

Codimension two complete intersections and Hilbert-Poincaré series

Gabriel Sticlaru

*Faculty of Mathematics and Informatics,
Ovidius University, Constanta, Romania*

**Short Research
Article**

Received: 22 December 2013

Accepted: XX January 2014

Online Ready: XX January 2014

Published: XX January 2014

Abstract

We investigate the relation between codimension two smooth complete intersections in a projective space and some naturally associated graded algebras. We give some examples of log-concave polynomials and we propose two conjectures for these algebras.

Keywords: Projective hypersurfaces; Singularities; Milnor algebra; Log concavity; Free resolutions; Hilbert-Poincaré series

2010 Mathematics Subject Classification: 13D40; 14J70; 14Q10; 32S25

1 Introduction

Let $S = \mathbb{C}[x_0, \dots, x_n]$ be the graded ring of polynomials in $x_0, \dots, x_n$ with complex coefficients and denote by S_r the vector space of homogeneous polynomials in S of degree r . For any polynomial $f \in S_r$ we define the *Jacobian ideal* $J_f \subset S$ as the ideal spanned by the partial derivatives $f_0, \dots, f_n$ of f with respect to $x_0, \dots, x_n$.

The Hilbert-Poincaré series of a graded S -module M of finite type is defined by

$$HP(M)(t) = \sum_{k \geq 0} \dim M_k \cdot t^k \quad (1)$$

and it is known, to be a rational function of the form

$$HP(M)(t) = \frac{P(M)(t)}{(1-t)^{n+1}} = \frac{Q(M)(t)}{(1-t)^d}. \quad (2)$$

For any polynomial $f \in S_r$ we define the corresponding graded *Milnor* (or *Jacobian*) algebra by

$$M(f) = S/J_f. \quad (3)$$

**Corresponding author: E-mail: gabrielsticlaru@yahoo.com*

The hypersurface $V(f) : f = 0$ is smooth of degree d if and only if $\dim M(f) < \infty$, and then it is known that

$$HP(M(f))(t) = \frac{(1 - t^{d-1})^{n+1}}{(1 - t)^{n+1}}. \quad (4)$$

Our examples and the second conjecture refer to log-concavity and unimodal property. Before listing these examples we recall some basic notions.

A sequence $a_0, a_1, \dots, a_m$ of real numbers is said to be *log-concave* (resp. *strictly log-concave*) if it verifies $a_k^2 \geq a_{k-1}a_{k+1}$ (resp. $a_k^2 > a_{k-1}a_{k+1}$) for $k = 1, 2, \dots, m-1$. An infinite sequence a_k , $k \in \mathbb{N}$ is (strictly) log-concave if any truncation of it is (strictly) log-concave.

Such sequences play an important role in Combinatorics and Algebraic Geometry, see for instance the recent paper [1]. Recall that a sequence $a_0, \dots, a_m$ of real numbers is said to be *unimodal* if there is an integer i between 0 and m such that

$$a_0 \leq a_1 \leq \dots \leq a_{i-1} \leq a_i \geq a_{i+1} \geq \dots \geq a_m.$$

A nonnegative log-concave sequence with no internal zeros (i.e. a sequence for which the indices of the nonzero elements form a set of consecutive integers), is known to be unimodal, see [1] and the references there.

A polynomial $P(t) \in \mathbb{R}[t]$, $P(t) = \sum_{j=0}^m a_j t^j$, with coefficients a_j for $0 \leq j \leq m$, is said to be log-concave (resp. unimodal) if the sequence of its coefficients is log-concave (resp. unimodal). For Singular language and applications, see [2] and [3].

2 Main results

Assume from now on that $V(f)$ is smooth of degree d and let $g \in S_e$ be another homogeneous polynomial of degree e . We say that $V(f, g) = V(f) \cap V(g)$ is a smooth complete intersection if any point $p \in V(f, g)$ is smooth on both $V(f)$ and $V(g)$ and the corresponding tangent spaces $T_p V(f)$ and $T_p V(g)$ are distinct. Our first result is the following.

