

# Evaluation of convolution sums as $\sum_{m=1}^{n-1} m\sigma_i(m)\sigma_j(n-m)$

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## Abstract

We calculate some special convolution sums related to the odd divisors multiplied by the even divisors. Also according to the Ramanujan's functions implying  $\sigma_1(n)$ ,  $\sigma_3(n)$ , and  $\sigma_5(n)$ , we obtain various convolution formulas.

*Keywords:* Convolution sum

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## 1 Introduction

The study of arithmetical identities is classical in number theory and such investigations have been carried out by several mathematicians including the legend Srinivasa Ramanujan.

For  $n, m \in \mathbb{N}$ ,  $s, r \in \mathbb{N} \cup \{0\}$ ,  $q \in \mathbb{C}$  with  $|q| < 1$ , we define some necessary divisor functions for later use, which also appear in many areas of number theory:

$$\begin{aligned}\sigma_s(n) &= \sum_{d|n} d^s, & \sigma_{s,r}(n; m) &= \sum_{\substack{d|n \\ d \equiv r \pmod{m}}} d^s, \\ \Delta(q) &:= \sum_{n=1}^{\infty} \tau(n)q^n = q \prod_{n=1}^{\infty} (1 - q^n)^{24}, \\ B(q) &:= \sum_{n=1}^{\infty} b(n)q^n = (\Delta(q)\Delta(q^2))^{\frac{1}{3}}, \\ C(q) &:= \sum_{n=1}^{\infty} c(n)q^n = (\Delta(q)^4 \Delta(q^2))^{\frac{1}{6}}, \\ D(q) &:= \sum_{n=1}^{\infty} d(n)q^n = (\Delta(q)^2 \Delta(q^2) \Delta(q^4)^2)^{\frac{1}{6}}.\end{aligned}\tag{1.1}$$

In general, it is satisfied that

$$b(n) = -8b\left(\frac{n}{2}\right) \quad (1.2)$$

for even  $n$  (see ([3], Remark 4.3)). We note that

$$\sigma_s(n) = \sigma_{s,1}(n; 2) + \sigma_{s,0}(n; 2) \quad (1.3)$$

and

$$\sigma_{s,1}(2n; 2) = \sigma_{s,1}(n; 2), \quad \sigma_{s,0}(2n; 2) = 2^s \sigma_s(n). \quad (1.4)$$

We also follow the definition of the convolution sum  $S_{e,f}(n)$  ( $e, f, n \in \mathbb{N}$ ) by

$$S_{e,f}(n) := \sum_{m=1}^{n-1} \sigma_e(m) \sigma_f(n-m), \quad (1.5)$$

which is found in ([1], p. 39). Ramanujan [8] and Lahiri [5], [6] have shown that  $S_{e,f}$  can be expressed as :

$$\begin{aligned} S_{1,1}(n) &= \frac{5}{12} \sigma_3(n) + \frac{(1-6n)}{12} \sigma_1(n), \\ S_{1,3}(n) &= \frac{7}{80} \sigma_5(n) + \frac{(1-3n)}{24} \sigma_3(n) - \frac{1}{240} \sigma_1(n), \\ S_{1,5}(n) &= \frac{5}{126} \sigma_7(n) + \frac{(1-2n)}{24} \sigma_5(n) + \frac{1}{504} \sigma_1(n), \\ S_{3,3}(n) &= \frac{1}{120} \sigma_7(n) - \frac{1}{120} \sigma_3(n), \\ S_{1,7}(n) &= \frac{11}{480} \sigma_9(n) + \frac{(2-3n)}{48} \sigma_7(n) - \frac{1}{480} \sigma_1(n), \\ S_{3,5}(n) &= \frac{11}{5040} \sigma_9(n) - \frac{1}{240} \sigma_5(n) + \frac{1}{504} \sigma_3(n), \\ S_{1,9}(n) &= \frac{455}{30404} \sigma_{11}(n) + \frac{(5-6n)}{120} \sigma_9(n) + \frac{1}{264} \sigma_1(n) - \frac{36}{3455} \tau(n), \\ S_{3,7}(n) &= \frac{91}{110560} \sigma_{11}(n) - \frac{1}{240} \sigma_7(n) - \frac{1}{480} \sigma_3(n) + \frac{15}{2764} \tau(n), \\ S_{5,5}(n) &= \frac{65}{174132} \sigma_{11}(n) + \frac{1}{252} \sigma_5(n) - \frac{3}{691} \tau(n), \\ S_{1,11}(n) &= \frac{691}{65520} \sigma_{13}(n) + \frac{(1-n)}{24} \sigma_{11}(n) - \frac{691}{65520} \tau(n), \\ S_{3,9}(n) &= \frac{1}{2640} \sigma_{13}(n) - \frac{1}{240} \sigma_9(n) + \frac{1}{264} \sigma_3(n), \\ S_{5,7}(n) &= \frac{1}{10080} \sigma_{13}(n) + \frac{1}{504} \sigma_7(n) - \frac{1}{480} \sigma_5(n). \end{aligned}$$

For  $e, f, m, n \in \mathbb{N}$  we set

$$T_{e,f}(n) := \sum_{m < \frac{n}{2}} \sigma_e(m) \sigma_f(n-2m) \quad (1.6)$$

and

$$I_{m,e,f}(n) := \sum_{m=1}^{n-1} m \sigma_e(m) \sigma_f(n-m). \quad (1.7)$$

And in ([9], p. 45–54) we can see that

$$\begin{aligned}
 T_{1,1}(n) &= \frac{1}{12}\sigma_3(n) + \frac{1}{3}\sigma_3\left(\frac{n}{2}\right) + \frac{(1-3n)}{24}\sigma_1(n) + \frac{(1-6n)}{24}\sigma_1\left(\frac{n}{2}\right), \\
 T_{3,1}(n) &= \frac{1}{240}\sigma_5(n) + \frac{1}{12}\sigma_5\left(\frac{n}{2}\right) + \frac{(1-3n)}{24}\sigma_3\left(\frac{n}{2}\right) - \frac{1}{240}\sigma_1(n), \\
 T_{1,3}(n) &= \frac{1}{48}\sigma_5(n) + \frac{1}{15}\sigma_5\left(\frac{n}{2}\right) + \frac{(2-3n)}{48}\sigma_3(n) - \frac{1}{240}\sigma_1\left(\frac{n}{2}\right), \\
 T_{5,1}(n) &= \frac{1}{2142}\sigma_7(n) + \frac{2}{51}\sigma_7\left(\frac{n}{2}\right) + \frac{(1-2n)}{24}\sigma_5\left(\frac{n}{2}\right) + \frac{1}{504}\sigma_1(n) - \frac{1}{408}b(n), \\
 T_{3,3}(n) &= \frac{1}{2040}\sigma_7(n) + \frac{2}{255}\sigma_7\left(\frac{n}{2}\right) - \frac{1}{240}\sigma_3(n) - \frac{1}{240}\sigma_3\left(\frac{n}{2}\right) + \frac{1}{272}b(n), \\
 T_{1,5}(n) &= \frac{1}{102}\sigma_7(n) + \frac{32}{1071}\sigma_7\left(\frac{n}{2}\right) + \frac{(1-n)}{24}\sigma_5(n) + \frac{1}{504}\sigma_1\left(\frac{n}{2}\right) - \frac{1}{102}b(n), \\
 T_{7,1}(n) &= \frac{1}{14880}\sigma_9(n) + \frac{17}{744}\sigma_9\left(\frac{n}{2}\right) + \frac{(2-3n)}{48}\sigma_7\left(\frac{n}{2}\right) - \frac{1}{480}\sigma_1(n) \\
 &\quad + \frac{1}{496}c(n) + \frac{2}{31}d(n), \\
 T_{5,3}(n) &= \frac{1}{31248}\sigma_9(n) + \frac{1}{465}\sigma_9\left(\frac{n}{2}\right) - \frac{1}{240}\sigma_5\left(\frac{n}{2}\right) + \frac{1}{504}\sigma_3(n) \\
 &\quad - \frac{1}{496}c(n) - \frac{2}{31}d(n), \\
 T_{3,5}(n) &= \frac{1}{7440}\sigma_9(n) + \frac{4}{1953}\sigma_9\left(\frac{n}{2}\right) - \frac{1}{240}\sigma_5(n) + \frac{1}{504}\sigma_3\left(\frac{n}{2}\right) \\
 &\quad + \frac{1}{248}c(n) + \frac{4}{31}d(n), \\
 T_{1,7}(n) &= \frac{17}{2976}\sigma_9(n) + \frac{8}{465}\sigma_9\left(\frac{n}{2}\right) + \frac{(4-3n)}{96}\sigma_7(n) - \frac{1}{480}\sigma_1\left(\frac{n}{2}\right) \\
 &\quad - \frac{1}{62}c(n) - \frac{16}{31}d(n), \\
 T_{9,1}(n) &= \frac{1}{91212}\sigma_{11}(n) + \frac{31}{2073}\sigma_{11}\left(\frac{n}{2}\right) + \frac{(5-6n)}{120}\sigma_9\left(\frac{n}{2}\right) + \frac{1}{264}\sigma_1(n) \\
 &\quad - \frac{21}{5528}\tau(n) - \frac{282}{3455}\tau\left(\frac{n}{2}\right), \\
 T_{7,3}(n) &= \frac{1}{331680}\sigma_{11}(n) + \frac{17}{20730}\sigma_{11}\left(\frac{n}{2}\right) - \frac{1}{240}\sigma_7\left(\frac{n}{2}\right) - \frac{1}{480}\sigma_3(n) \\
 &\quad + \frac{23}{11056}\tau(n) + \frac{91}{1382}\tau\left(\frac{n}{2}\right), \\
 T_{5,5}(n) &= \frac{1}{174132}\sigma_{11}(n) + \frac{16}{43533}\sigma_{11}\left(\frac{n}{2}\right) + \frac{1}{504}\sigma_5(n) + \frac{1}{504}\sigma_5\left(\frac{n}{2}\right) \\
 &\quad - \frac{11}{5528}\tau(n) - \frac{88}{691}\tau\left(\frac{n}{2}\right), \\
 T_{3,7}(n) &= \frac{17}{331680}\sigma_{11}(n) + \frac{8}{10365}\sigma_{11}\left(\frac{n}{2}\right) - \frac{1}{240}\sigma_7(n) - \frac{1}{480}\sigma_3\left(\frac{n}{2}\right) \\
 &\quad + \frac{91}{22112}\tau(n) + \frac{368}{691}\tau\left(\frac{n}{2}\right), \\
 T_{1,9}(n) &= \frac{31}{8292}\sigma_{11}(n) + \frac{256}{22803}\sigma_{11}\left(\frac{n}{2}\right) + \frac{(5-3n)}{120}\sigma_9(n) + \frac{1}{264}\sigma_1\left(\frac{n}{2}\right) \\
 &\quad - \frac{141}{6910}\tau(n) - \frac{2688}{691}\tau\left(\frac{n}{2}\right).
 \end{aligned}$$

In addition, Lahiri [5] gave the value of the sum

$$\sum_{m_1 + \dots + m_r = n} m_1^{a_1} \dots m_r^{a_r} \sigma_{b_1}(m_1) \dots \sigma_{b_r}(m_r) \quad (r \geq 3),$$

where the sum is over all positive integers  $m_1, \dots, m_r$  satisfying  $m_1 + \dots + m_r = n$ ,  $a_i \in \mathbb{N} \cup \{0\}$ , and  $b_i \in \mathbb{N}$ . For some of the history of the subject, and for a selection of these articles, we mention [1], [5] and [7], and especially [2] and [9].

