Folkman numbers and arrowing are defined in the same way as for edges.

Theorem 2.3 (Folkman's Theorem [Wood (2014)]). *For $q > max(s,t)$, Folkman numbers $F_e(a_1, ..., a_r; q)$ and $F_v(a_1, ..., a_r; q)$ exist.*

An alternate statement [Landman (2004)] of Folkman's theorem follows: for any $k, r \geq 1$, any r-colouring of $[1, F]$ contains a k-element subset T such that the sumset (T) of T is monochromatic. The sumset is defined as the set of all possible sums of distinct elements of T.

We can summarize edge Folkman numbers as Ramsey numbers with the additional condition that the graph may not contain a clique of some size. For instance, one of the most studied problems is related to $R(3,3)$, sparked by a question by Erdős in [Hajnal and Erdős (1967)]:

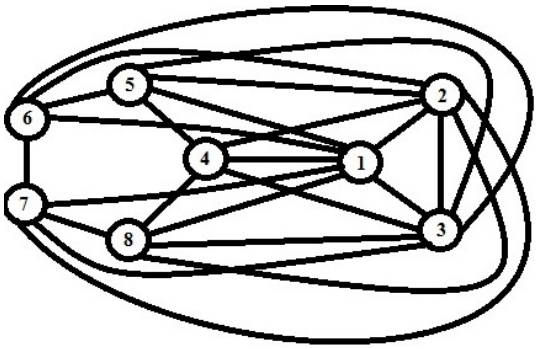
What is the order of the smallest K_4-free graph for which any 2-coloring of its edges must contain at least one monochromatic triangle?

This is asking, what is the value of the Folkman number $F(3,3;4)$? Unlike the other parameters, the larger the clique number $\omega(G)$ allowed, the easier the Folkman number is to determine. For instance consider:

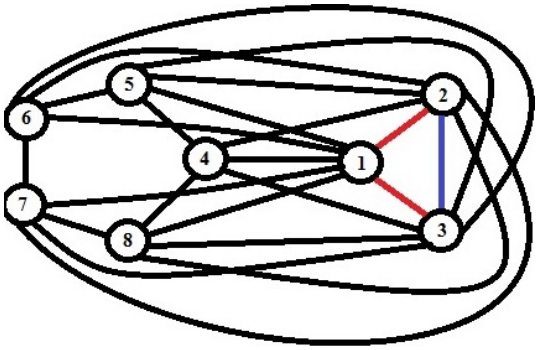
Example 2.3 ($F(3,3,7) = 6$). *The question is asking, what is the smallest order of a graph which does not contain K_7, and when 2-coloured must contain a monochromatic triangle? We already know that a 2-colouring of K_6 must contain a monochromatic triangle, and does not contain K_7. So $F(3,3,7) = R(3,3) = 6$.* □

From this we observe that in general, when $k > R(s,t)$, then $F(s,t;k) = R(s,t)$. For values of $k \leq R(s,t)$, the computation becomes increasingly difficult. Let us consider $F(3,3,6)$:

Example 2.4 ($F(3,3,6) = 8$ [Graham (1968)]). *What is the smallest order of a graph which does not contain K_6, and when 2-coloured must contain a monochromatic triangle? We have 7 as a lower bound already, because K_6 is the only graph of order 6 which contains a monochromatic triangle when 2-coloured.*



First, let us try to establish 8 as an upper bound. We must find a graph G of order 8 which does not contain K_6, but must contain a monochromatic triangle when 2-coloured. The graph we will use is $C_5 + K_3$, obtained by joining each vertex in C_5 to each vertex in K_3 by an edge. The result is a graph where the vertices of C_5 get degree 5, and the vertices of K_3 get degree 7. So how do we know this graph does not contain K_6? Well, as can be seen from the messy figure at right, there are still no triangles within C_5, so any occurrence of K_6 in the graph could use at most 2 of its vertices. But then there are only 3 other vertices available, so there cannot be a K_6 in our graph.

