

Hamiltonian Laceability in Ring Product and Cyclo Product of Graphs

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ABSTRACT

B. Alspach, C.C. Chen and Kevin McAvaney [1] have discussed the Hamiltonian laceability of the Brick product $C(2n, m, r)$ for even cycles. In [2], the authors have shown that the (m, r) -Brick Product $C(2n + 1, 1, 2)$ is Hamiltonian- t -laceable for $1 \leq t \leq \text{diam}n$. In [3] the authors have defined and discussed Hamiltonian- t -laceability properties of cyclic product $C(2n, m)$ cyclic product of graphs. In this paper we explore Hamiltonian- t^* -laceability of $(W_{1,n}, k)$ graph and Cyclo Product $C_y(n, mk)$ of graph.

Keywords: Brick product, Hamiltonian- t -laceable graph, Cyclo product.

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1 INTRODUCTION

Let G be a finite, simple, connected and undirected graph. Let u and v be two vertices in G . The distance between u and v denoted by, $d(u, v)$, is the length of a shortest $u - v$ path in G . A graph G is Hamiltonian- t -laceable [4] if there exists in G Hamiltonian path between every pair of vertices u and v with $d(u, v)=t$, $1 \leq t \leq \text{diam}G$, where t is a positive integer. A graph G is Hamiltonian- t^* -laceable [2] if there exists in G a Hamiltonian path between at least one pair of distinct vertices u and v such that $d(u, v)=t$, $1 \leq t \leq \text{diam}G$. In [1] B. Alspach, C.C. Chen and Kevin McAvaney have explored Hamiltonian Laceability in the Brick Products of even cycles. In [3], Leena Shenoy and R. Murali have discussed the (m, r) -Brick Product of odd cycles $C(2n + 1, m, r)$. Using this concept we define $(W_{1,n}, k)$ graph and Cyclo product, $C_y(n, mk)$ of graph and explore Hamiltonian laceability properties.

First we recall the following definitions.

2 HAMILTONIAN LACEABLE GRAPH

Definition 2.1. Let G be a finite, simple, connected undirected graph. A graph G is *Hamiltonian laceable* if there exists a Hamiltonian path between every pair of vertices at an odd distance in G .

Example: Hamiltonian laceable graph is shown in figure 1.

Definition 2.2. Let G be a finite, simple, connected undirected graph. The graph G is *Hamiltonian- t -laceable* if there exist a Hamiltonian path between every pair of distinct vertices a_i and a_j in G

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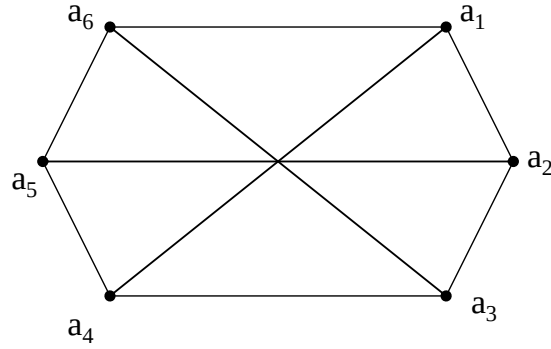


Figure 1: A Hamiltonian laceable graph

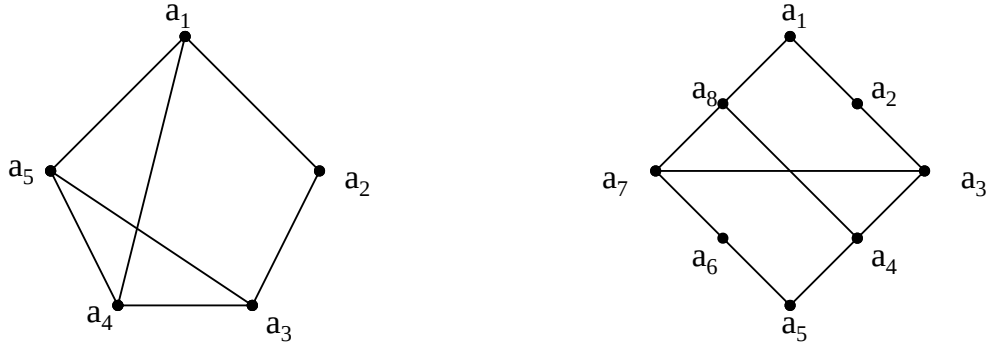


Figure 2: Hamiltonian-2*-laceable graph and Hamiltonian-3-laceable graph

with the property $d(a_i, a_j) = t$; $1 \leq t \leq \text{diam}G$ and *Hamiltonian - t^* - laceable* if there exists a Hamiltonian path between at least any one pair of distinct vertices a_i and a_j such that $d(a_i, a_j) = t$; $1 \leq t \leq \text{diam}G$.