In this paper, we obtain some formulas for the various convolution sums related to  $\sigma_{s,1}(n; 2)$ ,  $\sigma_{s,0}(n; 2)$  and  $I_{m,e,f}(n)$ . For instance, we introduce a main theorem as follows :

**Theorem 1.1.** *Let  $n \in \mathbb{N}$ . Then we have*

(a)

$$\begin{aligned} & \sum_{m=1}^{n-1} m \sigma_{3,1}(m; 2) \sigma_{1,1}(n-m; 2) \\ &= \frac{1}{120} \left[ n \left\{ \sigma_5(n) + 48 \sigma_5\left(\frac{n}{2}\right) - 5 \sigma_3(n) + 40 \sigma_3\left(\frac{n}{2}\right) \right\} + 4b(n) \right]. \end{aligned}$$

(b)

$$\begin{aligned} & \sum_{m=1}^{n-1} m \sigma_{3,0}(m; 2) \sigma_{1,1}(n-m; 2) \\ &= \frac{1}{45} \left[ n \left\{ \sigma_5(n) - 22 \sigma_5\left(\frac{n}{2}\right) - 15 \sigma_3\left(\frac{n}{2}\right) \right\} - b(n) \right]. \end{aligned}$$

(c)

$$\sum_{m=1}^{n-1} m \sigma_{3,0}(m; 2) \sigma_{1,0}(n-m; 2) = \frac{2}{15} n \left\{ 7 \sigma_5\left(\frac{n}{2}\right) - (6n-5) \sigma_3\left(\frac{n}{2}\right) \right\}.$$

(d)

$$\begin{aligned} & \sum_{m=1}^{n-1} m \sigma_{3,1}(m; 2) \sigma_{1,0}(n-m; 2) \\ &= \frac{1}{180} \left[ n \left\{ 5 \sigma_5(n) - 152 \sigma_5\left(\frac{n}{2}\right) - 3(6n-5) \sigma_{3,1}(n; 2) \right\} - 2b(n) \right]. \end{aligned}$$

## 2 The Convolution sums related to odd or even divisor function

**Corollary 2.1.** *Let  $n \in \mathbb{N}$ . Then we have*

(a)

$$\begin{aligned} & \sum_{m=1}^{n-1} \sigma_{1,1}(m; 2) \sigma_{3,0}(n-m; 2) \\ &= \frac{1}{30} \left\{ \sigma_5(n) - 22 \sigma_5\left(\frac{n}{2}\right) - 10 \sigma_3\left(\frac{n}{2}\right) - \sigma_1(n) + 2 \sigma_1\left(\frac{n}{2}\right) \right\}. \end{aligned}$$

(b)

$$\begin{aligned} & \sum_{m=1}^{n-1} \sigma_{1,1}(m; 2) \sigma_{5,0}(n-m; 2) \\ &= \frac{4}{1071} \left\{ 4\sigma_7(n) - 344\sigma_7\left(\frac{n}{2}\right) - 357\sigma_5\left(\frac{n}{2}\right) + 17\sigma_1(n) - 34\sigma_1\left(\frac{n}{2}\right) - 21b(n) \right\}. \end{aligned}$$

(c)

$$\begin{aligned} & \sum_{m=1}^{n-1} \sigma_{1,1}(m; 2) \sigma_{7,0}(n-m; 2) \\ &= \frac{4}{465} \left\{ \sigma_9(n) - 342\sigma_9\left(\frac{n}{2}\right) - 620\sigma_7\left(\frac{n}{2}\right) - 31\sigma_1(n) + 62\sigma_1\left(\frac{n}{2}\right) \right. \\ & \quad \left. + 30c(n) + 960d(n) \right\}. \end{aligned}$$

(d)

$$\begin{aligned} & \sum_{m=1}^{n-1} \sigma_{1,1}(m; 2) \sigma_{9,0}(n-m; 2) \\ &= \frac{64}{22803} \left\{ 2\sigma_{11}(n) - 2732\sigma_{11}\left(\frac{n}{2}\right) - 7601\sigma_9\left(\frac{n}{2}\right) + 691\sigma_1(n) \right. \\ & \quad \left. - 1382\sigma_1\left(\frac{n}{2}\right) - 693\tau(n) - 11088\tau\left(\frac{n}{2}\right) \right\}. \end{aligned}$$

(e)

$$\begin{aligned} & \sum_{m=1}^{n-1} \sigma_{3,1}(m; 2) \sigma_{3,0}(n-m; 2) \\ &= \frac{1}{510} \left\{ 2\sigma_7(n) - 240\sigma_7\left(\frac{n}{2}\right) - 17\sigma_3(n) + 255\sigma_3\left(\frac{n}{2}\right) + 15b(n) \right\}. \end{aligned}$$

(f)

$$\begin{aligned} & \sum_{m=1}^{n-1} \sigma_{3,1}(m; 2) \sigma_{5,0}(n-m; 2) \\ &= \frac{2}{9765} \left\{ 5\sigma_9(n) - 2392\sigma_9\left(\frac{n}{2}\right) + 4557\sigma_5\left(\frac{n}{2}\right) + 310\sigma_3(n) - 2480\sigma_3\left(\frac{n}{2}\right) \right. \\ & \quad \left. - 315c(n) - 10080d(n) \right\}. \end{aligned}$$

(g)

$$\begin{aligned} & \sum_{m=1}^{n-1} \sigma_{3,1}(m; 2) \sigma_{7,0}(n-m; 2) \\ &= \frac{4}{10365} \left\{ \sigma_{11}(n) - 1912\sigma_{11}\left(\frac{n}{2}\right) + 9674\sigma_7\left(\frac{n}{2}\right) - 691\sigma_3(n) + 5528\sigma_3\left(\frac{n}{2}\right) \right. \\ & \quad \left. + 690\tau(n) + 7440\tau\left(\frac{n}{2}\right) \right\}. \end{aligned}$$

(h)

$$\begin{aligned} & \sum_{m=1}^{n-1} \sigma_{5,1}(m; 2) \sigma_{5,0}(n-m; 2) \\ &= \frac{4}{43533} \left\{ 2\sigma_{11}(n) - 4032\sigma_{11}\left(\frac{n}{2}\right) + 691\sigma_5(n) - 43533\sigma_5\left(\frac{n}{2}\right) \right. \\ & \quad \left. - 693\tau(n) + 4032\tau\left(\frac{n}{2}\right) \right\}. \end{aligned}$$

*Proof.* For  $e, f \in \mathbb{N}$  and (1.4), we note that

$$\begin{aligned} & \sum_{m=1}^{n-1} \sigma_{e,1}(m; 2) \sigma_{f,0}(n-m; 2) = \sum_{m=1}^{n-1} \sigma_{e,1}(m; 2) \cdot 2^f \sigma_f\left(\frac{n-m}{2}\right) \\ &= 2^f \sum_{m < \frac{n}{2}} \sigma_{e,1}(n-2m; 2) \sigma_f(m) \\ &= 2^f \left\{ \sum_{m < \frac{n}{2}} \sigma_e(n-2m) \sigma_f(m) - \sum_{m < \frac{n}{2}} \sigma_{e,0}(n-2m; 2) \sigma_f(m) \right\} \\ &= 2^f \left\{ \sum_{m < \frac{n}{2}} \sigma_e(n-2m) \sigma_f(m) - 2^e \sum_{m < \frac{n}{2}} \sigma_e\left(\frac{n}{2}-m\right) \sigma_f(m) \right\} \\ &= 2^f \left\{ T_{f,e}(n) - 2^e S_{f,e}\left(\frac{n}{2}\right) \right\}. \end{aligned}$$

□

*Remark 2.1.* Finding the formula of  $\sum_{m=1}^{n-1} \sigma_{1,1}(m; 2) \sigma_{1,0}(n-m; 2)$ , it is similar to Corollary 2.1 except for using  $S_{1,1}(n)$  and  $T_{1,1}(n)$ . But here, we introduce another method. Now, we can see that

$$\sum_{m=1}^{n-1} m \sigma_1(m) \sigma_1(n-m) = \frac{5}{24} n \sigma_3(n) + \left( \frac{1}{24} n - \frac{1}{4} n^2 \right) \sigma_1(n) \quad (2.1)$$

in ([9], Theorem 12.7). And we expand the above equation as :

$$\begin{aligned} & \sum_{m=1}^{n-1} m \sigma_1(m) \sigma_1(n-m) \\ &= \sum_{m=1}^{n-1} m \{ \sigma_{1,1}(m; 2) + \sigma_{1,0}(m; 2) \} \{ \sigma_{1,1}(n-m; 2) + \sigma_{1,0}(n-m; 2) \} \\ &= \sum_{m=1}^{n-1} m \sigma_{1,1}(m; 2) \sigma_{1,1}(n-m; 2) + \sum_{m=1}^{n-1} m \sigma_{1,1}(m; 2) \sigma_{1,0}(n-m; 2) \\ & \quad + \sum_{m=1}^{n-1} m \sigma_{1,0}(m; 2) \sigma_{1,1}(n-m; 2) + \sum_{m=1}^{n-1} m \sigma_{1,0}(m; 2) \sigma_{1,0}(n-m; 2) \\ &= \sum_{m=1}^{n-1} m \sigma_{1,1}(m; 2) \sigma_{1,1}(n-m; 2) + n \sum_{m=1}^{n-1} \sigma_{1,1}(m; 2) \sigma_{1,0}(n-m; 2) \\ & \quad + 4 \sum_{m=1}^{n-1} m \sigma_1\left(\frac{m}{2}\right) \sigma_1\left(\frac{n-m}{2}\right), \end{aligned} \quad (2.2)$$

where we use

$$\sum_{m=1}^{n-1} m\sigma_{1,1}(m; 2)\sigma_{1,0}(n-m; 2) = \sum_{m=1}^{n-1} (n-m)\sigma_{1,1}(n-m; 2)\sigma_{1,0}(m; 2)$$

and  $\sigma_{1,0}(2n; 2) = 2\sigma_1(n)$  for the last line. Proceeding the Eq. (2.2), we obtain

$$\begin{aligned} & \sum_{m=1}^{n-1} m\sigma_1(m)\sigma_1(n-m) \\ &= \sum_{m=1}^{n-1} m\sigma_{1,1}(m; 2)\sigma_{1,1}(n-m; 2) + n \sum_{m=1}^{n-1} \sigma_{1,1}(m; 2)\sigma_{1,0}(n-m; 2) \\ & \quad + 8 \sum_{m < \frac{n}{2}} m\sigma_1(m)\sigma_1\left(\frac{n}{2}-m\right). \end{aligned}$$

So we refer to the following equation

$$\sum_{m=1}^{n-1} m\sigma_{1,1}(m; 2)\sigma_{1,1}(n-m; 2) = \frac{n}{48} \{11\sigma_3(n) - \sigma_3(2n) - 2\sigma_{1,1}(n; 2)\} \quad (2.3)$$

in ([4], Lemma 3.5) and (2.1). Therefore, we obtain

$$\begin{aligned} & \sum_{m=1}^{n-1} \sigma_{1,1}(m; 2)\sigma_{1,0}(n-m; 2) \\ &= \frac{1}{12} \left\{ 2 \left( \sigma_3(n) - 6\sigma_3\left(\frac{n}{2}\right) \right) - (3n-1)\sigma_1(n) + 3(2n-1)\sigma_1\left(\frac{n}{2}\right) \right\} \end{aligned}$$

for  $n \in \mathbb{N}$ .