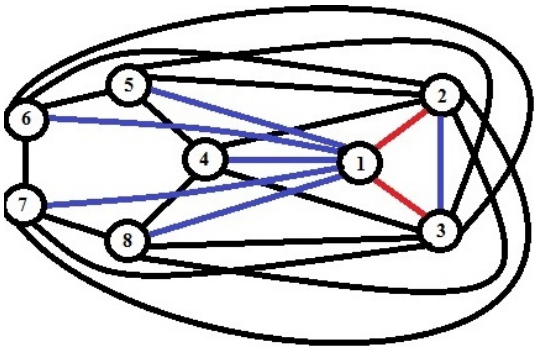


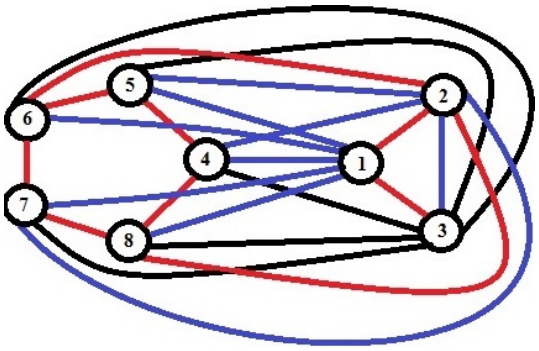
Now the trickier question: must there be a monochromatic triangle? There are clearly a large number of triangles in our graph, so we suspect it is the case. We proceed with a step-by-step colouring, attempting to prevent such a triangle from appearing. Let's start with the K_3. Without loss of generality, we can assume 2 of its edges must be red and the other blue, otherwise we have a monochromatic triangle.

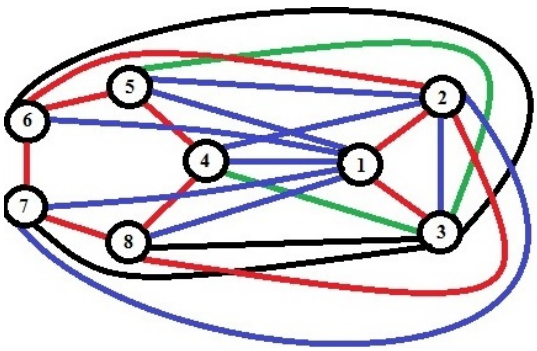
Note that at vertex 1, where 2 red edges meet, if we colour an edge from it to any vertex v in C_5 in red, then a red triangle would be formed by colouring an edge from vertices 2 or 3 to v red. So then the edges from both vertices 2 and 3 to v would need to be blue. But the edge between 2 and 3 is also blue, so we have formed a blue triangle. So we must assume that every edge from vertex 1 to C_5 is blue.

Then, if we colour any edge in C_5 blue, a blue triangle appears between it and vertex 1. So, every edge in C_5 must be red. All that remains is to colour the edges from vertices 2 and 3 to C_5. Without loss of generality we can deal with vertex 2 first. We can't make its edges to adjacent vertices in C_5 red without creating a red triangle. This results in its edges to at least 2 adjacent vertices (say 4 and 5) in C_5 being blue (since 5 is odd, and we must at least alternate red and blue).

But now we want to colour the edges between vertex 3 and those 2 adjacent vertices in C_5. If either of these edges is blue, we form a blue triangle with vertex 2. So they must both be red, but







this creates a red triangle with 4 and 5, and we have shown that no matter what, a monochromatic triangle must appear in a 2-colouring of this graph. So, $F(3,3,6) \leq 8$.

Next we want to do establish a lower bound of 8. We do this by showing that any K_6-free graph on 7 vertices can be 2-coloured such that it does not contain a monochromatic triangle. First off, such a graph must obviously contain in it more than 1 K_3. So we are looking to join K_3 to something. C_5 worked because it was an odd cycle graph, if we used C_4 it would fail at that point. We can't use C_3 because then we end up with K_6. And we can't use K_4 because again, then we end up with our graph containing a K_6. We have exhausted all reasonable options, so conclude that $F(3,3,6) \geq 8$.

