

The binomial combinatorial convolution sums

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Type of Article

Received: XX December 20XX

Accepted: XX December 20XX

Online Ready: XX December 20XX

Abstract

In [7] we can find some formulas of binomial combinatorial convolution sums. Starting from these formulas, we obtain various binomial combinatorial convolution sums.

Keywords: Divisor functions : Convolution sum

2010 Mathematics Subject Classification: 11A05

1 Introduction

Let $\mathbb{N}$ denote the set of positive integers. Further, let $N, d \in \mathbb{N}$ and $s, r, k \in \mathbb{N} \cup \{0\}$. Throughout this paper, we define

$$\sigma_s(N) := \sum_{d|N} d^s, \quad \sigma_{s,r}(N; k) := \sum_{\substack{d|N \\ d \equiv r \pmod{k}}} d^s, \quad \sigma_s^*(N) := \sum_{\substack{d|N \\ N/d \text{ odd}}} d^s.$$

We note that

$$\sigma_s(N) = \sigma_{s,1}(N; 2) + \sigma_{s,0}(N; 2) \tag{1.1}$$

and

$$\sigma_{s,1}(2N; 2) = \sigma_{s,1}(N; 2), \quad \sigma_{s,0}(2N; 2) = 2^s \sigma_s(N). \tag{1.2}$$

Moreover we can deduce the property

$$\sigma_s^*(2N) = 2^s \sigma_s^*(N) \quad \text{and} \quad \sigma_s^*(2N) = \sigma_s(2N) - \sigma_s(N). \tag{1.3}$$

For $a, b, n \in \mathbb{N}$, let us define the convolution sum

$$S_{a,b}(n) := \sum_{m=1}^{n-1} \sigma_a(m) \sigma_b(n-m).$$

Ramanujan showed that the sum $S_{a,b}(n)$ can be evaluated in terms of $\sigma_{a+b+1}(n), \sigma_{a+b-1}(n), \dots, \sigma_3(n), \sigma_1(n)$ for the nine pairs $(a, b) \in \mathbb{N}^2$ satisfying $a + b = 2, 4, 6, 8, 12, \quad a \leq b, \quad a \equiv b \equiv 1 \pmod{2}$. For example, explicitly, we know (see [6]) that

$$S_{1,11}(n) = \frac{691}{65520} \sigma_{13}(n) + \left(\frac{1}{24} - \frac{1}{24} n \right) \sigma_{11}(n) - \frac{691}{65520} \sigma_1(n) \quad (1.4)$$

and (see ([3], p. 35))

$$S_{3,9}(n) = \frac{1}{2640} \sigma_{13}(n) - \frac{1}{240} \sigma_9(n) + \frac{1}{264} \sigma_3(n). \quad (1.5)$$

From [4], we note that for any integer $n \geq 3$, we have

$$\begin{aligned} & \sum_{\substack{(m_1, m_2, m_3) \in \mathbb{N}^3 \\ m_1 + m_2 + m_3 = n}} m_1 m_2 \sigma_1(m_1) \sigma_1(m_2) \sigma_1(m_3) \\ &= \frac{1}{288} (n^2 \sigma_5(n) + (n^2 - 4n^3) \sigma_3(n) - (n^3 - 3n^4) \sigma_1(n)). \end{aligned} \quad (1.6)$$

For an elementary proof of (1.4) and (1.5), we refer to [7]. An another interesting arithmetical identity (which was stated by Ramanujan) see ([6], p. 146), for some analytical proofs of this identity, one may refer to ([1], p. 329), ([2], p. 136) and [5], also [7] is ; for $n \in \mathbb{N}$, we have

$$\sum_{m=0}^{n-1} \sigma_1(2m+1) \sigma_3(n-m) = \frac{1}{240} \sigma_5(2n+1) - \frac{1}{240} \sigma_3(2n+1). \quad (1.7)$$

We introduce the very important Proposition 1.1, which is the starting point of our paper.

Proposition 1.1. (See ([7], Theorem 12.3)) Let $k, N \in \mathbb{N}$. Then

$$\begin{aligned} & \sum_{s=0}^{k-1} \binom{2k}{2s+1} \left(\sum_{m=1}^{N-1} \sigma_{2k-2s-1}(m) \sigma_{2s+1}(N-m) \right) \\ &= \frac{2k+3}{4k+2} \sigma_{2k+1}(N) + \left(\frac{k}{6} - N \right) \sigma_{2k-1}(N) + \frac{1}{2k+1} \sum_{j=2}^k \binom{2k+1}{2j} B_{2j} \sigma_{2k+1-2j}(N). \end{aligned}$$

In Section 2 we obtain some results about the binomial combinatorial convolution sums similar to Proposition 1.1. For instance we have

$$\sum_{s=0}^{k-1} 2^{2s+1} \binom{2k}{2s+1} \sum_{m < \frac{N}{2}} \sigma_{2k-2s-1}(N-2m) \sigma_{2s+1}(m)$$

and

$$\sum_{s=0}^{k-1} 2^{2s+1} \binom{2k}{2s+1} \sum_{m < \frac{N}{2}} \sigma_{2k-2s-1,1}(N-2m; 2) \sigma_{2s+1}(m)$$

(see (2.7) and (2.8)).