**Corollary 2.2.** *Let  $n \in \mathbb{N}$ . Then we have*

(a)

$$\begin{aligned} & \sum_{m=1}^{n-1} \sigma_{1,0}(m; 2)\sigma_{3,1}(n-m; 2) \\ &= \frac{1}{120} \left\{ 5\sigma_5(n) - 152\sigma_5\left(\frac{n}{2}\right) - 15n\sigma_3(n) + 10\sigma_3(n) \right. \\ & \quad \left. + 40(3n-2)\sigma_3\left(\frac{n}{2}\right) + 7\sigma_1\left(\frac{n}{2}\right) \right\}. \end{aligned}$$

(b)

$$\begin{aligned} & \sum_{m=1}^{n-1} \sigma_{1,0}(m; 2)\sigma_{5,1}(n-m; 2) \\ &= \frac{1}{4284} \left\{ 84\sigma_7(n) - 10624\sigma_7\left(\frac{n}{2}\right) - 357n\sigma_5(n) + 357\sigma_5(n) + 11424n\sigma_5\left(\frac{n}{2}\right) \right. \\ & \quad \left. - 11424\sigma_5\left(\frac{n}{2}\right) - 527\sigma_1\left(\frac{n}{2}\right) - 84b(n) \right\}. \end{aligned}$$

(c)

$$\begin{aligned} & \sum_{m=1}^{n-1} \sigma_{1,0}(m; 2) \sigma_{7,1}(n-m; 2) \\ &= \frac{1}{7440} \left\{ 85\sigma_9(n) - 43392\sigma_9\left(\frac{n}{2}\right) - 465n\sigma_7(n) + 620\sigma_7(n) + 59520n\sigma_7\left(\frac{n}{2}\right) \right. \\ & \quad \left. - 79360\sigma_7\left(\frac{n}{2}\right) + 3937\sigma_1\left(\frac{n}{2}\right) - 240c(n) - 7680d(n) \right\}. \end{aligned}$$

(d)

$$\begin{aligned} & \sum_{m=1}^{n-1} \sigma_{1,0}(m; 2) \sigma_{9,1}(n-m; 2) \\ &= \frac{1}{456060} \left\{ 3410\sigma_{11}(n) - 6978560\sigma_{11}\left(\frac{n}{2}\right) - 22803n\sigma_9(n) + 38005\sigma_9(n) \right. \\ & \quad + 11675136n\sigma_9\left(\frac{n}{2}\right) - 19458560\sigma_9\left(\frac{n}{2}\right) - 1765505\sigma_1\left(\frac{n}{2}\right) - 18612\tau(n) \\ & \quad \left. + 1317888\tau\left(\frac{n}{2}\right) \right\}. \end{aligned}$$

(e)

$$\begin{aligned} & \sum_{m=1}^{n-1} \sigma_{3,0}(m; 2) \sigma_{5,1}(n-m; 2) \\ &= \frac{1}{19530} \left\{ 21\sigma_9(n) - 10592\sigma_9\left(\frac{n}{2}\right) - 651\sigma_5(n) + 20832\sigma_5\left(\frac{n}{2}\right) - 9610\sigma_3\left(\frac{n}{2}\right) \right. \\ & \quad \left. + 630c(n) + 20160d(n) \right\}. \end{aligned}$$

(f)

$$\begin{aligned} & \sum_{m=1}^{n-1} \sigma_{3,0}(m; 2) \sigma_{7,1}(n-m; 2) \\ &= \frac{1}{41460} \left\{ 17\sigma_{11}(n) - 34688\sigma_{11}\left(\frac{n}{2}\right) - 1382\sigma_7(n) + 176896\sigma_7\left(\frac{n}{2}\right) \right. \\ & \quad \left. + 87757\sigma_3\left(\frac{n}{2}\right) + 1365\tau(n) - 53760\tau\left(\frac{n}{2}\right) \right\}. \end{aligned}$$

*Proof.* For  $e, f \in \mathbb{N}$  and (1.4), we note that

$$\begin{aligned} & \sum_{m=1}^{n-1} \sigma_{e,0}(m; 2) \sigma_{f,1}(n-m; 2) = \sum_{m=1}^{n-1} 2^e \sigma_e\left(\frac{m}{2}\right) \sigma_{f,1}(n-m; 2) \\ &= 2^e \sum_{m < \frac{n}{2}} \sigma_e(m) \sigma_{f,1}(n-2m; 2) \\ &= 2^e \left\{ \sum_{m < \frac{n}{2}} \sigma_e(m) \sigma_f(n-2m) - \sum_{m < \frac{n}{2}} \sigma_e(m) \sigma_{f,0}(n-2m; 2) \right\} \\ &= 2^e \left\{ \sum_{m < \frac{n}{2}} \sigma_e(m) \sigma_f(n-2m) - 2^f \sum_{m < \frac{n}{2}} \sigma_e(m) \sigma_f\left(\frac{n}{2}-m\right) \right\} \\ &= 2^e \left\{ T_{e,f}(n) - 2^f S_{e,f}\left(\frac{n}{2}\right) \right\}. \end{aligned}$$

□

**Corollary 2.3.** *Let  $n \in \mathbb{N}$ . Then we have*

(a)

$$\begin{aligned} & \sum_{m=1}^{n-1} \sigma_{1,0}(m; 2) \sigma_{3,0}(n-m; 2) \\ &= \frac{1}{15} \left\{ 21\sigma_5\left(\frac{n}{2}\right) - 5(3n-2)\sigma_3\left(\frac{n}{2}\right) - \sigma_1\left(\frac{n}{2}\right) \right\}. \end{aligned}$$

(b)

$$\begin{aligned} & \sum_{m=1}^{n-1} \sigma_{1,0}(m; 2) \sigma_{5,0}(n-m; 2) \\ &= \frac{8}{63} \left\{ 20\sigma_7\left(\frac{n}{2}\right) - 21(n-1)\sigma_5\left(\frac{n}{2}\right) + \sigma_1\left(\frac{n}{2}\right) \right\}. \end{aligned}$$

(c)

$$\begin{aligned} & \sum_{m=1}^{n-1} \sigma_{1,0}(m; 2) \sigma_{7,0}(n-m; 2) \\ &= \frac{8}{15} \left\{ 11\sigma_9\left(\frac{n}{2}\right) - 5(3n-4)\sigma_7\left(\frac{n}{2}\right) - \sigma_1\left(\frac{n}{2}\right) \right\}. \end{aligned}$$

(d)

$$\begin{aligned} & \sum_{m=1}^{n-1} \sigma_{1,0}(m; 2) \sigma_{9,0}(n-m; 2) \\ &= \frac{128}{114015} \left\{ 13650\sigma_{11}\left(\frac{n}{2}\right) - 7601(3n-5)\sigma_9\left(\frac{n}{2}\right) + 3455\sigma_1\left(\frac{n}{2}\right) - 9504\tau\left(\frac{n}{2}\right) \right\}. \end{aligned}$$

(e)

$$\sum_{m=1}^{n-1} \sigma_{3,0}(m; 2) \sigma_{3,0}(n-m; 2) = \frac{8}{15} \left\{ \sigma_7\left(\frac{n}{2}\right) - \sigma_3\left(\frac{n}{2}\right) \right\}.$$

(f)

$$\begin{aligned} & \sum_{m=1}^{n-1} \sigma_{3,0}(m; 2) \sigma_{5,0}(n-m; 2) \\ &= \frac{16}{315} \left\{ 11\sigma_9\left(\frac{n}{2}\right) - 21\sigma_5\left(\frac{n}{2}\right) + 10\sigma_3\left(\frac{n}{2}\right) \right\}. \end{aligned}$$

(g)

$$\begin{aligned} & \sum_{m=1}^{n-1} \sigma_{3,0}(m; 2) \sigma_{7,0}(n-m; 2) \\ &= \frac{32}{10365} \left\{ 273\sigma_{11}\left(\frac{n}{2}\right) - 1382\sigma_7\left(\frac{n}{2}\right) - 691\sigma_3\left(\frac{n}{2}\right) + 1800\tau\left(\frac{n}{2}\right) \right\}. \end{aligned}$$

(h)

$$\begin{aligned} & \sum_{m=1}^{n-1} \sigma_{5,0}(m; 2) \sigma_{5,0}(n-m; 2) \\ &= \frac{256}{43533} \left\{ 65\sigma_{11}\left(\frac{n}{2}\right) + 691\sigma_5\left(\frac{n}{2}\right) - 756\tau\left(\frac{n}{2}\right) \right\}. \end{aligned}$$

*Proof.* For  $e, f \in \mathbb{N}$  and (1.4), we note that

$$\begin{aligned} \sum_{m=1}^{n-1} \sigma_{e,0}(m; 2) \sigma_{f,0}(n-m; 2) &= \sum_{m=1}^{n-1} 2^e \sigma_e\left(\frac{m}{2}\right) \cdot 2^f \sigma_f\left(\frac{n-m}{2}\right) \\ &= 2^{e+f} \sum_{m < \frac{n}{2}} \sigma_e(m) \sigma_f\left(\frac{n}{2} - m\right) = 2^{e+f} S_{e,f}\left(\frac{n}{2}\right). \end{aligned}$$