Theorem 1. *Let $m_2(f, g)$ be the ideal generated by all the 2×2 minors in the Jacobian matrix of the mapping $(f, g) : \mathbb{C}^{n+1} \rightarrow \mathbb{C}^2$. Define $A(f, g) = S/((f) + m_2(f, g))$ and $B(f, g) = S/((f) + (g) + m_2(f, g))$.*

With this notation, the following conditions are equivalent.

- (i) $V(f, g) = V(f) \cap V(g)$ is a smooth complete intersection.
- (ii) $\dim A(f, g) < \infty$.
- (iii) $\dim B(f, g) < \infty$.

If $f = x_0$, then $A(f, g) = B(f, g) = S'/J_{g'} = M(g')$, where $S' = \mathbb{C}[x_1, \dots, x_n]$ and $g'(x_1, \dots, x_n) = g(0, x_1, \dots, x_n)$. However, in general we have the following.

Proposition 2.1. *With the above notation, the ideals $I(f, g) = (f) + m_2(f, g)$ and $J(f, g) = (f) + (g) + m_2(f, g)$ have the same radical, but are distinct in general. When the equivalent conditions of Theorem 1 hold, this common radical is the maximal ideal $(x_0, \dots, x_n)$.*

Proof. (Theorem 1 and Proposition 2.1).

The equivalence of (i) and (iii) in Theorem 1 is a classical fact, see for instance Proposition (6.39) in [4]. To prove the equivalence of (ii) and (iii) in Theorem 1, it is enough to prove the first part of the claim in Proposition 2.1, namely that the ideals $I(f, g)$ and $J(f, g)$ have the same radical. To do this, it is enough by Hilbert's Nullstellensatz, see for instance Theorem (2.14) in [4]

to show that the corresponding zero sets $V(I(f, g))$ and $V(J(f, g))$ in $\mathbb{P}^n$ coincide. Moreover, it is clear that $V(J(f, g)) \subset V(I(f, g))$.

Conversely, note that a point p in $V(I(f, g))$ satisfies $f(p) = 0$ and the differentials $df(p)$ and $dg(p)$ are proportional. Since $df(p) \neq 0$ as $V(f)$ is supposed to be smooth, it follows that there is $\lambda \in \mathbb{C}$ such that $dg(p) = \lambda df(p)$. This implies via the Euler formula that $g(p) = 0$, hence $p \in V(J(f, g))$.

The second part of the claim in Proposition 2.1 is dealt with in the next section by means of examples computed using Singular. □

Remark 2.1. Note that the equivalent condition in Theorem 1 do not imply $\dim S/m_2(f, g) < \infty$. Take for instance $f = x_0$ and $g = x_0^2 + x_1^2 + \dots + x_n^2$.

As a consequence of Conjecture 3.1 we can prove the following formulas for the Hilbert-Poincaré series of the graded S -modules $A(f, g)$ and $B(f, g)$. The result is given in the form $HP(M)(t) = \frac{P(M)(t)}{(1-t)^4}$ and hence it is enough to give the polynomial $P(M)$.

Proposition 2.2. The Hilbert-Poincaré series of a graded S -module $A(f, g)$ and $B(f, g)$ depend only on the degrees d and e , when the equivalent conditions of Theorem 1 hold. More precisely, one has the following in $\mathbb{P}^3$.

$$(i) P(A(f, g))(t) = 1 - [t^d + 6t^{d+e-2}] + [4t^{d+2e-3} + 4t^{2d+e-3} + 6t^{2d+e-2}] - [t^{d+3e-4} + t^{2d+2e-4} + 4t^{2d+2e-3} + t^{3d+e-4} + 4t^{3d+e-3}] + [t^{2d+3e-4} + t^{3d+2e-4} + t^{4d+e-4}].$$

$$(ii) P(B(f, g))(t) = 1 - [t^e + t^d + 6t^{d+e-2}] + [4t^{d+e-1} + 4t^{d+2e-3} + 4t^{2d+e-3}] - [t^{d+3e-4} + t^{2d+2e-4} + 4t^{2d+2e-3} + t^{3d+e-4}] + [t^{2d+3e-4} + t^{3d+2e-4}].$$