Theorem 1.1. Let $k, N \in \mathbb{N}$. Then we have

$$\begin{aligned}
 & \sum_{s=0}^{k-1} 2^{2s+1} \binom{2k}{2s+1} \sum_{m < N} \sigma_{2k-2s-1,1}(N-m; 2) \sigma_{2s+1}(m) \\
 &= \frac{1}{24k+12} \left[(2k+1)(k-12N) \sigma_{2k-1}(2N) - 4^k (2k+1)(k-6N) \sigma_{2k-1}(N) \right. \\
 & \quad - 3 \left\{ 2^{2(k+1)} (k+1) \sigma_{2k+1}(N) - (2k+3) \sigma_{2k+1}(2N) \right. \\
 & \quad + 2^{2k+1} \sum_{j=2}^k \binom{2k+1}{2j} B_{2j} \sigma_{2k+1-2j}(N) - 2 \sum_{j=2}^k \binom{2k+1}{2j} B_{2j} \sigma_{2k+1-2j}(2N) \\
 & \quad \left. + \sum_{j=0}^{2k} \sum_{r=0}^{2k+1-j} (-1)^{j+r} 2^j B_j \binom{2k+1}{j} \binom{2k-j+1}{r} \sigma_{2k+1-j-r,1}(N; 2) \right\} \left. \right].
 \end{aligned}$$

Finally, in Section 3 we consider the convolution sums

$$\sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m < \frac{N}{2}} \sigma_{2k-2s-1}(N-2m) \sigma_{2s+1,1}(m; 2)$$

and

$$\sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m < \frac{N}{2}} \sigma_{2k-2s-1}(N-2m) \sigma_{2s+1}(2m)$$

for odd N .

2 The binomial combinatorial convolution sums

To prove Lemma 2.1 we need Proposition 2.1.

Proposition 2.1. (See ([7], p. 112)) Let $f : \mathbb{Z} \rightarrow \mathbb{C}$ be an even function. Let $N \in \mathbb{N}$. Then we obtain

$$\begin{aligned}
 & \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4 \\ ax+by=N \\ a,b \text{ odd}}} (f(a-b) - f(a+b)) \\
 &= f(0) (\sigma_1^*(N) - \sigma_0^*(N)) - \frac{1}{2} \sum_{\substack{d \in \mathbb{N} \\ d|N \\ d \text{ even}}} df(d) - 2 \sum_{\substack{d \in \mathbb{N} \\ d|N \\ d \text{ odd}}} \sum_{\substack{t \in \mathbb{N} \\ 1 \leq t < d \\ t \text{ even}}} f(t).
 \end{aligned}$$

Lemma 2.1. Let $k, N \in \mathbb{N}$. Then we have

(a)

$$\begin{aligned}
 & \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{N-1} \sigma_{2k-2s-1,1}(m; 2) \sigma_{2s+1,1}(N-m; 2) = 2^{2k-1} \sigma_{2k+1}\left(\frac{N}{2}\right) \\
 & \quad + \frac{1}{4k+2} \sum_{j=0}^{2k} \sum_{r=0}^{2k+1-j} (-1)^{j+r} \binom{2k+1}{j} \binom{2k+1-j}{r} 2^j B_j \sigma_{2k+1-j-r,1}(N; 2).
 \end{aligned}$$

(b)

$$\sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{N-1} m \sigma_{2k-2s-1,1}(m; 2) \sigma_{2s+1,1}(N-m; 2) = 2^{2(k-1)} N \sigma_{2k+1}\left(\frac{N}{2}\right) + \frac{N}{8k+4} \sum_{j=0}^{2k} \sum_{r=0}^{2k+1-j} (-1)^{j+r} \binom{2k+1}{j} \binom{2k+1-j}{r} 2^j B_j \sigma_{2k+1-j-r,1}(N; 2).$$

Proof. (a) We apply $f(x) = x^{2k}$ in Proposition 2.1. Then the left hand side is

$$\begin{aligned} \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4 \\ ax+by=N \\ a,b \text{ odd}}} (f(a-b) - f(a+b)) &= \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4 \\ ax+by=N \\ a,b \text{ odd}}} ((a-b)^{2k} - (a+b)^{2k}) \\ &= \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4 \\ ax+by=N \\ a,b \text{ odd}}} \left(\sum_{r=0}^{2k} \binom{2k}{r} (-1)^r a^{2k-r} b^r - \sum_{r=0}^{2k} \binom{2k}{r} a^{2k-r} b^r \right) \\ &= -2 \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4 \\ ax+by=N \\ a,b \text{ odd}}} \sum_{\substack{r=0 \\ r \text{ odd}}}^{2k} \binom{2k}{r} a^{2k-r} b^r \\ &= -2 \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4 \\ ax+by=N \\ a,b \text{ odd}}} a^{2k-2s-1} b^{2s+1}. \end{aligned} \tag{2.1}$$

Here we can observe that

$$\begin{aligned} \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4 \\ ax+by=N \\ a,b \text{ odd}}} a^{2k-2s-1} b^{2s+1} &= \sum_{m=1}^{N-1} \left(\sum_{\substack{a|m \\ a \text{ odd}}} a^{2k-2s-1} \right) \left(\sum_{\substack{b|N-m \\ b \text{ odd}}} b^{2s+1} \right) \\ &= \sum_{m=1}^{N-1} \sigma_{2k-2s-1,1}(m; 2) \sigma_{2s+1,1}(N-m; 2). \end{aligned}$$

So (2.1) becomes

$$-2 \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{N-1} \sigma_{2k-2s-1,1}(m; 2) \sigma_{2s+1,1}(N-m; 2). \tag{2.2}$$

And the right hand side of Proposition 2.1 is

$$-\frac{1}{2} \sum_{\substack{d|N \\ d \text{ even}}} d^{2k+1} - 2 \sum_{\substack{d|N \\ d \text{ odd}}} \sum_{\substack{1 \leq t < d \\ t \text{ even}}} t^{2k}. \tag{2.3}$$