□

**Corollary 2.4.** *Let  $n \in \mathbb{N}$ . Then we have*

(a)

$$\begin{aligned} &\sum_{m=1}^{n-1} \sigma_{1,1}(m; 2) \sigma_{3,1}(n-m; 2) \\ &= \frac{1}{240} \left\{ 3\sigma_5(n) + 144\sigma_5\left(\frac{n}{2}\right) - 10\sigma_3(n) + 80\sigma_3\left(\frac{n}{2}\right) + 7\sigma_1(n) - 14\sigma_1\left(\frac{n}{2}\right) \right\}. \end{aligned}$$

(b)

$$\begin{aligned} &\sum_{m=1}^{n-1} \sigma_{1,1}(m; 2) \sigma_{5,1}(n-m; 2) \\ &= \frac{1}{8568} \left\{ 44\sigma_7(n) + 10496\sigma_7\left(\frac{n}{2}\right) - 357\sigma_5(n) + 11424\sigma_5\left(\frac{n}{2}\right) \right. \\ &\quad \left. - 527\sigma_1(n) + 1054\sigma_1\left(\frac{n}{2}\right) + 840b(n) \right\}. \end{aligned}$$

(c)

$$\begin{aligned} &\sum_{m=1}^{n-1} \sigma_{1,1}(m; 2) \sigma_{7,1}(n-m; 2) \\ &= \frac{1}{14880} \left\{ 43\sigma_9(n) + 43264\sigma_9\left(\frac{n}{2}\right) - 620\sigma_7(n) + 79360\sigma_7\left(\frac{n}{2}\right) \right. \\ &\quad \left. + 3937\sigma_1(n) - 7874\sigma_1\left(\frac{n}{2}\right) - 3360c(n) - 107520d(n) \right\}. \end{aligned}$$

(d)

$$\begin{aligned} &\sum_{m=1}^{n-1} \sigma_{1,1}(m; 2) \sigma_{9,1}(n-m; 2) \\ &= \frac{1}{182424} \left\{ 342\sigma_{11}(n) + 1394688\sigma_{11}\left(\frac{n}{2}\right) - 7601\sigma_9(n) + 3891712\sigma_9\left(\frac{n}{2}\right) \right. \\ &\quad \left. - 353101\sigma_1(n) + 706202\sigma_1\left(\frac{n}{2}\right) + 360360\tau(n) + 7096320\tau\left(\frac{n}{2}\right) \right\}. \end{aligned}$$

(e)

$$\begin{aligned} &\sum_{m=1}^{n-1} \sigma_{3,1}(m; 2) \sigma_{3,1}(n-m; 2) \\ &= \frac{1}{2040} \left\{ \sigma_7(n) + 832\sigma_7\left(\frac{n}{2}\right) + 119\sigma_3(n) - 952\sigma_3\left(\frac{n}{2}\right) - 120b(n) \right\}. \end{aligned}$$

(f)

$$\begin{aligned} & \sum_{m=1}^{n-1} \sigma_{3,1}(m; 2) \sigma_{5,1}(n-m; 2) \\ &= \frac{1}{156240} \left\{ 13\sigma_9(n) + 73984\sigma_9\left(\frac{n}{2}\right) + 4557\sigma_5(n) - 145824\sigma_5\left(\frac{n}{2}\right) \right. \\ & \quad \left. - 9610\sigma_3(n) + 76880\sigma_3\left(\frac{n}{2}\right) + 5040c(n) + 161280d(n) \right\}. \end{aligned}$$

(g)

$$\begin{aligned} & \sum_{m=1}^{n-1} \sigma_{3,1}(m; 2) \sigma_{7,1}(n-m; 2) \\ &= \frac{1}{331680} \left\{ 9\sigma_{11}(n) + 242688\sigma_{11}\left(\frac{n}{2}\right) + 9674\sigma_7(n) - 1238272\sigma_7\left(\frac{n}{2}\right) \right. \\ & \quad \left. + 87757\sigma_3(n) - 702056\sigma_3\left(\frac{n}{2}\right) - 97440\tau(n) - 2365440\tau\left(\frac{n}{2}\right) \right\}. \end{aligned}$$

(h)

$$\begin{aligned} & \sum_{m=1}^{n-1} \sigma_{5,1}(m; 2) \sigma_{5,1}(n-m; 2) \\ &= \frac{1}{174132} \left\{ \sigma_{11}(n) + 62464\sigma_{11}\left(\frac{n}{2}\right) - 21421\sigma_5(n) + 685472\sigma_5\left(\frac{n}{2}\right) \right. \\ & \quad \left. + 21420\tau(n) + 645120\tau\left(\frac{n}{2}\right) \right\}. \end{aligned}$$

*Proof.* For  $e, f \in \mathbb{N}$ , we have

$$\begin{aligned} S_{e,f}(n) &= \sum_{m=1}^{n-1} \sigma_e(m) \sigma_f(n-m) \\ &= \sum_{m=1}^{n-1} \{ \sigma_{e,1}(m; 2) + \sigma_{e,0}(m; 2) \} \{ \sigma_{f,1}(n-m; 2) + \sigma_{f,0}(n-m; 2) \} \\ &= \sum_{m=1}^{n-1} \sigma_{e,1}(m; 2) \sigma_{f,1}(n-m; 2) + \sum_{m=1}^{n-1} \sigma_{e,1}(m; 2) \sigma_{f,0}(n-m; 2) \\ & \quad + \sum_{m=1}^{n-1} \sigma_{e,0}(m; 2) \sigma_{f,1}(n-m; 2) + \sum_{m=1}^{n-1} \sigma_{e,0}(m; 2) \sigma_{f,0}(n-m; 2). \end{aligned} \tag{2.4}$$

Thus, applying the proofs of Corollary 2.1, Corollary 2.2, and Corollary 2.3 into (2.4), we obtain

$$\begin{aligned} S_{e,f}(n) &= \sum_{m=1}^{n-1} \sigma_{e,1}(m; 2) \sigma_{f,1}(n-m; 2) + 2^f \left\{ T_{f,e}(n) - 2^e S_{f,e}\left(\frac{n}{2}\right) \right\} \\ & \quad + 2^e \left\{ T_{e,f}(n) - 2^f S_{e,f}\left(\frac{n}{2}\right) \right\} + 2^{e+f} S_{e,f}\left(\frac{n}{2}\right) \\ &= \sum_{m=1}^{n-1} \sigma_{e,1}(m; 2) \sigma_{f,1}(n-m; 2) + 2^f T_{f,e}(n) + 2^e T_{e,f}(n) - 2^{e+f} S_{e,f}\left(\frac{n}{2}\right). \end{aligned}$$

This leads to

$$\begin{aligned} & \sum_{m=1}^{n-1} \sigma_{e,1}(m;2)\sigma_{f,1}(n-m;2) \\ &= S_{e,f}(n) - 2^f T_{f,e}(n) - 2^e T_{e,f}(n) + 2^{e+f} S_{e,f}\left(\frac{n}{2}\right). \end{aligned}$$

□

**Corollary 2.5.** *Let  $n \in \mathbb{N}$ . Then we have*

(a)

$$\sum_{m=1}^n \sigma_1(2m-1)\sigma_3(2n-2m+1) = \sigma_5(n) - \sigma_5\left(\frac{n}{2}\right).$$

(b)

$$\begin{aligned} & \sum_{m=1}^n \sigma_1(2m-1;2)\sigma_5(2n-2m+1) \\ &= \frac{1}{17} \left[ 32 \left\{ \sigma_7(n) - \sigma_7\left(\frac{n}{2}\right) \right\} - 15b(n) \right]. \end{aligned}$$

(c)

$$\begin{aligned} & \sum_{m=1}^n \sigma_1(2m-1)\sigma_7(2n-2m+1) \\ &= \frac{1}{31} \left[ 8 \left\{ 17\sigma_9(n) - 17\sigma_9\left(\frac{n}{2}\right) - 28d(2n) + 28d(n) \right\} - 7c(2n) + 7c(n) \right]. \end{aligned}$$

(d)

$$\begin{aligned} & \sum_{m=1}^n \sigma_1(2m-1)\sigma_9(2n-2m+1) \\ &= \frac{1}{691} \left[ 7936\sigma_{11}(n) - 7936\sigma_{11}\left(\frac{n}{2}\right) + 105 \left\{ 13\tau(2n) + 243\tau(n) - 256\tau\left(\frac{n}{2}\right) \right\} \right]. \end{aligned}$$

(e)

$$\sum_{m=1}^n \sigma_3(2m-1)\sigma_3(2n-2m+1) = \frac{1}{17} \left\{ 8\sigma_7(n) - 8\sigma_7\left(\frac{n}{2}\right) + 9b(n) \right\}.$$

(f)

$$\begin{aligned} & \sum_{m=1}^n \sigma_3(2m-1)\sigma_5(2n-2m+1) \\ &= \frac{1}{31} \left[ 16 \left\{ \sigma_9(n) - \sigma_9\left(\frac{n}{2}\right) + 2d(2n) - 2d(n) \right\} + c(2n) - c(n) \right]. \end{aligned}$$

(g)

$$\begin{aligned} & \sum_{m=1}^n \sigma_3(2m-1)\sigma_7(2n-2m+1) \\ &= \frac{1}{691} \left[ 544\sigma_{11}(n) - 544\sigma_{11}\left(\frac{n}{2}\right) - 7 \left\{ 29\tau(2n) + 675\tau(n) - 704\tau\left(\frac{n}{2}\right) \right\} \right]. \end{aligned}$$

(h)

$$\begin{aligned} & \sum_{m=1}^n \sigma_5(2m-1)\sigma_5(2n-2m+1) \\ &= \frac{1}{691} \left[ 256\sigma_{11}(n) - 256\sigma_{11}\left(\frac{n}{2}\right) + 85\tau(2n) + 2475\tau(n) - 2560\tau\left(\frac{n}{2}\right) \right]. \end{aligned}$$

*Proof.* Let us consider the case  $n \rightarrow 2n$  in Corollary 2.4. Then we obtain

$$\begin{aligned} & \sum_{m=1}^{2n-1} \sigma_{e,1}(m;2)\sigma_{f,1}(2n-m;2) \\ &= \sum_{\substack{m=1 \\ 2 \nmid m}}^{2n-1} \sigma_{e,1}(m;2)\sigma_{f,1}(2n-m;2) + \sum_{\substack{m=1 \\ 2 \mid m}}^{2n-1} \sigma_{e,1}(m;2)\sigma_{f,1}(2n-m;2) \\ &= \sum_{m=1}^n \sigma_{e,1}(2m-1;2)\sigma_{f,1}(2n-2m+1;2) + \sum_{m=1}^{n-1} \sigma_{e,1}(m;2)\sigma_{f,1}(n-m;2) \end{aligned}$$

by  $\sigma_{e,1}(2n;2) = \sigma_{e,1}(n;2)$ . Thus we conclude that

$$\begin{aligned} & \sum_{m=1}^n \sigma_e(2m-1)\sigma_f(2n-2m+1) \\ &= \sum_{m=1}^n \sigma_{e,1}(2m-1;2)\sigma_{f,1}(2n-2m+1;2) \\ &= \sum_{m=1}^{2n-1} \sigma_{e,1}(m;2)\sigma_{f,1}(2n-m;2) - \sum_{m=1}^{n-1} \sigma_{e,1}(m;2)\sigma_{f,1}(n-m;2). \end{aligned}$$