Proof. We explain now briefly how to get Proposition 2.2 from Conjecture 3.1. To get the formulas, we start with the resolution and get $HP(M)(t) = HP(S)(t) - HP(R_1)(t) + HP(R_2)(t) - HP(R_3)(t) + HP(R_4)(t)$, where $HP(S)(t) = \frac{1}{(1-t)^4}$. Then we use the well-known formulas $HP(N_1 \oplus N_2)(t) = HP(N_1)(t) + HP(N_2)(t)$, $HP(N^p(-q))(t) = pt^q HP(N)(t)$, $HP(S(-k))(t) = \frac{t^k}{(1-t)^4}$. □

Remark 2.2. Note that if $f = x$ and g of degree e , we obtain the Hilbert-Poincaré series for any smooth hypersurfaces of degree e , in $\mathbb{P}^2$: $S(t) = \frac{(1-t^{e-1})^3}{(1-t)^3} = (1+t+t^2+\dots+t^{e-2})^3$.

3 Examples and conjectures

The examples and the conjectures concern codimension 2 complete intersections in $\mathbb{P}^3$ with coordinates $(x : y : z : w)$.

Example 3.1. The polynomial g does not belong to the ideal $I(f, g)$ in general, i.e. $I(f, g) \neq J(f, g)$. Let $V(f) : f = x^2 + y^2 + z^2 + w^2 = 0$ Fermat smooth surface of degree two and nodal surfaces $2A_1$ type, $V(g) : g = xzw + (z+w)y^2 + x^3 + x^2y + xy^2 + y^3 = 0$.

Indeed $\text{Radical}(I(f, g)) = \text{Radical}(J(f, g)) = \text{Ideal}(x, y, z, w)$. The ideal $I(f, g)$ has seven generators,

$I(f, g) = (G_1, G_2, G_3, G_4, G_5, G_6, G_7)$, where:

$$G_1 = 2x^3 - 2x^2y + 2xy^2 - 2y^3 + 4xyz + 4xyw - 2yzw,$$

$$G_2 = 2xy^2 - 6x^2z - 4xyz - 2y^2z + 2x^2w - 2z^2w,$$

$$G_3 = 2xy^2 + 2x^2z - 6x^2w - 4xyw - 2y^2w - 2zw^2,$$

$$G_4 = 2y^3 - 2x^2z - 4xyz - 6y^2z - 4yz^2 + 2xyw - 4yzw,$$

$$G_5 = 2y^3 + 2xyz - 2x^2w - 4xyw - 6y^2w - 4yzw - 4yw^2,$$

$$G_6 = 2y^2z + 2xz^2 - 2y^2w - 2xw^2,$$

$$G_7 = x^2 + y^2 + z^2 + w^2.$$

One can check the identity: $g = (7/2)G_1 + (1/2)G_2 + (-1/2)G_3 + 2G_4 + (-2)G_5 + 3G_6 + (-6x + 8y + 8z - 8w)G_7 + (-8z^3 - 28xyz + 2y^2w + xzw + 7yzw + 9z^2w + 12xw^2 - 16yw^2 - 9zw^2 + 8w^3)$ where Normal Form (remainder modulo $I(f, g)$) of g is not zero, so g does not belong to $I(f, g)$.

Example 3.2. The Hilbert-Poincaré series of a graded S -module $A(f, g)$ and $B(f, g)$ depend only on the degrees d and e , when the equivalent conditions of Theorem 1 hold.

The table below contain singular cubic projective surface in $\mathbb{P}^3$ displayed with the type of singularity.

Type	polynomial g of degree three
A_2	$(x + y + z)(x + 2y + 3z)w + xyz$
$2A_1$	$xzw + (z + w)y^2 + x^3 + x^2y + xy^2 + y^3$
$A_1 + A_2$	$x^3 + y^3 + x^2y + xy^2 + y^2z + xzw$
A_4	$y^2z + yx^2 - z^3 + xzw$
$3A_1$	$y^3 + y^2(x + z + w) + 4xzw$
$A_1 + A_3$	$wxz + (x + z)(y^2 - x^2)$
A_5	$wxz + y^2z + x^3 - z^3$
D_4	$w(x + y + z)^2 + xyz$
$2A_1 + A_2$	$wxz + y^2(x + y + z)$
$A_1 + A_4$	$wxz + y^2z + yx^2$
D_5	$wx^2 + xz^2 + y^2z$
$4A_1$	$w(xy + xz + yz) + xyz$
$A_1 + 2A_2$	$wxz + xy^2 + y^3$
$2A_1 + A_3$	$wxz + (x + z)y^2$
$A_1 + A_5$	$wxz + y^2z + x^3$
E_6	$wx^2 + xz^2 + y^3$
$3A_2$	$wxz + y^3$
$\tilde{E}_8$	$y^3 + 2z^3 + 4w^3$