Then we have

$$-\frac{1}{2} \sum_{\substack{d|N \\ d \text{ even}}} d^{2k+1} = -\frac{1}{2} \sum_{2l|N} (2l)^{2k+1} = -2^{2k} \sum_{l|\frac{N}{2}} l^{2k+1} = -2^{2k} \sigma_{2k+1}\left(\frac{N}{2}\right)$$

and

$$\begin{aligned} -2 \sum_{\substack{d|N \\ d \text{ odd}}} \sum_{\substack{1 \leq t \leq d \\ t \text{ even}}} t^{2k} &= -2 \sum_{\substack{d|N \\ d \text{ odd}}} \sum_{l=1}^{\frac{d-1}{2}} (2l)^{2k} = -2^{2k+1} \sum_{\substack{d|N \\ d \text{ odd}}} \sum_{l=1}^{\frac{d-1}{2}} l^{2k} \\ &= -2^{2k+1} \sum_{\substack{d|N \\ d \text{ odd}}} \frac{1}{2k+1} \sum_{j=0}^{2k} (-1)^j \binom{2k+1}{j} B_j \left(\frac{d-1}{2}\right)^{2k+1-j} \\ &= \sum_{j=0}^{2k} (-1)^{j+1} \binom{2k+1}{j} \frac{2^j B_j}{2k+1} \sum_{\substack{d|N \\ d \text{ odd}}} (d-1)^{2k+1-j} \\ &= \sum_{j=0}^{2k} (-1)^{j+1} \binom{2k+1}{j} \frac{2^j B_j}{2k+1} \sum_{\substack{d|N \\ d \text{ odd}}} \sum_{r=0}^{2k+1-j} \binom{2k+1-j}{r} (-1)^r d^{2k+1-j-r} \\ &= \sum_{j=0}^{2k} (-1)^{j+1} \binom{2k+1}{j} \frac{2^j B_j}{2k+1} \sum_{r=0}^{2k+1-j} \binom{2k+1-j}{r} (-1)^r \sigma_{2k+1-j-r,1}(N; 2) \\ &= \sum_{j=0}^{2k} \sum_{r=0}^{2k+1-j} (-1)^{j+r+1} \binom{2k+1}{j} \binom{2k+1-j}{r} \frac{2^j B_j}{2k+1} \sigma_{2k+1-j-r,1}(N; 2). \end{aligned} \quad (2.4)$$

Therefore (2.3) becomes

$$-2^{2k} \sigma_{2k+1}\left(\frac{N}{2}\right) + \sum_{j=0}^{2k} \sum_{r=0}^{2k+1-j} (-1)^{j+r+1} \binom{2k+1}{j} \binom{2k+1-j}{r} \frac{2^j B_j}{2k+1} \sigma_{2k+1-j-r,1}(N; 2). \quad (2.5)$$

Equating (2.2) and (2.5), we obtain

$$\begin{aligned} \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{N-1} \sigma_{2k-2s-1,1}(m; 2) \sigma_{2s+1,1}(N-m; 2) &= 2^{2k-1} \sigma_{2k+1}\left(\frac{N}{2}\right) \\ &+ \frac{1}{4k+2} \sum_{j=0}^{2k} \sum_{r=0}^{2k+1-j} (-1)^{j+r} \binom{2k+1}{j} \binom{2k+1-j}{r} 2^j B_j \sigma_{2k+1-j-r,1}(N; 2). \end{aligned} \quad (2.6)$$

(b) We have

$$\begin{aligned} \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{N-1} m \sigma_{2k-2s-1,1}(m; 2) \sigma_{2s+1,1}(N-m; 2) \\ = \frac{N}{2} \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{N-1} \sigma_{2k-2s-1,1}(N-m; 2) \sigma_{2s+1,1}(m; 2), \end{aligned}$$

since

$$\begin{aligned} & \sum_{m=1}^{N-1} m \sigma_{2k-2s-1,1}(m; 2) \sigma_{2s+1,1}(N-m; 2) \\ &= \sum_{m=1}^{N-1} (N-m) \sigma_{2k-2s-1,1}(N-m; 2) \sigma_{2s+1,1}(m; 2). \end{aligned}$$

Therefore, we use Lemma 2.1 (a). □

Lemma 2.2. *Let $k, N \in \mathbb{N}$. Then we have*

(a)

$$\begin{aligned} & \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{N-1} \sigma_{2k-2s-1,0}(m; 2) \sigma_{2s+1,0}(N-m; 2) \\ &= 2^{2k} \left\{ \frac{2k+3}{4k+2} \sigma_{2k+1}\left(\frac{N}{2}\right) + \left(\frac{k}{6} - \frac{N}{2}\right) \sigma_{2k-1}\left(\frac{N}{2}\right) \right. \\ & \quad \left. + \frac{1}{2k+1} \sum_{j=2}^k \binom{2k+1}{2j} B_{2j} \sigma_{2k+1-2j}\left(\frac{N}{2}\right) \right\}. \end{aligned}$$

(b)

$$\begin{aligned} & \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{N-1} m \sigma_{2k-2s-1,0}(m; 2) \sigma_{2s+1,0}(N-m; 2) \\ &= 2^{2k-1} N \left\{ \frac{2k+3}{4k+2} \sigma_{2k+1}\left(\frac{N}{2}\right) + \left(\frac{k}{6} - \frac{N}{2}\right) \sigma_{2k-1}\left(\frac{N}{2}\right) \right. \\ & \quad \left. + \frac{1}{2k+1} \sum_{j=2}^k \binom{2k+1}{2j} B_{2j} \sigma_{2k+1-2j}\left(\frac{N}{2}\right) \right\}. \end{aligned}$$