Now note that  $\sigma_{e,1}(\text{odd};2) = \sigma_e(\text{odd})$ . Especially, to complete the proof of Corollary 2.5 (b) and (e), we refer to the Eq. (1.2). □

**Lemma 2.6.** Let  $n \in \mathbb{N}$ . Then we have

(a)

$$\begin{aligned} & \sum_{m=1}^{n-1} \sigma_1(2m)\sigma_3(2n-2m) \\ &= \frac{1}{240} \left\{ 453\sigma_5(n) - 432\sigma_5\left(\frac{n}{2}\right) - 10(6n-1)\sigma_3(2n) - \sigma_1(2n) \right\}. \end{aligned}$$

(b)

$$\begin{aligned} & \sum_{m=1}^{n-1} \sigma_1(2m)\sigma_5(2n-2m) \\ &= \frac{1}{8568} \left\{ 27732\sigma_7(n) - 27392\sigma_7\left(\frac{n}{2}\right) - 1428n\sigma_5(2n) + 357\sigma_5(2n) \right. \\ & \quad \left. + 17\sigma_1(2n) + 7560b(n) \right\}. \end{aligned}$$

(c)

$$\begin{aligned} & \sum_{m=1}^{n-1} \sigma_1(2m)\sigma_7(2n-2m) \\ &= \frac{1}{14880} \left\{ 109653\sigma_9(n) - 109312\sigma_9\left(\frac{n}{2}\right) - 1860n\sigma_7(2n) \right. \\ & \quad \left. + 620\sigma_7(2n) - 31\sigma_1(2n) + 107520d(2n) - 107520d(n) \right. \\ & \quad \left. + 3360c(2n) - 3360c(n) \right\}. \end{aligned}$$

(d)

$$\begin{aligned} & \sum_{m=1}^{n-1} \sigma_1(2m)\sigma_9(2n-2m) \\ &= \frac{1}{912120} \left[ 6 \left\{ 291555\sigma_{11}(n) - 2913280\sigma_{11}\left(\frac{n}{2}\right) \right\} - 7601(12n-5)\sigma_9(2n) \right. \\ & \quad \left. + 3455\sigma_1(2n) - 792 \left\{ 2287\tau(2n) + 42525\tau(n) - 44800\tau\left(\frac{n}{2}\right) \right\} \right]. \end{aligned}$$

(e)

$$\begin{aligned} & \sum_{m=1}^{n-1} \sigma_3(2m)\sigma_3(2n-2m) \\ &= \frac{1}{2040} \left\{ 1233\sigma_7(n) - 1216\sigma_7\left(\frac{n}{2}\right) - 17\sigma_3(2n) - 1080b(n) \right\}. \end{aligned}$$

(f)

$$\begin{aligned} & \sum_{m=1}^{n-1} \sigma_3(2m)\sigma_5(2n-2m) \\ &= \frac{1}{156240} \left\{ 94293\sigma_9(n) - 93952\sigma_9\left(\frac{n}{2}\right) - 651\sigma_5(2n) + 310\sigma_3(2n) \right. \\ & \quad \left. - 161280d(2n) + 161280d(n) - 5040c(2n) + 5040c(n) \right\}. \end{aligned}$$

(g)

$$\begin{aligned} & \sum_{m=1}^{n-1} \sigma_3(2m)\sigma_7(2n-2m) \\ &= \frac{1}{331680} \left[ 3 \left\{ 99419\sigma_{11}(n) - 99328\sigma_{11}\left(\frac{n}{2}\right) \right\} - 1382\sigma_7(2n) - 691\sigma_3(2n) \right. \\ & \quad \left. + 3 \left\{ 33080\tau(2n) + 756000\tau(n) - 788480\tau\left(\frac{n}{2}\right) \right\} \right]. \end{aligned}$$

(h)

$$\begin{aligned} & \sum_{m=1}^{n-1} \sigma_5(2m)\sigma_5(2n-2m) \\ &= \frac{1}{174132} \left\{ 68673\sigma_{11}(n) - 68608\sigma_{11}\left(\frac{n}{2}\right) + 691\sigma_5(2n) - 22176\tau(2n) \right. \\ & \quad \left. - 623700\tau(n) + 645120\tau\left(\frac{n}{2}\right) \right\}. \end{aligned}$$

*Proof.* From the definition of  $S_{e,f}(n)$  we deduce that

$$\begin{aligned} S_{e,f}(2n) &= \sum_{m=1}^{2n-1} \sigma_e(m) \sigma_f(2n-m) \\ &= \sum_{\substack{m=1 \\ 2 \nmid m}}^{2n-1} \sigma_e(m) \sigma_f(2n-m) + \sum_{\substack{m=1 \\ 2 \mid m}}^{2n-1} \sigma_e(m) \sigma_f(2n-m) \\ &= \sum_{m=1}^n \sigma_e(2m-1) \sigma_f(2n-2m+1) + \sum_{m=1}^{n-1} \sigma_e(2m) \sigma_f(2n-2m). \end{aligned}$$

This shows that

$$\sum_{m=1}^{n-1} \sigma_e(2m) \sigma_f(2n-2m) = S_{e,f}(2n) - \sum_{m=1}^n \sigma_e(2m-1) \sigma_f(2n-2m+1).$$

So we use Corollary 2.5. □

*Remark 2.2.* Note that

$$\sum_{m=1}^{2n} \sigma_{3,1}(m; 2) \sigma_{3,1}(2n+1-m; 2) = 2 \sum_{m=1}^n \sigma_{3,1}(2m-1; 2) \sigma_{3,1}(2(n-m+1); 2).$$

Hence we obtain

$$\begin{aligned} &\sum_{m=1}^n \sigma_{3,1}(2m-1; 2) \sigma_{3,1}(2(n-m+1); 2) \\ &= \frac{1}{2} \sum_{m=1}^{2n} \sigma_{3,1}(m; 2) \sigma_{3,1}(2n+1-m; 2) \\ &= \frac{1}{4080} \{ \sigma_7(2n+1) + 119\sigma_3(2n+1) - 120b(2n+1) \} \end{aligned}$$

by replacing  $n \rightarrow 2n+1$  in Corollary 2.4 (e). Therefore we have

$$\begin{aligned} &\sum_{m=1}^n \sigma_{3,1}(2m-1; 2) \sigma_{3,1}(2(n-m+1); 2) \\ &= \frac{1}{4080} \{ \sigma_7(2n+1) + 119\sigma_3(2n+1) - 120b(2n+1) \}. \end{aligned}$$

### 3 About the convolution sum $\sum_{m=1}^{n-1} m \sigma_i(m) \sigma_j(n-m)$

We can see that

$$\sum_{m=1}^{n-1} m \sigma_3(m) \sigma_1(n-m) = \frac{7}{120} n \sigma_5(n) + \left( \frac{1}{24} n - \frac{1}{10} n^2 \right) \sigma_3(n) \quad (3.1)$$

in ([9], p. 157), which gives us inspiration to induce some convolution sums as

$$\sum_{m=1}^{n-1} m\sigma_i(m)\sigma_j(n-m) \quad \text{for some odd } i, j.$$

In ([1], p. 42), N. Cheng and K. S. Williams begin by recalling Ramanujan's functions as :

$$L(q) = 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n)q^n, \quad |q| < 1, \quad (3.2)$$

$$M(q) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n)q^n, \quad |q| < 1, \quad (3.3)$$

and

$$N(q) = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n)q^n, \quad |q| < 1. \quad (3.4)$$

Also they show the following equations in Proposition 3.1 which is needed to find some formulas of the various convolution sums.

**Proposition 3.1.** (See ([1], p. 42–43))

$$L^2(q) = 1 - 2^5 \cdot 3^2 \sum_{n=1}^{\infty} n\sigma_1(n)q^n + 2^4 \cdot 3 \cdot 5 \sum_{n=1}^{\infty} \sigma_3(n)q^n, \quad (3.5)$$

$$M^2(q) = 1 + 2^5 \cdot 3 \cdot 5 \sum_{n=1}^{\infty} \sigma_7(n)q^n, \quad (3.6)$$

$$L(q)M(q) = 1 + 2^4 \cdot 3^2 \cdot 5 \sum_{n=1}^{\infty} n\sigma_3(n)q^n - 2^3 \cdot 3^2 \cdot 7 \sum_{n=1}^{\infty} \sigma_5(n)q^n, \quad (3.7)$$

$$L(q)M^2(q) = 1 + 2^4 \cdot 3^2 \cdot 5 \sum_{n=1}^{\infty} n\sigma_7(n)q^n - 2^3 \cdot 3 \cdot 11 \sum_{n=1}^{\infty} \sigma_9(n)q^n, \quad (3.8)$$

$$L(q)N(q) = 1 - 2^4 \cdot 3^2 \cdot 7 \sum_{n=1}^{\infty} n\sigma_5(n)q^n + 2^5 \cdot 3 \cdot 5 \sum_{n=1}^{\infty} \sigma_7(n)q^n, \quad (3.9)$$

$$M(q)N(q) = 1 - 2^3 \cdot 3 \cdot 11 \sum_{n=1}^{\infty} \sigma_9(n)q^n, \quad (3.10)$$

**Theorem 3.1.** Let  $n \in \mathbb{N}$ . Then we have

$$I_{m,5,1}(n) = \sum_{m=1}^{n-1} m\sigma_5(m)\sigma_1(n-m) = \frac{1}{168}n \{5\sigma_7(n) - (12n-7)\sigma_5(n)\}.$$

*Proof.* By (3.2), (3.6), and (3.9), we obtain

$$\begin{aligned} \sum_{n=1}^{\infty} I_{m,5,1}(n)q^n &= \sum_{n=1}^{\infty} \left( \sum_{m=1}^{n-1} m\sigma_5(m)\sigma_1(n-m) \right) q^n \\ &= \sum_{l=1}^{\infty} l\sigma_5(l)q^l \sum_{m=1}^{\infty} \sigma_1(m)q^m = \left( \frac{M^2(q) - L(q)N(q)}{2^4 \cdot 3^2 \cdot 7} \right) \left( \frac{1 - L(q)}{24} \right) \\ &= \frac{1}{24 \cdot 2^4 \cdot 3^2 \cdot 7} \{M^2(q) - M^2(q)L(q) - L(q)N(q) + L^2(q)N(q)\}. \end{aligned} \quad (3.11)$$