Since $F(3,3,6) \leq 8$ and $F(3,3,6) \geq 8$, $F(3,3,6) = 8$ □

2.3 Van der Waerden Numbers

Next we introduce another Ramsey-type number, this one dealing with sequences of integers rather than graphs:

Definition 2.3. The *Van der Waerden number* [Landman (2004)] $W = w(k; r)$ is the least positive integer such that any r-colouring of $[1, W]$ contains a monochromatic arithmetic progression of length k.

And like Folkman and Ramsey numbers, there is a corresponding theorem on their existence:

Theorem 2.4 (Van der Waerden's Theorem [Landman (2004)])**.** *For all positive integers k and r, $w(k; r)$ exists.*

Example 2.5 ($w(2; r) = r + 1$)**.** *We are looking for a 2-term monochromatic arithmetic progression in an r-colouring of $[1, w(2; r)]$. Since any 2 integers form a 2 term arithmetic progression, we require only that 2 terms have the same colour. By the pigeonhole principle, clearly we need $r + 1$ many terms then if there are r many colours.* □

We can denote Van der Waerden numbers as Ramsey numbers with the property we are looking for being an arithmetic progression, as below:

Example 2.6 ($w(3; 2), or R(AP, 3; 2) = 9$)**.** *First to establish a lower bound, we must find a 2-colouring of [1, 8] which does not contain a 3-term monochromatic arithmetic progression. Such an example is given: $\{1, 4, 5, 8\} \in A$, $\{2, 3, 6, 7\} \in B$. So $w(3; 2) \geq 9$*

Next to establish an upper bound, we must show that any 2-colouring of [1, 9] must contain a 3-term monochromatic arithmetic progression. We attempt to colour [1, 9] without admitting such a progression. Assume 3 and 5 have the same colour. Then since $\{1, 3, 5\}$, $\{3, 4, 5\}$, and $\{3, 5, 7\}$ all form 3-term arithmetic progressions, 1, 4, and 7 must all have the other colour. But since they themselves form an arithmetic progression, we have failed. So then 3 and 5 must have different colours. Likewise, $\{4, 6\}$ and $\{5, 7\}$ cannot be monochromatic. So 5 must be a different colour from 3 and 7.

Without loss of generality, we can assume $\{4, 5\} \in A$, and $\{3, 6, 7\} \in B$. This forces $8 \in A$ (because of $\{6, 7, 8\}$), which in turn forces $2 \in B$ (because of $\{2, 5, 8\}$). Then $1 \in A$ (because of $\{1, 2, 3\}$). Clearly, if $9 \in B$ we have the progression $\{3, 6, 9\}$, and if $9 \in A$ we have the progression $\{1, 5, 9\}$. So it is impossible to 2-colour [1, 9] without admitting a 3-term arithmetic progression, so $w(3; 2) \leq 9$.

Since $w(3; 2) \geq 9$ and $w(3; 2) \leq 9$, $w(3; 2) = 9$. □

2.4 Schur Numbers

Definition 2.4. The *Schur number* [Landman (2004)] $s = s(r)$ is the least positive integer such that any r-colouring of $[1, s]$ contains a monochromatic solution to $x + y = z$

Note this is the case $k = 2$ of Folkman numbers for the sumset definition.

Theorem 2.5 (Schur's Theorem [Landman (2004)])**.** *For any $r \geq 1, s(r)$ exists.*

Example 2.7 ($s(1) = 2$)**.** *Obviously, if we use only one colour, in [1, 2] we have 1 + 1 = 2.* □

Example 2.8 ($s(2) = 5$)**.** *First we establish a lower bound of 5 by finding an example of a 2-colouring of [1, 4] which contains no monochromatic solution to $x + y = z$. Such an example is $\{1, 4\} \in A, \{2, 3\} \in B$, so $s(2) \geq 5$.*

To establish an upper bound of 5, we must show that any 2-colouring of [1, 5] must contain a monochromatic solution to $x + y = z$. So we attempt to colour it while avoiding such a solution. Clearly if 1 and 2 are the same colour, then they constitute a solution of 1 + 1 = 2, so without loss of generality, assume $\{1\} \in A, \{2\} \in B$. Since 2 + 2 = 4 is also a solution, 4 must be a different colour from 2, so $\{1, 4\} \in A$. Then, 1 + 3 = 4, so $\{2, 3\} \in B$. 1 + 4 = 5 and 2 + 3 = 5, so whichever colour we assign to 5, we get a monochromatic solution to $x + y = z$. So $s(2) \leq 5$.