Proof. (a) From (1.2) we obtain

$$\begin{aligned} & \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{N-1} \sigma_{2k-2s-1,0}(m; 2) \sigma_{2s+1,0}(N-m; 2) \\ &= \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{N-1} 2^{2k-2s-1} \sigma_{2k-2s-1}\left(\frac{m}{2}\right) \cdot 2^{2s+1} \sigma_{2s+1}\left(\frac{N-m}{2}\right) \\ &= 2^{2k} \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{N-1} \sigma_{2k-2s-1}(m) \sigma_{2s+1}\left(\frac{N}{2} - m\right). \end{aligned}$$

So we use Proposition 1.1.

(b) We obtain

$$\begin{aligned} & \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{N-1} m \sigma_{2k-2s-1,0}(m; 2) \sigma_{2s+1,0}(N-m; 2) \\ &= \frac{N}{2} \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{N-1} \sigma_{2k-2s-1,0}(N-m; 2) \sigma_{2s+1,0}(m; 2), \end{aligned}$$

since

$$\begin{aligned} & \sum_{m=1}^{N-1} m \sigma_{2k-2s-1,0}(m; 2) \sigma_{2s+1,0}(N-m; 2) \\ &= \sum_{m=1}^{N-1} (N-m) \sigma_{2k-2s-1,0}(N-m; 2) \sigma_{2s+1,0}(m; 2). \end{aligned}$$

Therefore, we refer to Lemma 2.2 (a).

□

Corollary 2.3. *Let $k, N \in \mathbb{N}$. Then we have*

(a)

$$\begin{aligned} & \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{N-1} \sigma_{2k-2s-1}(m) \sigma_{2s+1,0}(N-m; 2) \\ &= \frac{1}{24k+12} \left[(2k+1)(k-6N) \sigma_{2k-1}(N) + 4^k (2k+1)(k-3N) \sigma_{2k-1}\left(\frac{N}{2}\right) \right. \\ & \quad + 3 \left\{ 2^{2k+1} \sigma_{2k+1}\left(\frac{N}{2}\right) + (2k+3) \sigma_{2k+1}(N) \right. \\ & \quad + 2^{2k+1} \sum_{j=2}^k \binom{2k+1}{2j} B_{2j} \sigma_{2k+1-2j}\left(\frac{N}{2}\right) + 2 \sum_{j=2}^k \binom{2k+1}{2j} B_{2j} \sigma_{2k+1-2j}(N) \\ & \quad \left. \left. - \sum_{j=0}^{2k} \sum_{r=0}^{2k+1-j} (-1)^{j+r} 2^j B_j \binom{2k+1}{j} \binom{2k-j+1}{r} \sigma_{2k+1-j-r,1}(N; 2) \right\} \right]. \end{aligned}$$

(b)

$$\begin{aligned}
 & \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{N-1} \sigma_{2k-2s-1,1}(m;2) \sigma_{2s+1,0}(N-m;2) \\
 &= \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{N-1} \sigma_{2k-2s-1,0}(m;2) \sigma_{2s+1,1}(N-m;2) \\
 &= \frac{1}{24k+12} \left[(2k+1)(k-6N) \sigma_{2k-1}(N) - 4^k (2k+1)(k-3N) \sigma_{2k-1}\left(\frac{N}{2}\right) \right. \\
 &\quad - 3 \left\{ 2^{2(k+1)} (k+1) \sigma_{2k+1}\left(\frac{N}{2}\right) - (2k+3) \sigma_{2k+1}(N) \right. \\
 &\quad + 2^{2k+1} \sum_{j=2}^k \binom{2k+1}{2j} B_{2j} \sigma_{2k+1-2j}\left(\frac{N}{2}\right) - 2 \sum_{j=2}^k \binom{2k+1}{2j} B_{2j} \sigma_{2k+1-2j}(N) \\
 &\quad \left. \left. + \sum_{j=0}^{2k} \sum_{r=0}^{2k+1-j} (-1)^{j+r} 2^j B_j \binom{2k+1}{j} \binom{2k-j+1}{r} \sigma_{2k+1-j-r,1}(N;2) \right\} \right].
 \end{aligned}$$

Proof. (a) From (1.1), we can write

$$\begin{aligned}
 & \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{N-1} \sigma_{2k-2s-1,1}(m;2) \sigma_{2s+1,1}(N-m;2) \\
 &= \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{N-1} \{ \sigma_{2k-2s-1}(m) - \sigma_{2k-2s-1,0}(m;2) \} \\
 &\quad \times \{ \sigma_{2s+1}(N-m) - \sigma_{2s+1,0}(N-m;2) \} \\
 &= \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{N-1} \sigma_{2k-2s-1}(m) \sigma_{2s+1}(N-m) \\
 &\quad - 2 \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{N-1} \sigma_{2k-2s-1}(m) \sigma_{2s+1,0}(N-m;2) \\
 &\quad + \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{N-1} \sigma_{2k-2s-1,0}(m;2) \sigma_{2s+1,0}(N-m;2),
 \end{aligned}$$

since

$$\begin{aligned}
 & \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{N-1} \sigma_{2k-2s-1}(m) \sigma_{2s+1,0}(N-m;2) \\
 &= \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{N-1} \sigma_{2k-2s-1,0}(m;2) \sigma_{2s+1}(N-m).
 \end{aligned}$$

Thus we use Proposition 1.1, Lemma 2.1 (a), and Lemma 2.2 (a).

