

Traveling Wave Solutions for a Coupled KdV Equations *

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Abstract

In this paper, we use a method in order to find exact explicit traveling solutions in the subspace of the phase space for a coupled KdV equations. The key idea is to remove the coupled relation for a given system such that the new systems can be solved. The existence of solitary wave solutions is obtained. It is shown that the bifurcation theory of dynamical systems provides a powerful mathematical tool for solving a great many nonlinear partial differential equations in mathematical physics.

Key Words: Solitary wave solution; bifurcation theory of dynamical systems; coupled KdV equations

1 Introduction

Studies of various physical structures of nonlinear dispersive equations have attracted much attention in connection with the important problems that arise in scientific applications. Mathematically, these physical structures have been studied by using various analytical methods, such as inverse scattering method [1], Darboux transformation

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method [2,3], Hirota bilinear method [4], Lie group method [5,6], sine-cosine method [7,8], tanh function method [9], Fan-expansion method [10,11] and so on. Practically, there is no unified technique that can be employed to handle all types of nonlinear differential equations.

In 1981, Hirota [4] presented the following coupled KdV equations,

$$\begin{cases} u_t + 6\alpha uu_x - 2bv v_x + \alpha u_{xxx} = 0, \\ v_t + 3\beta uv_x + \beta v_{xxx} = 0, \end{cases} \quad (1.1)$$

which are derived to model the interaction of water waves. Here α, β, b are constants. Liu et al. in [12] based on a subtle balance method, a given function expansion is applied to Eq. (1.1), a series of periodic solutions, solitary wave solutions and singular solutions are obtained by the aid of symbolic computation.

For $b = 3, \alpha = \beta$, Eq. (1.1) becomes to

$$\begin{cases} u_t + 6\alpha uu_x - 6vv_x + \alpha u_{xxx} = 0, \\ v_t + 3\alpha uv_x + \alpha v_{xxx} = 0. \end{cases} \quad (1.2)$$

Recently, Wang et al. [13] considered the solitary wave solutions of Eq. (1.2), by using a homogeneous balance method.

In this paper, we consider the dynamical bifurcation behavior for the traveling wave solutions of (1.2). Let $u(x, t) = u(\xi)$, $v(x, t) = v(\xi)$, $\xi = x - ct$, where c is the wave speed. Then (1.2) becomes to

$$\begin{cases} -cu' + 6\alpha uu' - 6vv' + \alpha u''' = 0, \\ -cv' + 3\alpha uv' + \alpha v''' = 0, \end{cases} \quad (1.3)$$

where " $'$ " is the derivative with respect to ξ . To remove the coupled relationship of (1.3), let $u = a + bv^2$, substituting it into (1.3) and integrating once, we obtain

$$\begin{cases} -ac + 3\alpha a^2 + (6\alpha ab - bc - 3)v^2 + 3\alpha b^2 v^4 + 2b\alpha(vv'' + (v')^2) = g_1, \\ -cv + 3\alpha v(a + \frac{1}{3}bv^2) + \alpha v'' = g_2, \end{cases} \quad (1.4)$$

where g_1 and g_2 are integral constants.

Thus, we have two Hamiltonian systems

$$\frac{dv}{d\xi} = y, \quad \frac{dy}{d\xi} = \frac{g_1 + ac - 3\alpha a^2 + (bc - 6\alpha ab + 3)v^2 - 3\alpha b^2 v^4 - 2b\alpha y^2}{2b\alpha v}, \quad (1.5)$$

$$\frac{dv}{d\xi} = y, \quad \frac{dy}{d\xi} = \frac{1}{\alpha} \left(g_2 + (c - 3\alpha a)v - \alpha b v^3 \right) \quad (1.6)$$

with the first integral

$$y^2 = \frac{\frac{1}{2}(g_1 + ac - 3\alpha a^2)v^2 + \frac{1}{4}(bc - 6\alpha ab - 3)v^4 - \frac{1}{2}\alpha b^2 v^6 + h_1}{\alpha b v^2}, \quad (1.7)$$