Example: *Hamiltonian - 3 - laceable* and *Hamiltonian - 2^* - laceable* graphs are shown in figure 2.

3 THE GRAPH $(W_{1,n}, k)$

Let $W_{1,n}$ be a wheel graph. We shall denote the vertices of $W_{1,n}$ by $\langle a_i \rangle$, $1 \leq i \leq n$ and a root vertex a_0 .

Let $G = (W_{1,n}, k)$ be a graph obtained by taking disjoint union of k copies of cycle C_k with the vertices $a_{k1}, a_{k2}, a_{k3}, a_{k3}, \dots, a_{kn}$.

If $k=1$, For $1 \leq i \leq n$, draw an edge connecting vertices a_i of $W_{1,n}$ to a_{1i} of C_1 .

If $k \geq 2$, starting from $k=2$ proceed recursively joining the vertices $a_{(k-1)_i}$ to a_{k_i} by an edge. Where $1 \leq i \leq n$.

Example: The graph $(W_{1,n}, k)$ is shown in figure 3.

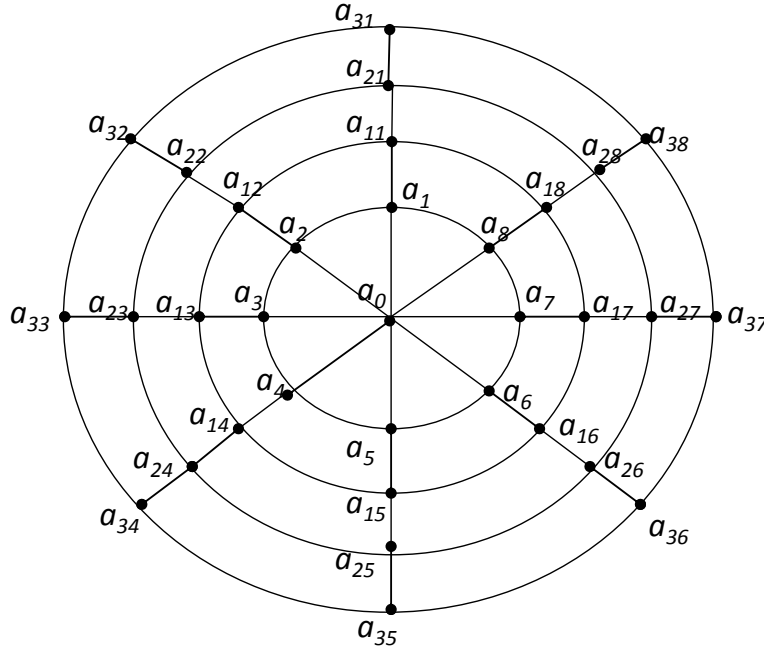


Figure 3: Hamiltonian laceable graph $(W_{1,8}, 3)$

Theorem 3.1. The graph $G = (W_{1,n}, k)$, $n \geq 3, k \geq 1$ is Hamiltonian- t^* -laceable for $1 \leq t \leq 3$.

Proof. Let $G = (W_{1,n}, k)$. The vertices of interior cycle $(W_{1,n})$ be $a_1, a_2, a_3, a_4, \dots, a_{n-1}, a_n$ and vertices on the k^{th} cycle be $a_{k_1}, a_{k_2}, a_{k_3}, a_{k_4}, \dots, a_{k_{n-1}}, a_{k_n}$. The graph G has $n(k+1)+1$ number of vertices and $2(k+1)n$ number of edges. To establish the result, we consider the following cases,