(b) Since

$$\begin{aligned}
 & \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{N-1} \sigma_{2k-2s-1,1}(m; 2) \sigma_{2s+1,0}(N-m; 2) \\
 &= \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{N-1} \sigma_{2k-2s-1}(m) \sigma_{2s+1,0}(N-m; 2) \\
 & \quad - \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{N-1} \sigma_{2k-2s-1,0}(m; 2) \sigma_{2s+1,0}(N-m; 2),
 \end{aligned}$$

therefore we refer to Lemma 2.2 (a) and Corollary 2.3 (a).

□

Remark 2.1. We can rewrite Corollary 2.3 (a) and (b) as follows :

$$\begin{aligned}
 & \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{N-1} \sigma_{2k-2s-1}(m) \sigma_{2s+1,0}(N-m; 2) \\
 &= \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{N-1} \sigma_{2k-2s-1}(m) \cdot 2^{2s+1} \sigma_{2s+1}\left(\frac{N-m}{2}\right) \\
 &= \sum_{s=0}^{k-1} 2^{2s+1} \binom{2k}{2s+1} \sum_{m < \frac{N}{2}} \sigma_{2k-2s-1}(N-2m) \sigma_{2s+1}(m).
 \end{aligned} \tag{2.7}$$

Similarly,

$$\begin{aligned}
 & \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{N-1} \sigma_{2k-2s-1,1}(m; 2) \sigma_{2s+1,0}(N-m; 2) \\
 &= \sum_{s=0}^{k-1} 2^{2s+1} \binom{2k}{2s+1} \sum_{m < \frac{N}{2}} \sigma_{2k-2s-1,1}(N-2m; 2) \sigma_{2s+1}(m).
 \end{aligned} \tag{2.8}$$

Corollary 2.4. Let $k, N \in \mathbb{N}$. Then we have

(a)

$$\begin{aligned}
 & \sum_{s=0}^{k-1} 2^{2s+1} \binom{2k}{2s+1} \sum_{m < \frac{N}{2}} \sigma_{2k-2s-1,0}(N-2m; 2) \sigma_{2s+1}(m) \\
 &= \frac{2^{2k-1}}{3(2k+1)} \left\{ (2k+1)(k-3N) \sigma_{2k-1}\left(\frac{N}{2}\right) + (6k+9) \sigma_{2k+1}\left(\frac{N}{2}\right) \right. \\
 & \quad \left. + 6 \sum_{j=2}^k \binom{2k+1}{2j} B_{2j} \sigma_{2k+1-2j}\left(\frac{N}{2}\right) \right\}.
 \end{aligned}$$

(b)

$$\begin{aligned}
 & \sum_{s=0}^{k-1} 2^{2s+1} \binom{2k}{2s+1} \sum_{m < \frac{N}{2}} \sigma_{2k-2s-1,0}(N-2m; 2) \sigma_{2s+1,0}(m; 2) \\
 &= \frac{4^k}{24k+12} \left[(2k+1)(k-3N) \sigma_{2k-1}\left(\frac{N}{2}\right) + 4^k (2k+1) \left(k - \frac{3}{2}N\right) \sigma_{2k-1}\left(\frac{N}{4}\right) \right. \\
 & \quad + 3 \left\{ 2^{2k+1} \sigma_{2k+1}\left(\frac{N}{4}\right) + (2k+3) \sigma_{2k+1}\left(\frac{N}{2}\right) \right. \\
 & \quad + 2^{2k+1} \sum_{j=2}^k \binom{2k+1}{2j} B_{2j} \sigma_{2k+1-2j}\left(\frac{N}{4}\right) + 2 \sum_{j=2}^k \binom{2k+1}{2j} B_{2j} \sigma_{2k+1-2j}\left(\frac{N}{2}\right) \\
 & \quad \left. \left. - \sum_{j=0}^{2k} \sum_{r=0}^{2k+1-j} (-1)^{j+r} 2^j B_j \binom{2k+1}{j} \binom{2k-j+1}{r} \sigma_{2k+1-j-r,1}\left(\frac{N}{2}; 2\right) \right\} \right].
 \end{aligned}$$

(c)

$$\begin{aligned}
 & \sum_{s=0}^{k-1} 2^{2s+1} \binom{2k}{2s+1} \sum_{m < \frac{N}{2}} \sigma_{2k-2s-1,0}(N-2m; 2) \sigma_{2s+1,1}(m; 2) \\
 &= \frac{1}{48k+24} \left[16^k \{2k(3N-1) + 3N - 4k^2\} \sigma_{2k-1}\left(\frac{N}{4}\right) \right. \\
 & \quad + 2^{2k+1} \left\{ (2k+1)(k-3N) \sigma_{2k-1}\left(\frac{N}{2}\right) - 3 \cdot 2^{2k+1} \sigma_{2k+1}\left(\frac{N}{4}\right) + 3(2k+3) \sigma_{2k+1}\left(\frac{N}{2}\right) \right. \\
 & \quad - 3 \cdot 2^{2k+1} \sum_{j=2}^k \binom{2k+1}{2j} B_{2j} \sigma_{2k+1-2j}\left(\frac{N}{4}\right) + 6 \sum_{j=2}^k \binom{2k+1}{2j} B_{2j} \sigma_{2k+1-2j}\left(\frac{N}{2}\right) \\
 & \quad \left. \left. + 3 \sum_{j=0}^{2k} \sum_{r=0}^{2k+1-j} (-1)^{j+r} 2^j B_j \binom{2k+1}{j} \binom{2k-j+1}{r} \sigma_{2k+1-j-r,1}\left(\frac{N}{2}; 2\right) \right\} \right].
 \end{aligned}$$