Then, for the last term of (3.11), we can use (3.4) and (3.5) so that we deduce

$$\begin{aligned}
 L^2(q)N(q) &= \left\{ 1 - 2^5 \cdot 3^2 \sum_{n=1}^{\infty} n\sigma_1(n)q^n + 2^4 \cdot 3 \cdot 5 \sum_{n=1}^{\infty} \sigma_3(n)q^n \right\} \left\{ 1 - 504 \sum_{m=1}^{\infty} \sigma_5(m)q^m \right\} \\
 &= 1 - 504 \sum_{m=1}^{\infty} \sigma_5(m)q^m - 2^5 \cdot 3^2 \sum_{n=1}^{\infty} n\sigma_1(n)q^n \\
 &\quad + 2^5 \cdot 3^2 \cdot 504 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} n\sigma_1(n)\sigma_5(m)q^{n+m} \\
 &\quad + 2^4 \cdot 3 \cdot 5 \sum_{n=1}^{\infty} \sigma_3(n)q^n - 2^4 \cdot 3 \cdot 5 \cdot 504 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sigma_3(n)\sigma_5(m)q^{n+m}.
 \end{aligned} \tag{3.12}$$

Here the fourth term of (3.12) can be written as

$$\begin{aligned}
 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} n\sigma_1(n)\sigma_5(m)q^{n+m} &= \sum_{K=2}^{\infty} \left\{ \sum_{m=1}^{K-1} (K-m)\sigma_1(K-m)\sigma_5(m) \right\} q^K \\
 &= \sum_{K=1}^{\infty} \left\{ \sum_{m=1}^{K-1} (K-m)\sigma_1(K-m)\sigma_5(m) \right\} q^K \\
 &= \sum_{K=1}^{\infty} \left\{ K \sum_{m=1}^{K-1} \sigma_1(K-m)\sigma_5(m) - I_{m,5,1}(K) \right\} q^K,
 \end{aligned}$$

so we refer to  $S_{1,5}(n)$ . And the last term of (3.12) is

$$\begin{aligned}
 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sigma_3(n)\sigma_5(m)q^{n+m} &= \sum_{K=2}^{\infty} \left\{ \sum_{m=1}^{K-1} \sigma_3(K-m)\sigma_5(m) \right\} q^K \\
 &= \sum_{K=1}^{\infty} \left\{ \sum_{m=1}^{K-1} \sigma_3(K-m)\sigma_5(m) \right\} q^K,
 \end{aligned}$$

thus we refer to  $S_{3,5}(n)$ . Therefore, (3.12) is equal to

$$\begin{aligned}
 L^2(q)N(q) &= 1 - 24 \sum_{n=1}^{\infty} \{ 11\sigma_9(n) - 240n\sigma_7(n) + 252n(2n-1)\sigma_5(n) + 6048I_{m,5,1}(n) \} q^n.
 \end{aligned} \tag{3.13}$$

Finally, by applying (3.6), (3.8), (3.9), and (3.13), we can complete the Eq. (3.11). □

**Lemma 3.2.** *We have*

$$\begin{aligned}
 L^2(q)N(q) &= 1 - 2^6 \cdot 3^3 \sum_{n=1}^{\infty} n^2\sigma_5(n)q^n + 2^5 \cdot 3^2 \cdot 5 \sum_{n=1}^{\infty} n\sigma_7(n)q^n \\
 &\quad - 2^3 \cdot 3 \cdot 11 \sum_{n=1}^{\infty} \sigma_9(n)q^n.
 \end{aligned}$$

*Proof.* It is obvious by (3.13) and Theorem 3.1. □

**Theorem 3.3.** *Let  $n \in \mathbb{N}$ . Then we have*

$$I_{m,1,5}(n) = \sum_{m=1}^{n-1} m\sigma_1(m)\sigma_5(n-m) = \frac{1}{504}n \{5\sigma_7(n) - 6n\sigma_5(n) + \sigma_1(n)\}.$$

*Proof.* Since, by (3.3), (3.4), and (3.5),

$$\begin{aligned} \sum_{n=1}^{\infty} I_{m,1,5}(n)q^n &= \sum_{n=1}^{\infty} \left( \sum_{m=1}^{n-1} m\sigma_1(m)\sigma_5(n-m) \right) q^n \\ &= \sum_{l=1}^{\infty} l\sigma_1(l)q^l \sum_{m=1}^{\infty} \sigma_5(m)q^m = \left( \frac{M(q) - L^2(q)}{2^5 \cdot 3^2} \right) \left( \frac{1 - N(q)}{504} \right) \\ &= \frac{1}{504 \cdot 2^5 \cdot 3^2} \{M(q) - M(q)N(q) - L^2(q) + L^2(q)N(q)\}, \end{aligned}$$

we use (3.3), (3.5), (3.10), and Lemma 3.2. □

**Remark 3.1.** From the definition of (1.7) let us investigate as follows :

$$\begin{aligned} I_{m,e,f}(n) &= \sum_{m=1}^{n-1} m\sigma_e(m)\sigma_f(n-m) \\ &= \sum_{m=1}^{n-1} (n-m)\sigma_e(n-m)\sigma_f(m) \\ &= n \sum_{m=1}^{n-1} \sigma_e(n-m)\sigma_f(m) - \sum_{m=1}^{n-1} m\sigma_e(n-m)\sigma_f(m) \\ &= nS_{f,e}(n) - I_{m,f,e}(n) \end{aligned}$$

so this leads to

$$I_{m,f,e}(n) = nS_{f,e}(n) - I_{m,e,f}(n). \quad (3.14)$$

For example, Theorem 3.3 is obtained easily since

$$I_{m,1,5}(n) = \sum_{m=1}^{n-1} m\sigma_1(m)\sigma_5(n-m) = nS_{1,5}(n) - I_{m,5,1}(n)$$

by (3.14). So we use  $S_{1,5}(n)$  and Theorem 3.1.

**Lemma 3.4.** *We have*

$$\begin{aligned} L(q)M(q)N(q) &= 1 - \frac{2^4 \cdot 3^2 \cdot 11}{5} \sum_{n=1}^{\infty} n\sigma_9(n)q^n + \frac{2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13}{691} \sum_{n=1}^{\infty} \sigma_{11}(n)q^n \\ &\quad - \frac{2^8 \cdot 3^4 \cdot 11}{5 \cdot 691} \sum_{n=1}^{\infty} \tau(n)q^n. \end{aligned}$$

*Proof.* The formula of  $L(q)M(q)N(q)$  is found in ([1], (2.12)) but it has a typo. So we search the exact identity as follows :

By (3.2) and (3.10), we obtain

$$\begin{aligned} L(q)M(q)N(q) &= \left\{ 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n)q^n \right\} \left\{ 1 - 2^3 \cdot 3 \cdot 11 \sum_{m=1}^{\infty} \sigma_9(m)q^m \right\} \\ &= 1 - 2^3 \cdot 3 \cdot 11 \sum_{m=1}^{\infty} \sigma_9(m)q^m - 24 \sum_{n=1}^{\infty} \sigma_1(n)q^n \\ &\quad + 24 \cdot 2^3 \cdot 3 \cdot 11 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sigma_1(n)\sigma_9(m)q^{n+m}. \end{aligned}$$

Thus we use  $S_{1,9}(n)$ .

□

**Theorem 3.5.** Let  $n \in \mathbb{N}$ . Then we have

$$\begin{aligned} I_{m,7,1}(n) &= \sum_{m=1}^{n-1} m\sigma_7(m)\sigma_1(n-m) \\ &= \frac{1}{1800} \{33n\sigma_9(n) - 25n(4n-3)\sigma_7(n) - 8\tau(n)\}. \end{aligned}$$

*Proof.* From (3.2), (3.8), and (3.10), we have

$$\begin{aligned} \sum_{n=1}^{\infty} I_{m,7,1}(n)q^n &= \sum_{n=1}^{\infty} \left( \sum_{m=1}^{n-1} m\sigma_7(m)\sigma_1(n-m) \right) q^n \\ &= \sum_{l=1}^{\infty} l\sigma_7(l)q^l \sum_{m=1}^{\infty} \sigma_1(m)q^m = \left( \frac{L(q)M^2(q) - M(q)N(q)}{2^4 \cdot 3^2 \cdot 5} \right) \left( \frac{1 - L(q)}{24} \right) \\ &= \frac{1}{24 \cdot 2^4 \cdot 3^2 \cdot 5} \{L(q)M^2(q) - L^2(q)M^2(q) - M(q)N(q) + M(q)N(q)L(q)\}. \end{aligned} \quad (3.15)$$

Then, to calculate the second term of (3.15) we use (3.5) and (3.6) :

$$\begin{aligned} L^2(q)M^2(q) &= \left\{ 1 - 2^5 \cdot 3^2 \sum_{n=1}^{\infty} n\sigma_1(n)q^n + 2^4 \cdot 3 \cdot 5 \sum_{n=1}^{\infty} \sigma_3(n)q^n \right\} \left\{ 1 + 2^5 \cdot 3 \cdot 5 \sum_{m=1}^{\infty} \sigma_7(m)q^m \right\} \\ &= 1 + 2^5 \cdot 3 \cdot 5 \sum_{m=1}^{\infty} \sigma_7(m)q^m - 2^5 \cdot 3^2 \sum_{n=1}^{\infty} n\sigma_1(n)q^n \\ &\quad - 2^{10} \cdot 3^3 \cdot 5 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} n\sigma_1(n)\sigma_7(m)q^{n+m} \\ &\quad + 2^4 \cdot 3 \cdot 5 \sum_{n=1}^{\infty} \sigma_3(n)q^n + 2^9 \cdot 3^2 \cdot 5^2 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sigma_3(n)\sigma_7(m)q^{n+m}. \end{aligned} \quad (3.16)$$