- Let $d = 2, e = 3$, $V(f) : f = x^2 + y^2 + z^2 + w^2 = 0$ and $V(g) : g = 0$ any singular cubic projective surface in $\mathbb{P}^3$ from the table above. In all cases, g not belongs to $I(f, g)$, $I(f, g)$ and $J(f, g)$ have the same radical $\text{Ideal}(x, y, z, w)$, but $A(f, g)$ and $B(f, g)$ are distinct, with different Hilbert-Poincaré finite series, log-concave polynomials.

$$HP(A(f, g))(t) = 1 + 4t + 9t^2 + 10t^3 + 5t^4 + t^5.$$

$$HP(B(f, g))(t) = 1 + 4t + 9t^2 + 9t^3 + 5t^4 + t^5.$$

- Let $d = 2, e = 3$, $V(f) : f = x^2 + y^2 + z^2 + w^2 = 0$ and $V(g) : g = 0$ any singular cubic projective surface in $\mathbb{P}^3$ from the list below.

$$A_3 : g = xzw + (x + z)(y^2 - x^2 - z^2) = 0,$$

$$2A_2 : g = x^3 + y^3 + x^2y + xy^2 + xzw = 0.$$

In these cases, g not belongs to $I(f, g)$. $I(f, g)$ and $J(f, g)$ have the same radical other than $\text{Ideal}(x, y, z, w)$, but $A(f, g)$ and $B(f, g)$ are distinct, with different Hilbert-Poincaré infinite series.

$$HP(A(f, g))(t) = 1 + 4t + 9t^2 + 10t^3 + 5t^4 + 2(t^5 + t^6 + \dots),$$

$$HP(B(f, g))(t) = 1 + 4t + 9t^2 + 9t^3 + 5t^4 + 2(t^5 + t^6 + \dots).$$

- Let $d = 3, e = 3$, $V(f) : f = x^3 + y^3 + z^3 + w^3 = 0$ and $V(g) : g = 0$ any singular cubic projective surface in $\mathbb{P}^3$ from the table above. In all cases, g not belongs to $I(f, g)$, $I(f, g)$ and $J(f, g)$ have the same radical $\text{Ideal}(x, y, z, w)$, but $A(f, g)$ and $B(f, g)$ are distinct, with different Hilbert-Poincaré finite series, log-concave polynomials.

$$HP(A(f, g))(t) = 1 + 4t + 10t^2 + 19t^3 + 25t^4 + 22t^5 + 12t^6 + 3t^7.$$

$$HP(B(f, g))(t) = 1 + 4t + 10t^2 + 18t^3 + 21t^4 + 16t^5 + 8t^6 + 2t^7.$$

4. Let $d = 1, e = 3$, $V(f) : f = x = 0$ and $V(g) : g = 0$ any singular cubic projective surface in $\mathbb{P}^3$ from the table above. We obtain the Hilbert-Poincaré series for any smooth hypersurfaces of degree 3 in $\mathbb{P}^2$: $S(t) = \frac{(1-t^2)^3}{(1-t)^3} = 1 + 3t + 3t^2 + t^3$.

These examples motivate our following conjectures.