Proof. (a) We can see that

$$\begin{aligned}
 & \sum_{s=0}^{k-1} 2^{2s+1} \binom{2k}{2s+1} \sum_{m < \frac{N}{2}} \sigma_{2k-2s-1,0}(N-2m; 2) \sigma_{2s+1}(m) \\
 &= \sum_{s=0}^{k-1} 2^{2s+1} \binom{2k}{2s+1} \sum_{m < \frac{N}{2}} \sigma_{2k-2s-1}(N-2m) \sigma_{2s+1}(m) \\
 & \quad - \sum_{s=0}^{k-1} 2^{2s+1} \binom{2k}{2s+1} \sum_{m < \frac{N}{2}} \sigma_{2k-2s-1,1}(N-2m; 2) \sigma_{2s+1}(m).
 \end{aligned}$$

Thus we refer to (2.7) and (2.8).

(b) Since

$$\begin{aligned}
 & \sum_{s=0}^{k-1} 2^{2s+1} \binom{2k}{2s+1} \sum_{m < \frac{N}{2}} \sigma_{2k-2s-1,0}(N-2m; 2) \sigma_{2s+1,0}(m; 2) \\
 &= 2^{2k} \sum_{s=0}^{k-1} 2^{2s+1} \binom{2k}{2s+1} \sum_{m < \frac{N}{2}} \sigma_{2k-2s-1}(\frac{N}{2}-m) \sigma_{2s+1}(\frac{m}{2}) \\
 &= 2^{2k} \sum_{s=0}^{k-1} 2^{2s+1} \binom{2k}{2s+1} \sum_{m < \frac{N}{4}} \sigma_{2k-2s-1}(\frac{N}{2}-2m) \sigma_{2s+1}(m),
 \end{aligned}$$

therefore we use (2.7).

(c) We note that

$$\begin{aligned}
 & \sum_{s=0}^{k-1} 2^{2s+1} \binom{2k}{2s+1} \sum_{m < \frac{N}{2}} \sigma_{2k-2s-1,0}(N-2m; 2) \sigma_{2s+1,1}(m; 2) \\
 &= \sum_{s=0}^{k-1} 2^{2s+1} \binom{2k}{2s+1} \sum_{m < \frac{N}{2}} \sigma_{2k-2s-1,0}(N-2m; 2) \sigma_{2s+1}(m) \\
 &\quad - \sum_{s=0}^{k-1} 2^{2s+1} \binom{2k}{2s+1} \sum_{m < \frac{N}{2}} \sigma_{2k-2s-1,0}(N-2m; 2) \sigma_{2s+1,0}(m; 2).
 \end{aligned}$$

So we apply Corollary 2.4 (a) and (b). □

Proof of Theorem 1.1. Let $N = 2L$ in (2.8). Then, by (1.2), we have

$$\begin{aligned}
 & \sum_{s=0}^{k-1} 2^{2s+1} \binom{2k}{2s+1} \sum_{m < L} \sigma_{2k-2s-1,1}(2L-2m; 2) \sigma_{2s+1}(m) \\
 &= \sum_{s=0}^{k-1} 2^{2s+1} \binom{2k}{2s+1} \sum_{m < L} \sigma_{2k-2s-1,1}(L-m; 2) \sigma_{2s+1}(m).
 \end{aligned}$$

So we refer to Corollary 2.3 (b). □

Corollary 2.5. Let $k, N \in \mathbb{N}$. Then we have

(a)

$$\begin{aligned}
 & \sum_{s=0}^{k-1} 2^{2s+1} \binom{2k}{2s+1} \sum_{m=1}^{N-1} \sigma_{2k-2s-1}(2m-1) \sigma_{2s+1}(N-m) \\
 &= \frac{1}{24k+12} [(2k+1)(k-6(2N-1)) \sigma_{2k-1}(2N-1) \\
 &\quad + 3 \left\{ (2k+3) \sigma_{2k+1}(2N-1) + 2 \sum_{j=2}^k \binom{2k+1}{2j} B_{2j} \sigma_{2k+1-2j}(2N-1) \right. \\
 &\quad \left. - \sum_{j=0}^{2k} \sum_{r=0}^{2k+1-j} (-1)^{j+r} 2^j B_j \binom{2k+1}{j} \binom{2k-j+1}{r} \sigma_{2k+1-j-r}(2N-1) \right\}].
 \end{aligned}$$

(b)

$$\begin{aligned}
 & \sum_{s=0}^{k-1} 2^{2s+1} \binom{2k}{2s+1} \sum_{m=1}^{N-1} \sigma_{2k-2s-1}(2m) \sigma_{2s+1}(N-m) \\
 &= \frac{1}{24k+12} \left[(2k+1)(k-12N) \sigma_{2k-1}(2N) + 4^k (2k+1)(k-6N) \sigma_{2k-1}(N) \right. \\
 & \quad + 3 \left\{ 2^{2k+1} \sigma_{2k+1}(N) + (2k+3) \sigma_{2k+1}(2N) \right. \\
 & \quad + 2^{2k+1} \sum_{j=2}^k \binom{2k+1}{2j} B_{2j} \sigma_{2k+1-2j}(N) + 2 \sum_{j=2}^k \binom{2k+1}{2j} B_{2j} \sigma_{2k+1-2j}(2N) \\
 & \quad \left. \left. - \sum_{j=0}^{2k} \sum_{r=0}^{2k+1-j} (-1)^{j+r} 2^j B_j \binom{2k+1}{j} \binom{2k-j+1}{r} \sigma_{2k+1-j-r,1}(N; 2) \right\} \right].
 \end{aligned}$$