Case(i): For $t=1$

In G , let $d(a_0, a_1) = 1$ then the path

$P : (a_0, a_2) \cup \{(a_2, a_3) \cup (a_3, a_4) \cup \dots \cup (a_{n-1}, a_n)\} \cup (a_n, a_{1_n}) \cup \{(a_{1_n}, a_{1_{(n-1)}}) \cup (a_{1_{(n-1)}}, a_{1_{(n-2)}}) \cup \dots \cup (a_{1_3}, a_{1_2})\} \cup (a_{1_2}, a_{2_2}) \cup \{(a_{2_2}, a_{2_3}) \cup (a_{2_3}, a_{2_4}) \cup (a_{2_4}, a_{2_5}) \cup \dots \cup (a_{2_{(n-1)}}, a_{2_n})\} \cup (a_{2_n}, a_{3_n}) \cup \{(a_{3_n}, a_{3_{n-1}}) \cup (a_{3_{n-1}}, a_{3_{n-2}}) \cup (a_{3_{n-2}}, a_{3_{n-3}}) \cup \dots \cup (a_{3_3}, a_{3_2})\} \cup (a_{3_2}, a_{4_2}) \cup \dots \cup T \cup (a_{k_1}, a_{(k-1)_1}) \cup (a_{(k-1)_1}, a_{(k-2)_1}) \cup \dots \cup (a_{2_1}, a_{1_1}) \cup (a_{1_1}, a_1)$ is a Hamiltonian path from a_0 to a_1 .

Where,

$$T = \begin{cases} (a_{k_n}, a_{k_{n-1}}) \cup (a_{k_{n-1}}, a_{k_{n-2}}) \cup \dots \cup (a_{k_3}, a_{k_2}) \cup (a_{k_2}, a_{k_1}), & \text{if } k \text{ is odd;} \\ (a_{k_2}, a_{k_3}) \cup (a_{k_3}, a_{k_4}) \cup \dots \cup (a_{k_{n-1}}, a_{k_n}) \cup (a_{k_n}, a_{k_1}), & \text{if } k \text{ is even.} \end{cases}$$

Case(ii): For $t=2$

In G , let $d(a_0, a_1) = 2$ then the path

$P : (a_0, a_1) \cup \{(a_1, a_2) \cup (a_2, a_3) \cup (a_3, a_4) \cup \dots \cup (a_{n-2}, a_{n-1}) \cup (a_{n-1}, a_n)\} \cup (a_n, a_{1_n}) \cup \{(a_{1_n}, a_{1_{(n-1)}}) \cup (a_{1_{(n-1)}}, a_{1_{(n-2)}}) \cup (a_{1_{n-2}}, a_{1_{(n-3)}}) \cup \dots \cup (a_{1_4}, a_{1_3}) \cup (a_{1_3}, a_{1_2})\} \cup (a_{1_2}, a_{2_2}) \cup$

$$T = \begin{cases} (a_{k_n}, a_{k_{(n-1)}}) \cup (a_{k_{(n-1)}}, a_{k_{(n-2)}}) \cup \dots \cup (a_{k_3}, a_{k_2}) \cup (a_{k_2}, a_{k_1}), & \text{if } k \text{ is odd;} \\ (a_{k_2}, a_{k_3}) \cup (a_{k_3}, a_{k_4}) \cup \dots \cup (a_{k_{(n-1)}}, a_{k_n}) \cup (a_{k_n}, a_{k_1}), & \text{if } k \text{ is even.} \end{cases}$$

In G , let $d(a_3, a_{1_1}) = 3$ then the path

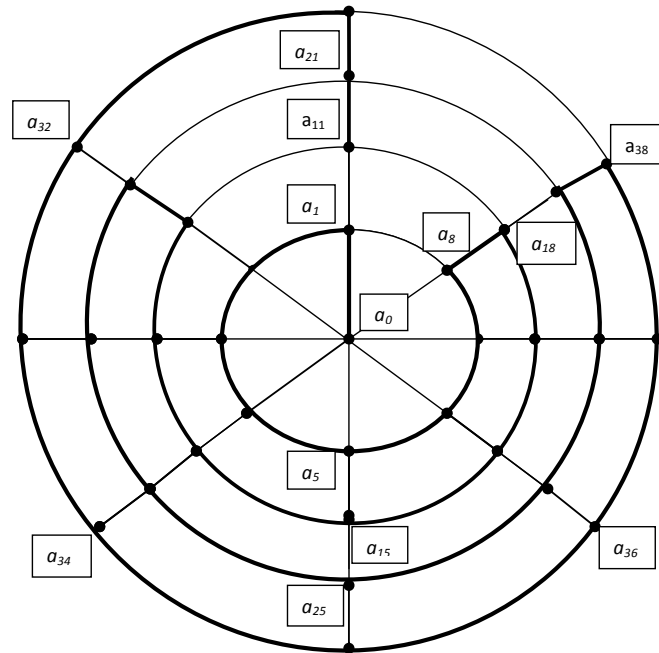
$$T = \begin{cases} (a_{k_2}, a_{k_3}) \cup (a_{k_3}, a_{k_4}) \cup \dots \cup (a_{k_{(n-1)}}, a_{k_n}) \cup (a_{k_n}, a_{k_1}), & \text{if } k \text{ is odd;} \\ (a_{k_n}, a_{k_{(n-1)}}) \cup (a_{k_{(n-1)}}, a_{k_{(n-2)}}) \cup \dots \cup (a_{k_3}, a_{k_2}) \cup (a_{k_2}, a_{k_1}), & \text{if } k \text{ is even.} \end{cases}$$
☐