$$y^2 = \frac{2g_2 v + (c - 3\alpha a)v^2 - \frac{\alpha b}{2}v^4 + h_2}{\alpha} \quad (1.8)$$

and

$$H_1(v, y) = \alpha b v^2 y^2 - \left[\frac{1}{2}(g_1 + ac - 3\alpha a^2)v^2 + \frac{1}{4}(bc - 6\alpha ab - 3)v^4 - \frac{1}{2}\alpha b^2 v^6 \right] = h_1, \quad (1.9)$$

$$H_2(v, y) = \alpha y^2 - \left[2g_2 v + (c - 3\alpha a)v^2 - \frac{\alpha b}{2}v^4 \right] = h_2 \quad (1.10)$$

For

$$h_1 = h_2 = g_2 = 0, \quad a = \frac{c \pm \sqrt{c^2 + 12g_1\alpha}}{6\alpha}, \quad b = \frac{1}{c - 2\alpha a}, \quad c^2 + 12g_1\alpha \geq 0, \quad (1.11)$$

system (1.5) and system (1.6) are equivalent to each other. In this paper, we always assume that equations (1.2) satisfy (1.11). Therefore, under condition (1.11), we only consider the system (1.6) and we have two Hamiltonian systems

$$\frac{dv}{d\xi} = y, \quad \frac{dy}{d\xi} = \frac{v}{2\alpha} \left(c - \sqrt{c^2 + 12g_1\alpha} - \frac{3\alpha v^2}{c - \frac{1}{2}\sqrt{c^2 + 12g_1\alpha}} \right), \quad (1.12)$$

$$\frac{dv}{d\xi} = y, \quad \frac{dy}{d\xi} = \frac{v}{2\alpha} \left(c + \sqrt{c^2 + 12g_1\alpha} - \frac{3\alpha v^2}{c + \frac{1}{2}\sqrt{c^2 + 12g_1\alpha}} \right) \quad (1.13)$$

with the first integral

$$y^2 = \frac{1}{2\alpha} \left(c - \sqrt{c^2 + 12g_1\alpha} \right) v^2 - \frac{3v^4}{4 \left(2c - \sqrt{c^2 + 12g_1\alpha} \right)}, \quad (1.14)$$

$$y^2 = \frac{1}{2\alpha} \left(c + \sqrt{c^2 + 12g_1\alpha} \right) v^2 - \frac{3v^4}{4 \left(2c + \sqrt{c^2 + 12g_1\alpha} \right)} \quad (1.15)$$

and

$$H_{21}(v, y) = y^2 - \frac{1}{2\alpha} \left(c - \sqrt{c^2 + 12g_1\alpha} \right) v^2 + \frac{3v^4}{4 \left(2c - \sqrt{c^2 + 12g_1\alpha} \right)} = 0. \quad (1.16)$$

$$H_{22}(v, y) = y^2 - \frac{1}{2\alpha} \left(c + \sqrt{c^2 + 12g_1\alpha} \right) v^2 + \frac{3v^4}{4 \left(2c + \sqrt{c^2 + 12g_1\alpha} \right)} = 0. \quad (1.17)$$

2 Exact Explicit traveling solutions of system (1.2)

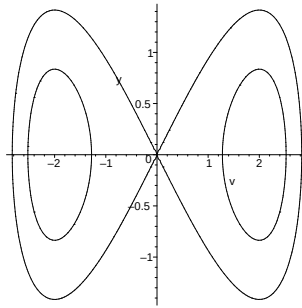
2.1 $a = \frac{c - \sqrt{c^2 + 12g_1\alpha}}{6\alpha}$