Proof. (a) Let $N = 2L - 1$ in Corollary 2.3 (a). Then we have

$$\begin{aligned}
 & \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{2L-1} \sigma_{2k-2s-1}(m) \sigma_{2s+1,0}(2L-1-m; 2) \\
 &= \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{L-1} \sigma_{2k-2s-1}(2m-1) \sigma_{2s+1,0}(2L-2m; 2) \\
 &= \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{L-1} \sigma_{2k-2s-1}(2m-1) \cdot 2^{2s+1} \sigma_{2s+1}(L-m).
 \end{aligned}$$

(b) Let $N = 2L$ in Corollary 2.3 (a). Then we have

$$\begin{aligned}
 & \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{2L-1} \sigma_{2k-2s-1}(m) \sigma_{2s+1,0}(2L-m; 2) \\
 &= \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{L-1} \sigma_{2k-2s-1}(2m) \sigma_{2s+1,0}(2L-2m; 2) \\
 &= \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{L-1} \sigma_{2k-2s-1}(2m) \cdot 2^{2s+1} \sigma_{2s+1}(L-m).
 \end{aligned}$$

□

3 Convolution sums for odd N

In this Section we find some convolution sum formulas for odd $N \in \mathbb{N}$.

Theorem 3.1. *Let N be a odd positive integer. Then we have*

$$\begin{aligned}
 & \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m < \frac{N}{2}} \sigma_{2k-2s-1}(N-2m) \sigma_{2s+1,1}(m; 2) \\
 &= \frac{1}{8k+4} \sum_{j=0}^{2k} \sum_{r=0}^{2k+1-j} 2^j (-1)^{j+r} B_j \binom{2k+1}{j} \binom{2k+1-j}{r} \sigma_{2k+1-j-r}(N).
 \end{aligned}$$

Proof. For odd N , we have

$$\begin{aligned}
 & \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4 \\ ax+by=N \\ a,b \text{ odd}}} (f(a-b) - f(a+b)) \\
 &= \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4 \\ ax+by=N \\ a,b \text{ odd} \\ x,y \text{ odd}}} (f(a-b) - f(a+b)) + 2 \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4 \\ ax+by=N \\ a,b \text{ odd} \\ x \text{ odd} \\ y \text{ even}}} (f(a-b) - f(a+b)) \\
 & \quad + \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4 \\ ax+by=N \\ a,b \text{ odd} \\ x,y \text{ even}}} (f(a-b) - f(a+b)) \\
 &= 2 \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4 \\ ax+by=N \\ a,b \text{ odd} \\ x \text{ odd} \\ y \text{ even}}} (f(a-b) - f(a+b)),
 \end{aligned}$$

since there are no elements in $A(N)$ and $B(N)$ where

$$A(N) := \{(a, b, x, y) \in \mathbb{N}^4 \mid ax + by = N \text{ with odd } a, b, x, y, \text{ and odd } N\}$$

and

$$B(N) := \{(a, b, x, y) \in \mathbb{N}^4 \mid ax + by = N \text{ with odd } a, b, \text{ and even } x, y, \text{ and odd } N\}.$$

This concludes that

$$2 \sum_{\substack{ax+by=N \\ a,b \text{ odd} \\ x \text{ odd} \\ y \text{ even} \\ N \text{ odd}}} (f(a-b) - f(a+b)) = \sum_{\substack{ax+by=N \\ a,b \text{ odd} \\ N \text{ odd}}} (f(a-b) - f(a+b)), \quad (3.1)$$

so we need Proposition 2.1 and let $f(x) = x^{2k}$ in (3.1). As the same manner in (2.1), the left hand side of Eq. (3.1) becomes

$$\begin{aligned}
 & 2 \sum_{\substack{ax+by=N \\ a,b \text{ odd} \\ x \text{ odd} \\ y \text{ even} \\ N \text{ odd}}} (f(a-b) - f(a+b)) = -4 \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{\substack{ax+by=N \\ a,b \text{ odd} \\ x \text{ odd} \\ y \text{ even} \\ N \text{ odd}}} a^{2k-2s-1} b^{2s+1} \\
 &= -4 \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{\substack{m < \frac{N}{2} \\ N \text{ odd}}} \sigma_{2k-2s-1}(N-2m) \sigma_{2s+1,1}(m; 2),
 \end{aligned} \quad (3.2)$$