Figure 4, shows a Hamiltonian path between a_0 to a_{11} in $(W_{1,8}, 3)$. This path is

$$P : (a_0, a_1) \cup (a_1, a_2) \cup (a_2, a_3) \cup \dots \cup (a_7, a_8) \cup (a_8, a_{18}) \cup (a_{18}, a_{17}) \cup (a_{17}, a_{16}) \cup \dots \cup (a_{13}, a_{12}) \cup (a_{12}, a_{22}) \cup (a_{22}, a_{23}) \cup (a_{23}, a_{24}) \cup \dots \cup (a_{27}, a_{28}) \cup (a_{28}, a_{38}) \cup (a_{38}, a_{37}) \cup (a_{37}, a_{36}) \cup (a_{31}, a_{21}) \cup (a_{21}, a_{11}).$$

4 CYCLO PRODUCT

Definition 4.1. Let n, k be positive integers and $m \geq 2$. The *cycloproduct* $C_y(n, mk)$ is defined by joining each vertex a_i in C_n to a_{i+mk} under modulo n .

Cyclo product $C_y(15, 3k)$ under modulo 15 is shown in fig 5.

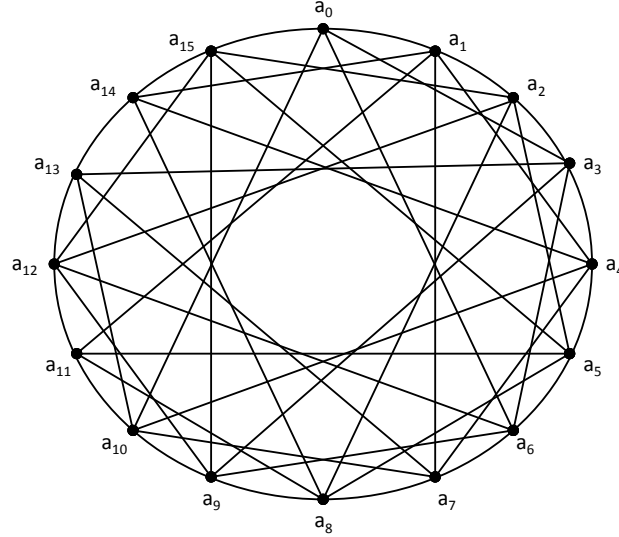


Figure 5: Cyclo product $C_y(15, 3k)$ under modulo 15

Theorem 4.1. The graph $C_y(n, 3k)$, $n \geq 6$ is Hamiltonian- t -laceable for $t = 1, 2$.

Proof. Let $G = C_y(n, 3k)$ be a graph with n no. vertices. Let this vertex set be $V = \{a_0, a_1, a_2, a_3, a_4, a_5, \dots, a_{n-2}, a_{n-1}\}$. We consider the following cases

Case(i): For $t=1$.

Let $d(a_i, a_j)=1$ and $|i - j| = 1$ then we find a path P in G such that $P : (a_i, a_{i-1}) \cup (a_{i-1}, a_{i-2}) \cup (a_{i-2}, a_{i-3}) \cup (a_{i-3}, a_{i-4}) \cup \dots \cup (a_{j+2}, a_{j+1}) \cup (a_{j+1}, a_j)$ is a Hamiltonian path.

Hence G is Hamiltonian-1-laceable.