Since $s(2) \geq 5$ and $s(2) \leq 5$, $s(2) = 5$. □

2.5 Erdős-Szekeres Theorem

Theorem 2.6 (Erdős-Szekeres Theorem)**.** *Any sequence of real numbers of size greater than $(m - 1)(n - 1)$ contains either an increasing subsequence of length m or a decreasing subsequence of length n.*

Proof [Seidenberg (1959)]. Consider a sequence of $(m-1)(n-1)+1$ numbers. Let each element s_i of the sequence correspond to a pair (a_i, b_i), where a_i refers to the longest increasing subsequence ending with the term s_i, and b_i refers to the longest decreasing subsequence ending with s_i. Now consider the term s_{i+1}. If $s_{i+1} > s_i$, then $a_{i+1} = a_i + 1$, and if $s_{i+1} < s_i$, then $b_{i+1} = b_i + 1$. That is to say, any pair of terms in the original sequence have different corresponding pairs (a_i, b_i), since the one which comes later has either a longer increasing or decreasing subsequence ending with it. But there are only $(m - 1)(n - 1)$ many unique pairs where a_i is not greater than $(m - 1)$ and b_i is not greater than $(n - 1)$. So by the pigeonhole principle, in a sequence of $(m - 1)(n - 1) + 1$ terms, at least one of these terms must have a corresponding pair (a_i, b_i) where either $a_i \geq m$ or $b_i \geq n$, as was to be shown. □

Example 2.9 ($m = 3, n = 2$)**.** *Clearly, any arrangement of $(3 - 1)(2 - 1) + 1 = 3$ distinct numbers contains either an increasing subsequence of length 3 or a decreasing subsequence of length 2, since $\{$1, 2, 3$\}$ is the only arrangement which does not contain a 2-term decreasing subsequence.* □

2.6 Criticality

Returning to Ramsey numbers for a moment, consider how we calculate their values. We first find one or more examples of complete graphs on n vertices to establish $(n+1)$ as a lower bound for that Ramsey number. We then try to show that any colouring of K_{n+1} must yield a desired monochromatic subgraph. That is, if we add one more vertex to our colouring of K_n, it forces such an subgraph to appear.

This notion of adding or removing a single vertex or edge from a graph which results in some property changing is generally called *criticality*. The main use of criticality follows:

Definition 2.5. A *critical element* of a graph is a vertex or edge which, when removed, reduces that graph's chromatic number by one.

Definition 2.6. A *k-critical graph* G is a graph with chromatic number k where every vertex and edge is a critical element. A k-vertex-critical graph has only all its vertices as critical elements.

Clearly, a k-critical graph G must be connected. Since G has chromatic number k, that means at least one of its components must be coloured with k many colours. If there is only one such component, then removing an element from one of the other components does not reduce the graph's chromatic number, while if there is more than one, then we would have to remove more than one element to reduce the chromatic number.

Also, the degree of every vertex is at least $(k-1)$. Suppose a vertex has degree $(k-2)$ (or less). Removing this vertex should result in a graph with chromatic number $(k-1)$. But since its neighbours

are coloured with at most $(k-2)$ many colours, we could already colour this vertex with the $(k-1)^{st}$ colour, removing it does not change the chromatic number.

Definition 2.7. A *factor-critical graph* [Cornuéjols and Pulleyblank (1983)] G is a graph where the removal of any one of its vertices results in a *perfect matching* , that is, a subset of the remaining edges has every vertex as the endpoint of exactly one edge.

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