Suppose that $g_1 < 0$, $\alpha > 0$, $c > 2\sqrt{-3g_1\alpha}$ or $g_1 < 0$, $\alpha < 0$, $c > 2\sqrt{g_1\alpha}$. Then, origin of system (1.12) is a saddle point. Corresponding to the level curves defined by $H_{21}(v, y) = 0$, two homoclinic orbits of (1.12) have the following parametric representations (see Fig. 1)

$$v = \pm \sqrt{\frac{(c - \sqrt{c^2 + 12g_1\alpha})(2c - \sqrt{c^2 + 12g_1\alpha})}{3\alpha}} \operatorname{sech} \sqrt{\frac{c - \sqrt{c^2 + 12g_1\alpha}}{2\alpha}} \xi. \quad (2.1)$$

Thus, we obtain the parametric representation of Eq. (1.2) a smooth solitary wave solution as follows:

$$\begin{cases} u(x, t) = \frac{c - \sqrt{c^2 + 12g_1\alpha}}{6\alpha} + \left(\frac{6\alpha}{2c - \sqrt{c^2 + 12g_1\alpha}} \right) v^2, \\ v(x, t) = \pm \sqrt{\frac{(c - \sqrt{c^2 + 12g_1\alpha})(2c - \sqrt{c^2 + 12g_1\alpha})}{3\alpha}} \operatorname{sech} \sqrt{\frac{c - \sqrt{c^2 + 12g_1\alpha}}{2\alpha}} (x - ct). \end{cases} \quad (2.2)$$



$$g_1 < 0, \alpha > 0, c > 2\sqrt{-3g_1\alpha} \text{ or } g_1 < 0, \alpha < 0, c > 2\sqrt{g_1\alpha}.$$

Fig. 1 The phase portraits of (2.12) for $H_{21}(v, y) = 0$.

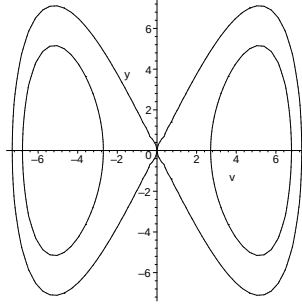
2.2 $a = \frac{c + \sqrt{c^2 + 12g_1\alpha}}{6\alpha}$

Suppose that $g_1 < 0$, $\alpha > 0$, $c > 2\sqrt{-3g_1\alpha}$ or $g_1 > 0$, $\alpha > 0$, $c > 0$. Then, origin of system (1.13) is a saddle point. Corresponding to the level curves defined by $H_{22}(v, y) = 0$, two homoclinic orbits of (1.13) have the following parametric representations (see Fig. 2)

$$v = \pm \sqrt{\frac{(c + \sqrt{c^2 + 12g_1\alpha})(2c + \sqrt{c^2 + 12g_1\alpha})}{3\alpha}} \operatorname{sech} \sqrt{\frac{c + \sqrt{c^2 + 12g_1\alpha}}{2\alpha}} \xi. \quad (2.3)$$

Thus, we obtain the parametric representation of Eq. (1.2) a smooth solitary wave solution as follows:

$$\begin{cases} u(x, t) = \frac{c + \sqrt{c^2 + 12g_1\alpha}}{6\alpha} + \left(\frac{6\alpha}{2c + \sqrt{c^2 + 12g_1\alpha}} \right) v^2, \\ v(x, t) = \pm \sqrt{\frac{(c + \sqrt{c^2 + 12g_1\alpha})(2c + \sqrt{c^2 + 12g_1\alpha})}{3\alpha}} \operatorname{sech} \sqrt{\frac{c + \sqrt{c^2 + 12g_1\alpha}}{2\alpha}} (x - ct). \end{cases} \quad (2.4)$$



$g_1 > 0$, $\alpha > 0$, $c > 0$ or $g_1 < 0$, $\alpha < 0$, $c > 2\sqrt{-3g_1\alpha}$.

Fig. 2 The phase portraits of (2.13) for $H_{22}(v, y) = 0$.

Remark. Obviously, (2.2) and (2.4) is different from (3.22) and (3.23) of [13].

3 Conclusion

In this paper, we have considered traveling wave solutions for the coupled KdV equations of (1.2) in its subspace of parameter space, by using the method of dynamical systems. We obtain parametric representations for solitary wave solutions of (1.1) in different parameter regions of the parameter space.

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