Here the fourth term of (3.16) can be written as

$$\begin{aligned} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} n \sigma_1(n) \sigma_7(m) q^{n+m} &= \sum_{K=2}^{\infty} \left\{ \sum_{m=1}^{K-1} (K-m) \sigma_1(K-m) \sigma_7(m) \right\} q^K \\ &= \sum_{K=1}^{\infty} \left\{ \sum_{m=1}^{K-1} (K-m) \sigma_1(K-m) \sigma_7(m) \right\} q^K \\ &= \sum_{K=1}^{\infty} \left\{ K \sum_{m=1}^{K-1} \sigma_1(K-m) \sigma_7(m) - I_{m,7,1}(K) \right\} q^K, \end{aligned}$$

so we refer to  $S_{1,7}(n)$ . Similarly, the last term of (3.16) is

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sigma_3(n) \sigma_7(m) q^{n+m} = \sum_{K=1}^{\infty} \left\{ \sum_{m=1}^{K-1} \sigma_3(K-m) \sigma_7(m) \right\} q^K,$$

thus we use  $S_{3,7}(n)$ . Therefore, (3.16) is equal to

$$\begin{aligned} L^2(q) M^2(q) &= 1 + \frac{144}{691} \sum_{n=1}^{\infty} \{ 455 \sigma_{11}(n) - 15202n \sigma_9(n) + 13820n(3n-2) \sigma_7(n) \\ &\quad 3000\tau(n) + 663360 I_{m,7,1}(n) \} q^n. \end{aligned} \quad (3.17)$$

Lastly, by applying (3.8), (3.10), (3.17), and Lemma 3.4, we can complete the Eq. (3.15).  $\square$

**Lemma 3.6.** *We have*

$$\begin{aligned} L^2(q) M^2(q) &= 1 + 2^6 \cdot 3 \cdot 5 \sum_{n=1}^{\infty} n^2 \sigma_7(n) q^n - \frac{2^5 \cdot 3^2 \cdot 11}{5} \sum_{n=1}^{\infty} n \sigma_9(n) q^n \\ &\quad + \frac{2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13}{691} \sum_{n=1}^{\infty} \sigma_{11}(n) q^n + \frac{2^7 \cdot 3 \cdot 97}{5 \cdot 691} \sum_{n=1}^{\infty} \tau(n) q^n. \end{aligned}$$

*Proof.* It is obvious by (3.17) and Theorem 3.5.  $\square$

**Theorem 3.7.** *Let  $n \in \mathbb{N}$ . Then we have*

$$I_{m,5,3}(n) = \sum_{m=1}^{n-1} m \sigma_5(m) \sigma_3(n-m) = \frac{1}{8400} \{ 11n \sigma_9(n) - 35n \sigma_5(n) + 24\tau(n) \}.$$

*Proof.* Multiplying (3.9) with (3.3), we obtain

$$\begin{aligned}
 & M(q)L(q)N(q) \\
 &= \left\{ 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n)q^n \right\} \left\{ 1 - 2^4 \cdot 3^2 \cdot 7 \sum_{m=1}^{\infty} m\sigma_5(m)q^m + 2^5 \cdot 3 \cdot 5 \sum_{m=1}^{\infty} \sigma_7(m)q^m \right\} \\
 &= 1 - 2^4 \cdot 3^2 \cdot 7 \sum_{m=1}^{\infty} m\sigma_5(m)q^m + 2^5 \cdot 3 \cdot 5 \sum_{m=1}^{\infty} \sigma_7(m)q^m + 240 \sum_{n=1}^{\infty} \sigma_3(n)q^n \\
 &\quad - 240 \cdot 2^4 \cdot 3^2 \cdot 7 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sigma_3(n) \cdot m\sigma_5(m)q^{n+m} \\
 &\quad + 240 \cdot 2^5 \cdot 3 \cdot 5 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sigma_3(n)\sigma_7(m)q^{n+m} \\
 &= 1 - 2^4 \cdot 3^2 \cdot 7 \sum_{n=1}^{\infty} n\sigma_5(n)q^n + 2^5 \cdot 3 \cdot 5 \sum_{n=1}^{\infty} \sigma_7(n)q^n + 240 \sum_{n=1}^{\infty} \sigma_3(n)q^n \\
 &\quad - 240 \cdot 2^4 \cdot 3^2 \cdot 7 \sum_{n=1}^{\infty} I_{m,5,3}(n)q^n + 240 \cdot 2^5 \cdot 3 \cdot 5 \sum_{n=1}^{\infty} S_{3,7}(n)q^n.
 \end{aligned}$$

This leads to

$$\begin{aligned}
 M(q)L(q)N(q) &= 1 + \frac{65520}{691} \sum_{n=1}^{\infty} \sigma_{11}(n)q^n - 1008 \sum_{n=1}^{\infty} n\sigma_5(n)q^n \\
 &\quad + \frac{432000}{691} \sum_{n=1}^{\infty} \tau(n)q^n - 241920 \sum_{n=1}^{\infty} I_{m,5,3}(n)q^n.
 \end{aligned} \tag{3.18}$$

Equating (3.18) with Lemma 3.4, we complete the proof.  $\square$

**Theorem 3.8.** *Let  $n \in \mathbb{N}$ . Then we have*

(a)

$$\begin{aligned}
 I_{m,1,7}(n) &= \sum_{m=1}^{n-1} m\sigma_1(m)\sigma_7(n-m) \\
 &= \frac{1}{7200} \{ 33n\sigma_9(n) - 50n^2\sigma_7(n) - 15n\sigma_1(n) + 32\tau(n) \}.
 \end{aligned}$$

(b)

$$\begin{aligned}
 I_{m,3,5}(n) &= \sum_{m=1}^{n-1} m\sigma_3(m)\sigma_5(n-m) \\
 &= \frac{1}{12600} \{ 11n\sigma_9(n) + 25n\sigma_3(n) - 36\tau(n) \}.
 \end{aligned}$$

*Proof.* It is obvious by (3.14).  $\square$

To prove some theorems ahead we need the following proposition and Corollary 3.9.

**Proposition 3.2.** (a) (See ([1], Theorem 4.1.(i)))

$$L(q)M(q^2) = 2L(q^2)M(q^2) + \frac{1}{21}N(q) - \frac{22}{21}N(q^2).$$

(b) (See ([1], Theorem 4.1.(ii)))

$$M(q)L(q^2) = \frac{1}{2}L(q)M(q) - \frac{11}{42}N(q) + \frac{16}{21}N(q^2).$$

(c) (See ([1], Theorem 5.1.(i)))

$$L(q)N(q^2) = 2L(q^2)N(q^2) + \frac{1}{85}M^2(q) - \frac{86}{85}M^2(q^2) - \frac{504}{17}B(q).$$

(d) (See ([1], Theorem 5.1.(iii)))

$$N(q)L(q^2) = \frac{1}{2}L(q)N(q) - \frac{43}{170}M^2(q) + \frac{64}{85}M^2(q^2) - \frac{2016}{17}B(q).$$

**Corollary 3.9.** We have

(a)

$$\begin{aligned} L(q^2)L(q)M(q) &= \frac{1}{2}L^2(q)M(q) - \frac{11}{42}L(q)N(q) + \frac{32}{21}L(q^2)N(q^2) \\ &\quad + \frac{16}{1785}M^2(q) - \frac{1376}{1785}M^2(q^2) - \frac{384}{17}B(q). \end{aligned}$$

(b)

$$\begin{aligned} L(q)L(q^2)M(q^2) &= 2L^2(q^2)M(q^2) + \frac{1}{42}L(q)N(q) - \frac{43}{3570}M^2(q) \\ &\quad + \frac{64}{1785}M^2(q^2) - \frac{22}{21}L(q^2)N(q^2) - \frac{96}{17}B(q). \end{aligned}$$

**Proof.** (a) From Proposition 3.2 (b) we have

$$\begin{aligned} L(q^2)L(q)M(q) &= L(q) \cdot L(q^2)M(q) \\ &= L(q) \left\{ \frac{1}{2}L(q)M(q) - \frac{11}{42}N(q) + \frac{16}{21}N(q^2) \right\} \\ &= \frac{1}{2}L^2(q)M(q) - \frac{11}{42}L(q)N(q) + \frac{16}{21}L(q)N(q^2). \end{aligned}$$

This leads

$$\begin{aligned} &L(q^2)L(q)M(q) \\ &= \frac{1}{2}L^2(q)M(q) - \frac{11}{42}L(q)N(q) \\ &\quad + \frac{16}{21} \left\{ 2L(q^2)N(q^2) + \frac{1}{85}M^2(q) - \frac{86}{85}M^2(q^2) - \frac{504}{17}B(q) \right\} \\ &= \frac{1}{2}L^2(q)M(q) - \frac{11}{42}L(q)N(q) + \frac{32}{21}L(q^2)N(q^2) + \frac{16}{1785}M^2(q) - \frac{1376}{1785}M^2(q^2) \\ &\quad - \frac{384}{17}B(q), \end{aligned}$$

by Proposition 3.2 (c).

(b) It is similar to Corollary 3.9 (a). That is, we note that

$$\begin{aligned} L(q)L(q^2)M(q^2) &= L(q^2) \cdot L(q)M(q^2) \\ &= L(q^2) \left\{ 2L(q^2)M(q^2) + \frac{1}{21}N(q) - \frac{22}{21}N(q^2) \right\} \\ &= 2L^2(q^2)M(q^2) + \frac{1}{21}L(q^2)N(q) - \frac{22}{21}L(q^2)N(q^2), \end{aligned}$$

by Proposition 3.2 (a). Again applying Proposition 3.2 (d) into the above equation we have the proof.  $\square$

**Lemma 3.10.** *Let  $n \in \mathbb{N}$ . We have*

$$\begin{aligned} &\sum_{m=1}^{n-1} m\sigma_3(m)\sigma_{1,1}(n-m; 2) \\ &= \frac{1}{360} \left[ n \left\{ 11\sigma_5(n) - 32\sigma_5\left(\frac{n}{2}\right) - 15\sigma_3(n) \right\} + 4b(n) \right]. \end{aligned}$$

*Proof.* First by (3.2), we have

$$\begin{aligned} L(q) &= 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n)q^n = 1 - 24 \sum_{n=1}^{\infty} (\sigma_{1,1}(n; 2) + \sigma_{1,0}(n; 2)) q^n \\ &= 1 - 24 \sum_{n=1}^{\infty} \sigma_{1,1}(n; 2)q^n - 24 \sum_{n=1}^{\infty} 2\sigma_1\left(\frac{n}{2}\right)q^n \\ &= 1 - 24 \sum_{n=1}^{\infty} \sigma_{1,1}(n; 2)q^n + 2 \left\{ 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n)q^{2n} \right\} - 2. \end{aligned}$$

Thus we obtain

$$24 \sum_{n=1}^{\infty} \sigma_{1,1}(n; 2)q^n = 2L(q^2) - 1 - L(q). \quad (3.19)$$

Second by (3.4) and (3.7) we have

$$2^4 \cdot 3^2 \cdot 5 \sum_{n=1}^{\infty} n\sigma_3(n)q^n = L(q)M(q) - N(q). \quad (3.20)$$