Case(ii): For $t=2$.

Let $d(a_i, a_j)=2$ and $|i - j| = 2$ then we find a path P in G such that $P : (a_i, a_{i+1}) \cup (a_{i+1}, a_{i-5}) \cup (a_{i-5}, a_{i-4}) \cup (a_{i-4}, a_{i-1}) \cup (a_{i-1}, a_{i-2}) \cup (a_{i-2}, a_{i-3}) \cup (a_{i-3}, a_{i-6}) \cup (a_{i-6}, a_{i-7}) \cup (a_{i-7}, a_{i-8}) \cup (a_{i-8}, a_{i-9}) \cup (a_{i-9}, a_{i-10}) \cup \dots \cup (a_{j+2}, a_{j+1}) \cup (a_{j+1}, a_j)$ is a Hamiltonian path from a_i to a_j .

Hence G is Hamiltonian-2-laceable.

Hence the proof. $\square$

In figure 6, a Hamiltonian path in $G = C_y(15, 3k)$ under modulo 15, between the vertices a_1 to a_3 is shown. This path is $P : (a_1, a_2) \cup (a_2, a_{12}) \cup (a_{12}, a_{13}) \cup (a_{13}, a_0) \cup (a_0, a_{15}) \cup (a_{15}, a_{14}) \cup (a_{14}, a_{11}) \cup (a_{11}, a_{10}) \cup (a_{10}, a_9) \cup (a_9, a_8) \cup (a_8, a_7) \cup (a_7, a_6) \cup (a_6, a_5) \cup (a_5, a_4) \cup (a_4, a_3)$.

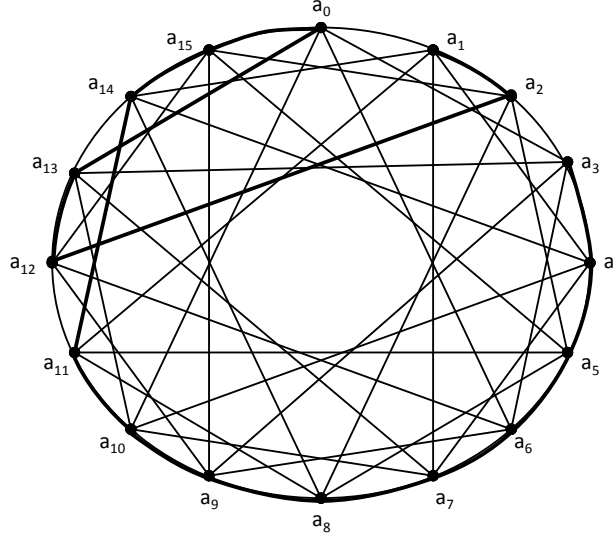


Figure 6: Hamiltonian path from a_1 to a_3 is shown by dark lines in $C_y(15, 3k)$

Theorem 4.2. *The graph $G = C_y(n, 2k)$, $n \geq 6$ is Hamiltonian- t -laceable for $t=1,2$.*

Proof. Let $G = C_y(n, 3k)$ be the graph of order n , ($n \geq 6$). Let the vertex set $V = \{a_0, a_1, a_2, a_3, a_4, a_5, \dots, a_{n-2}, a_n\}$

Case(i): For $t=1$

Let $d(a_i, a_j)=1$ and $|i-j|=2, 0 \leq i, j \leq n$. There exists a path P in G such that $P : (a_i, a_{i+3}) \cup (a_{i+3}, a_{i+5}) \cup (a_{i+5}, a_{i+7}) \cup (a_{i+7}, a_{i+9}) \cup (a_{i+9}, a_{i+11}) \cup (a_{i+11}, a_{i+12}) \cup \dots \cup (a_{i-1}, a_{j+1}) \cup (a_{j+1}, a_{i-2}) \cup (a_{i-2}, a_{i-4}) \cup (a_{i-4}, a_{i-8}) \cup \dots \cup (a_{i-1}, a_{j+1}) \cup (a_{j+1}, a_{i-2}) \cup (a_{i-2}, a_{i-4}) \cup (a_{i-4}, a_{i-8}) \cup \dots \cup (a_{j+4}, a_{j+2}) \cup (a_{j+2}, a_j)$ is a Hamiltonian path from a_i to a_j under modulo n . Hence G is Hamiltonian-1-laceable.