The best way to understand the graded S -modules $A(f, g)$ and $B(f, g)$ is to construct their free resolutions. Based on the free resolution, we can compute the Hilbert-Poincaré series for each graded algebra. We propose the following conjecture, concern codimension 2 complete intersections in $\mathbb{P}^3$

Conjecture 3.1. Let $A(f, g)$ and $B(f, g)$ with $\dim A(f, g)$ and $\dim B(f, g) < \infty$. Then the minimal graded free resolutions of these algebras are the following.

$$0 \rightarrow R_4 \rightarrow R_3 \rightarrow R_2 \rightarrow R_1 \rightarrow S \rightarrow M \rightarrow 0 \quad (5)$$

(i) If $M=A(f, g)$,

$$\begin{aligned} R_1 &= S(-d) \oplus S^6[-(d+e-2)], \\ R_2 &= S^4[-(d+2e-3)] \oplus S^4[-(2d+e-3)] \oplus S^6[-(2d+e-2)], \\ R_3 &= S[-(d+3e-4)] \oplus S[-(2d+2e-4)] \oplus S^4[-(2d+2e-3)] \oplus S[-(3d+e-4)] \\ &\oplus S^4[-(3d+e-3)], \\ R_4 &= S[-(2d+3e-4)] \oplus S[-(3d+2e-4)] \oplus S[-(4d+e-4)]. \end{aligned}$$

(ii) If $M=B(f, g)$,

$$\begin{aligned} R_1 &= S(-e) \oplus S(-d) \oplus S^6[-(d+e-2)], \\ R_2 &= S^4[-(d+e-1)] \oplus S^4[-(d+2e-3)] \oplus S^4[-(2d+e-3)], \\ R_3 &= S[-(d+3e-4)] \oplus S[-(2d+2e-4)] \oplus S^4[-(2d+2e-3)] \oplus S[-(3d+e-4)], \\ R_4 &= S[-(2d+3e-4)] \oplus S[-(3d+2e-4)]. \end{aligned}$$

Conjecture 3.2. When the equivalent conditions of Theorem 1 hold, the Hilbert-Poincaré finite series $HP(A(f, g))$ and $HP(B(f, g))$ are log-concave polynomials with no internal zeros, in particular they are unimodal.

4 Conclusion

The main results of this paper are Theorem 1 and Proposition 2.1 in $\mathbb{P}^n$ and Proposition 2.2 in $\mathbb{P}^3$.

By analogy with Milnor algebra associated with one homogeneous polynomial in $\mathbb{P}^n$, we define two graded algebras $A(f, g)$ and $B(f, g)$ associated with homogeneous polynomials f and g , where projective hypersurface $V(f) : f = 0$ is smooth. We replace the *Jacobian ideal* with two ideals $I(f, g)$ and $J(f, g)$, generated by f, g and 2×2 minors in the Jacobian matrix (f, g) . For these graded algebras we consider Hilbert-Poincaré series that encapsulates information about the dimensions of homogeneous components. When $f = x_0$, the Hilbert-Poincaré series corresponds to the smooth hypersurface in $\mathbb{P}^{n-1}$, as in the Milnor algebra case.

When the equivalent conditions of Theorem 1 hold, the Hilbert-Poincaré series of a graded module $A(f, g)$ and $B(f, g)$ depend only on the degrees of f and g , as in Milnor algebra smooth case, and are log-concave polynomials.

The main conclusion of the given examples 3.1 and 3.2 are Conjectures 3.1 and 3.2 in $\mathbb{P}^3$.

Finally, we consider this construction as a new method to generate many, infinite families of log-concave polynomials.

Acknowledgment

The author is grateful to A. Dimca for suggesting this problem.

References

- [1] Huh J. Milnor numbers of projective hypersurfaces and the chromatic polynomial of graphs. J. Amer. Math. Soc. 25, 2012, 907-927.
- [2] Greuel G.M. Pfister G. A Singular Introduction to Commutative Algebra, Springer-Verlag, 2007.
- [3] Decker W. Greuel G.M. Pfister G. Schönemann H. A computer algebra system for polynomial computations, 2013.
Available: <http://www.singular.uni-kl.de>.
- [4] Dimca A. Topics on Real and Complex Singularities. Vieweg und Sohn, Braunschweig, 1987, 242+xvii pp.

©2014 Gabriel Sticlaru; This is an Open Access article distributed under the terms of the Creative Commons Attribution License <http://creativecommons.org/licenses/by/2.0>, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Peer-review history:

The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)
www.sciencedomain.org/review-history.php?iid=204&id=6&aid=1064