Multiplying (3.19) by (3.20) we obtain

$$\begin{aligned} &2^7 \cdot 3^3 \cdot 5 \left( \sum_{m=1}^{\infty} m\sigma_3(m)q^m \right) \left( \sum_{n=1}^{\infty} \sigma_{1,1}(n; 2)q^n \right) \\ &= (L(q)M(q) - N(q)) (2L(q^2) - 1 - L(q)) \\ &= 2L(q)M(q)L(q^2) - L(q)M(q) - L^2(q)M(q) - 2N(q)L(q^2) \\ &\quad + N(q) + N(q)L(q) \\ &= -\frac{11}{21}L(q)N(q) + \frac{64}{21}L(q^2)N(q^2) + \frac{11}{21}M^2(q) - \frac{64}{21}M^2(q^2) \\ &\quad - L(q)M(q) + N(q) + 192B(q), \end{aligned}$$

where we use Proposition 3.2 (d) and Corollary 3.9 (a) for the last line. Therefore we refer to (1.1), (3.4), (3.6), (3.7), and (3.9) so that we complete the proof.  $\square$

**Proof of the Theorem 1.1.** (a) Now by (3.4) and (3.7), we have

$$\begin{aligned} L(q)M(q) &= N(q) + 2^4 \cdot 3^2 \cdot 5 \sum_{n=1}^{\infty} n\sigma_3(n)q^n \\ &= N(q) + 2^4 \cdot 3^2 \cdot 5 \sum_{n=1}^{\infty} n\sigma_{3,1}(n; 2)q^n + 2^4 \cdot 3^2 \cdot 5 \sum_{n=1}^{\infty} 16n\sigma_3(n)q^{2n} \\ &= N(q) + 2^4 \cdot 3^2 \cdot 5 \sum_{n=1}^{\infty} n\sigma_{3,1}(n; 2)q^n + 16 \{L(q^2)M(q^2) - N(q^2)\}, \end{aligned}$$

since

$$L(q^2)M(q^2) = N(q^2) + 2^4 \cdot 3^2 \cdot 5 \sum_{n=1}^{\infty} n\sigma_3(n)q^{2n}.$$

So we obtain

$$\begin{aligned} &2^4 \cdot 3^2 \cdot 5 \sum_{n=1}^{\infty} n\sigma_{3,1}(n; 2)q^n \\ &= L(q)M(q) - N(q) - 16 \{L(q^2)M(q^2) - N(q^2)\}. \end{aligned} \tag{3.21}$$

Combining (3.19) and (3.21) we note that

$$\begin{aligned} &2^7 \cdot 3^3 \cdot 5 \left( \sum_{n=1}^{\infty} \sigma_{1,1}(n; 2)q^n \right) \left( \sum_{m=1}^{\infty} m\sigma_{3,1}(m; 2)q^m \right) \\ &= \{2L(q^2) - 1 - L(q)\} \{L(q)M(q) - N(q) - 16L(q^2)M(q^2) + 16N(q^2)\} \\ &= 2L(q^2)L(q)M(q) - 2L(q^2)N(q) - 32L^2(q^2)M(q^2) + 32L(q^2)N(q^2) \\ &\quad - L(q)M(q) + N(q) + 16L(q^2)M(q^2) - 16N(q^2) - L^2(q)M(q) \\ &\quad + L(q)N(q) + 16L(q)L(q^2)M(q^2) - 16L(q)N(q^2) \\ &= -\frac{1}{7}L(q)N(q) - \frac{96}{7}L(q^2)N(q^2) + \frac{1}{7}M^2(q) + \frac{96}{7}M^2(q^2) \\ &\quad - L(q)M(q) + N(q) + 16L(q^2)M(q^2) - 16N(q^2) + 576B(q), \end{aligned}$$

where we use Proposition 3.2 (c), (d), and Corollary 3.9 for the last line. Finally, we refer to (1.1), (3.4), (3.6), (3.7), and (3.9).

(b) We note that

$$\begin{aligned}
 & \sum_{m=1}^{n-1} m\sigma_3(m)\sigma_{1,1}(n-m;2) \\
 &= \sum_{m=1}^{n-1} m\{\sigma_{3,1}(m;2) + \sigma_{3,0}(m;2)\}\sigma_{1,1}(n-m;2) \\
 &= \sum_{m=1}^{n-1} m\sigma_{3,1}(m;2)\sigma_{1,1}(n-m;2) + \sum_{m=1}^{n-1} m\sigma_{3,0}(m;2)\sigma_{1,1}(n-m;2).
 \end{aligned}$$

So we use Lemma 3.10 and Theorem 1.1 (a).

(c) By (1.4), we have

$$\begin{aligned}
 \sum_{m=1}^{n-1} m\sigma_{3,0}(m;2)\sigma_{1,0}(n-m;2) &= \sum_{m=1}^{n-1} m \cdot 2^3\sigma_3\left(\frac{m}{2}\right) \cdot 2\sigma_1\left(\frac{n-m}{2}\right) \\
 &= 32 \sum_{m < \frac{n}{2}} m\sigma_3(m)\sigma_1\left(\frac{n}{2}-m\right).
 \end{aligned}$$

So we refer to (3.1).

(d) Since

$$\begin{aligned}
 & \sum_{m=1}^{n-1} m\sigma_3(m)\sigma_1(n-m) \\
 &= \sum_{m=1}^{n-1} m\{\sigma_{3,1}(m;2) + \sigma_{3,0}(m;2)\}\{\sigma_{1,1}(n-m;2) + \sigma_{1,0}(n-m;2)\} \\
 &= \sum_{m=1}^{n-1} m\sigma_{3,1}(m;2)\sigma_{1,1}(n-m;2) + \sum_{m=1}^{n-1} m\sigma_{3,1}(m;2)\sigma_{1,0}(n-m;2) \\
 &\quad + \sum_{m=1}^{n-1} m\sigma_{3,0}(m;2)\sigma_{1,1}(n-m;2) + \sum_{m=1}^{n-1} m\sigma_{3,0}(m;2)\sigma_{1,0}(n-m;2),
 \end{aligned}$$

we use Theorem 1.1 (a), (b), and (c). □

**Theorem 3.11.** *Let  $n \in \mathbb{N}$ . Then we have*

(a)

$$\begin{aligned}
 & \sum_{m=1}^{n-1} m\sigma_{1,1}(m;2)\sigma_{3,1}(n-m;2) \\
 &= \frac{1}{240} \left[ n \left\{ \sigma_5(n) + 48\sigma_5\left(\frac{n}{2}\right) + 7\sigma_1(n) - 14\sigma_1\left(\frac{n}{2}\right) \right\} - 8b(n) \right].
 \end{aligned}$$

(b)

$$\begin{aligned}
 & \sum_{m=1}^{n-1} m\sigma_{1,1}(m;2)\sigma_{3,0}(n-m;2) \\
 &= \frac{1}{90} \left[ n \left\{ \sigma_5(n) - 22\sigma_5\left(\frac{n}{2}\right) - 3\sigma_1(n) + 6\sigma_1\left(\frac{n}{2}\right) \right\} + 2b(n) \right].
 \end{aligned}$$

(c)

$$\sum_{m=1}^{n-1} m\sigma_{1,0}(m;2)\sigma_{3,0}(n-m;2) = \frac{1}{15}n \left\{ 7\sigma_5\left(\frac{n}{2}\right) - 3n\sigma_3\left(\frac{n}{2}\right) - \sigma_1\left(\frac{n}{2}\right) \right\}.$$

(d)

$$\begin{aligned} & \sum_{m=1}^{n-1} m\sigma_{1,0}(m;2)\sigma_{3,1}(n-m;2) \\ &= \frac{1}{360} \left[ n \left\{ 5\sigma_5(n) - 152\sigma_5\left(\frac{n}{2}\right) - 9n\sigma_{3,1}(n;2) + 21\sigma_1\left(\frac{n}{2}\right) \right\} + 4b(n) \right]. \end{aligned}$$

*Proof.* It is the similar manner to (3.14). In other words, for  $i, j \in \{0, 1\}$  we observe that

$$\sum_{m=1}^{n-1} m\sigma_{3,i}(m;2)\sigma_{1,j}(n-m;2) = \sum_{m=1}^{n-1} (n-m)\sigma_{3,i}(n-m;2)\sigma_{1,j}(m;2)$$

and so we use Theorem 1.1 and Corollary 2.4 (a), Corollary 2.1 (a), Corollary 2.3 (a), and Corollary 2.2 (a), respectively. □

## 4 Appendix

The first twenty values of  $b(n)$  are given in the Table 1,

| $n$ | $b(n)$ | $n$ | $b(n)$ | $n$ | $b(n)$ | $n$ | $b(n)$ |
|-----|--------|-----|--------|-----|--------|-----|--------|
| 1   | 1      | 6   | −96    | 11  | 1092   | 16  | 4096   |
| 2   | −8     | 7   | 1016   | 12  | 768    | 17  | 14706  |
| 3   | 12     | 8   | −512   | 13  | 1382   | 18  | 16344  |
| 4   | 64     | 9   | −2043  | 14  | −8128  | 19  | −39940 |
| 5   | −210   | 10  | 1680   | 15  | −2520  | 20  | −13440 |

TABLE 1.  $b(n)$  for  $n$  ( $1 \leq n \leq 20$ )

similarly the first twenty values of  $c(n)$  and  $d(n)$  are listed in the following tables.

| $n$ | $c(n)$ | $n$ | $c(n)$ | $n$ | $c(n)$  | $n$ | $c(n)$  |
|-----|--------|-----|--------|-----|---------|-----|---------|
| 1   | 1      | 6   | 2496   | 11  | −38996  | 16  | −65536  |
| 2   | −16    | 7   | −4536  | 12  | 39936   | 17  | 311442  |
| 3   | 100    | 8   | −4096  | 13  | 37806   | 18  | −74448  |
| 4   | −256   | 9   | 23085  | 14  | 15232   | 19  | 128244  |
| 5   | −154   | 10  | −13920 | 15  | −146472 | 20  | −222720 |

TABLE 2.  $c(n)$  for  $n$  ( $1 \leq n \leq 20$ )

| $n$ | $d(n)$ | $n$ | $d(n)$ | $n$ | $d(n)$ | $n$ | $d(n)$ |
|-----|--------|-----|--------|-----|--------|-----|--------|
| 1   | 0      | 6   | −156   | 11  | −536   | 16  | 4096   |
| 2   | 1      | 7   | 112    | 12  | −2496  | 17  | −17472 |
| 3   | −8     | 8   | 256    | 13  | 4384   | 18  | 4653   |
| 4   | 16     | 9   | −576   | 14  | −952   | 19  | 5848   |
| 5   | 32     | 10  | 870    | 15  | 336    | 20  | 13920  |

TABLE 3.  $d(n)$  for  $n$  ( $1 \leq n \leq 20$ )

## 5 CONCLUSIONS

In this paper, we deduce various convolution sums formulas with even and odd divisor functions and in fact these results are available by Cheng and Williams' paper [1], especially.

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