Case(ii): For $t=2$.

Let $d(a_i, a_j)=2$ and $j-i=1, 0 \leq i, j \leq n$. There exists a path P in G such that $P : (a_i, a_{i+1}) \cup (a_{i+1}, a_{i+2}) \cup (a_{i+2}, a_{i-2}) \cup (a_{i-2}, a_{i+1}) \cup (a_{i+1}, a_{i-3}) \cup (a_{i-3}, a_{i-4}) \cup (a_{i-4}, a_{i-5}) \cup (a_{i-5}, a_{i-6}) \cup (a_{i-6}, a_{i-7}) \cup \dots \cup (a_{j+3}, a_{j+2}) \cup (a_{j+2}, a_{j+1}) \cup (a_{j+1}, a_j)$ is a Hamiltonian path

Hence Hamiltonian-2-laceable. $\square$

In figure 7, a Hamiltonian path in $G = C_y(15, 2k)$ under modulo 15, between the vertices a_1 to a_4 is shown. This path is $P : (a_1, a_2) \cup (a_2, a_3) \cup (a_3, a_{14}) \cup (a_{14}, a_0) \cup (a_0, a_{13}) \cup (a_{13}, a_{12}) \cup (a_{12}, a_{11}) \cup (a_{11}, a_{10}) \cup (a_{10}, a_9) \cup (a_9, a_8) \cup (a_8, a_7) \cup (a_7, a_6) \cup (a_6, a_5) \cup (a_5, a_4)$.

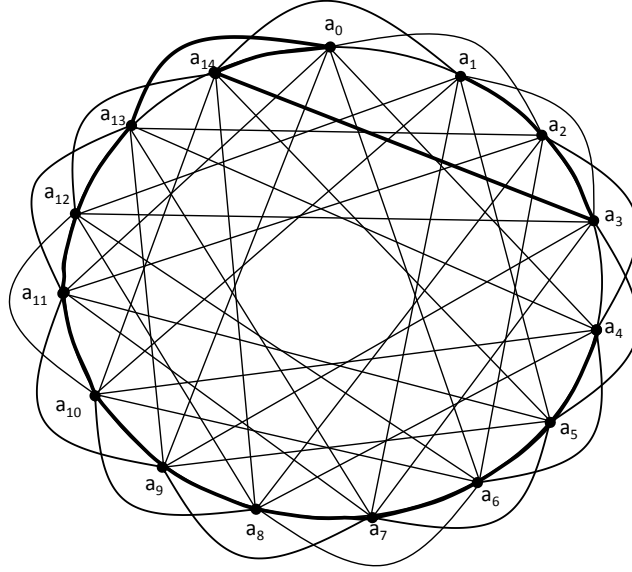


Figure 7: Hamiltonian path from a_1 to a_4 is shown by dark lines in $C_y(15, 3k)$

5 LACEABILITY IN SQUARE OF A GRAPH

Let G be a simple connected graph with n vertices. G^2 of G is the graph obtained by inserting edge between every two vertices u and v at a distance $d(u, v)=2$.

Theorem 5.1. *If $G = C(2n, 1)$ then $G^2 - G$ is Hamiltonian- t^* -laceable for $t=1, 2$.*

Proof. Let $G = C(2n, 1)$ be a graph of order n . Let $G^2 - G$ is the graph having same vertex as in G . Consider the following cases

Case(i): For $t=1$

Let $d(a_i, a_j)=1$ where $i=0$ and $j = \frac{n+2}{2}$ then there exists a path $P : a_i J[P]^{\frac{n-4}{2}} J J[P]^{\frac{n-4}{2}}$ is a Hamiltonian path from a_i to a_j . Hence G_1 is Hamiltonian-1*-laceable.

Case(ii): For $t=2$

Let $d(a_i, a_j)=2$ where $i=0$ and $j = \frac{n-4}{2}$

$P : a_i J P^{\frac{n-4}{2}} J J[P]^{\frac{n}{2}} K J$ is a Hamiltonian path from a_i to a_j . Hence G_1 is Hamiltonian-2*-laceable. $\square$

6 CONCLUSIONS

In this paper we have explored Hamiltonian properties of Cyclo product, the graph $G = (W_{1,n}, k)$ and square of a graph. Further, Hamiltonian- t -laceability properties of these graphs for higher values of t .

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