since

$$\begin{aligned}
 \sum_{\substack{ax+by=N \\ a,b \text{ odd} \\ x \text{ odd} \\ y \text{ even} \\ N \text{ odd}}} a^{2k-2s-1} b^{2s+1} &= \sum_{\substack{ax+2by=N \\ a,b,x \text{ odd} \\ N \text{ odd}}} a^{2k-2s-1} b^{2s+1} \\
 &= \sum_{\substack{m < \frac{N}{2} \\ N \text{ odd}}} \left(\sum_{\substack{a|N-2m \\ a \text{ odd} \\ \frac{N-2m}{a} \text{ odd}}} a^{2k-2s-1} \right) \left(\sum_{\substack{b|m \\ b \text{ odd}}} b^{2s+1} \right) \\
 &= \sum_{\substack{m < \frac{N}{2} \\ N \text{ odd}}} \sigma_{2k-2s-1,1}^*(N-2m; 2) \sigma_{2s+1,1}(m; 2) \\
 &= \sum_{\substack{m < \frac{N}{2} \\ N \text{ odd}}} \left(\sigma_{2k-2s-1,1}(N-2m; 2) - \sigma_{2k-2s-1,1}\left(\frac{N-2m}{2}; 2\right) \right) \sigma_{2s+1,1}(m; 2) \\
 &= \sum_{\substack{m < \frac{N}{2} \\ N \text{ odd}}} \sigma_{2k-2s-1}(N-2m) \sigma_{2s+1,1}(m; 2),
 \end{aligned}$$

where we use (1.3) and $\sigma_{s,1}(\text{odd}; 2) = \sigma_s(\text{odd})$. And by Proposition 2.1 and (2.4), the right hand side of (3.1) becomes

$$\begin{aligned}
 \sum_{\substack{ax+by=N \\ a,b \text{ odd} \\ N \text{ odd}}} (f(a-b) - f(a+b)) &= -2 \sum_{\substack{d|N \\ d \text{ odd} \\ N \text{ odd}}} \sum_{\substack{1 \leq t < d \\ t \text{ even}}} t^{2k} \\
 &= \sum_{j=0}^{2k} \sum_{r=0}^{2k+1-j} (-1)^{j+r+1} \binom{2k+1}{j} \binom{2k+1-j}{r} \frac{2^j B_j}{2k+1} \sigma_{2k+1-j-r}(N).
 \end{aligned} \tag{3.3}$$

Equating (3.1) and (3.3), we obtain the proof. $\square$

Corollary 3.2. *Let N be a odd positive integer. Then we have*

$$\begin{aligned}
 &\sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m < \frac{N}{2}} \sigma_{2k-2s-1}(N-2m) \sigma_{2s+1}(2m) \\
 &= \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m < \frac{N}{2}} \sigma_{2k-2s-1}(2m-1) \sigma_{2s+1}(N-2m+1) \\
 &= \frac{1}{24k+12} \{ (2k+1)(k-6N) \sigma_{2k-1}(N) + 3(2k+3) \sigma_{2k+1}(N) \\
 &\quad + 6 \sum_{j=2}^k \binom{2k+1}{2j} B_{2j} \sigma_{2k+1-2j}(N) \}.
 \end{aligned}$$

Proof. By (1.1) and (1.2), we can rewrite Theorem 3.1 as

$$\begin{aligned}
 & \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m < \frac{N}{2}} \sigma_{2k-2s-1}(N-2m) \sigma_{2s+1,1}(m; 2) \\
 &= \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m < \frac{N}{2}} \sigma_{2k-2s-1}(N-2m) \sigma_{2s+1,1}(2m; 2) \\
 &= \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m < \frac{N}{2}} \sigma_{2k-2s-1}(N-2m) \{ \sigma_{2s+1}(2m) - \sigma_{2s+1,0}(2m; 2) \} \\
 &= \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m < \frac{N}{2}} \sigma_{2k-2s-1}(N-2m) \sigma_{2s+1}(2m) \\
 &\quad + \sum_{s=0}^{k-1} 2^{2s+1} \binom{2k}{2s+1} \sum_{m < \frac{N}{2}} \sigma_{2k-2s-1}(N-2m) \sigma_{2s+1}(m).
 \end{aligned}$$

Therefore we use (2.7).

□

4 CONCLUSIONS

Expanding K. S. Williams' result

$$\begin{aligned}
 & \sum_{s=0}^{k-1} \binom{2k}{2s+1} \left(\sum_{m=1}^{N-1} \sigma_{2k-2s-1}(m) \sigma_{2s+1}(N-m) \right) \\
 &= \frac{2k+3}{4k+2} \sigma_{2k+1}(N) + \left(\frac{k}{6} - N \right) \sigma_{2k-1}(N) + \frac{1}{2k+1} \sum_{j=2}^k \binom{2k+1}{2j} B_{2j} \sigma_{2k+1-2j}(N)
 \end{aligned}$$

(see Proposition 1.1), we obtain some similar formulas for odd and even divisor functions, mainly we show

$$\begin{aligned}
 & \sum_{s=0}^{k-1} 2^{2s+1} \binom{2k}{2s+1} \sum_{m < N} \sigma_{2k-2s-1,1}(N-m; 2) \sigma_{2s+1}(m) \\
 &= \frac{1}{24k+12} \left[(2k+1)(k-12N) \sigma_{2k-1}(2N) - 4^k (2k+1)(k-6N) \sigma_{2k-1}(N) \right. \\
 &\quad - 3 \left\{ 2^{2(k+1)} (k+1) \sigma_{2k+1}(N) - (2k+3) \sigma_{2k+1}(2N) \right. \\
 &\quad + 2^{2k+1} \sum_{j=2}^k \binom{2k+1}{2j} B_{2j} \sigma_{2k+1-2j}(N) - 2 \sum_{j=2}^k \binom{2k+1}{2j} B_{2j} \sigma_{2k+1-2j}(2N) \\
 &\quad \left. \left. + \sum_{j=0}^{2k} \sum_{r=0}^{2k+1-j} (-1)^{j+r} 2^j B_j \binom{2k+1}{j} \binom{2k-j+1}{r} \sigma_{2k+1-j-r,1}(N; 2) \right\} \right]
 \end{aligned}$$

in Theorem 1.1.

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