

Applications to the quadratic forms and product-to-sum formulae

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Abstract

In this paper, we attempt to express two values

$$G(q) := \sum_{n=1}^{\infty} g(n)q^n = 2^4 q \prod_{n=1}^{\infty} \frac{(1+q^n)^{32} (1-q^n)^{16}}{(1+q^{2n})^{12}},$$

$$H(q) := \sum_{n=1}^{\infty} h(n)q^n = 2^{12} q^3 \prod_{n=1}^{\infty} (1+q^{2n})^4 (1-q^{4n})^{16}$$

into sums of squares and obtain

$$g(n) = 8 \sum_{\substack{(x_1, x_2, \dots, x_{11}, x_{12}) \in \mathbb{Z}^{12} \\ x_1^2 + x_2^2 + \dots + x_9^2 + x_{10}^2 + 2(x_{11}^2 + x_{12}^2) = n}} (x_{10}^2 - 2x_{12}^2),$$

$$h(n) = 1024 \sum_{\substack{(x_1, x_2, x_3, x_4) \in \mathbb{Z}^4 \\ 2(x_1^2 + x_2^2 + 2x_4^2) + x_3^2 = n}} (-1)^{x_1^2 + x_2^2 + 1} (x_1^4 - 3x_1^2 x_2^2) (x_3^2 - 4x_4^2)$$

(see Theorem 1.1 and Theorem 1.2, respectively). Finally, we obtain some relations $\varphi_1(q)$ according to $\xi_1(q)$, $\varphi(q)$, and $\psi(q)$.

Keywords: Quadratic forms; Theta functions; Eisenstein series; Infinite products; Product-to-sum formulae

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1 Introduction

The set of positive integers is denoted by $\mathbb{N}$ and the set of nonnegative integers by $\mathbb{N}_0$. Throughout this paper $q \in \mathbb{C}$ is taken to satisfy $|q| < 1$, for such q we define

$$E_k = E_k(q) := \prod_{n \in \mathbb{N}} (1 - q^{kn}), \quad k \in \mathbb{N}.$$

As following the K. S. Williams' paper [6], we define

$$\tilde{\sigma}_k(n) := \begin{cases} \sum_{\substack{d \in \mathbb{N} \\ d|n \\ n/d \text{ odd}}} d^k, & \text{if } n \in \mathbb{N}, \\ 0, & \text{if } n \in \mathbb{Q}, n \notin \mathbb{N}, \end{cases}$$

for $k \in \mathbb{N}_0$ and $n \in \mathbb{Q}$. The Eisenstein series $\xi_k(q)$ is defined for $k \in \mathbb{N}$ with $k \equiv 1 \pmod{2}$ by

$$\xi_k(q) := \sum_{n=1}^{\infty} \tilde{\sigma}_k(n) q^n = \sum_{n=1}^{\infty} \frac{n^k q^n}{1 - q^{2n}}. \quad (1.1)$$

The one-dimensional theta function $\varphi_k(q)$ is defined for $k \in \mathbb{N}_0$ by

$$\varphi_k(q) := \sum_{n=-\infty}^{\infty} n^{2k} q^{n^2} \quad \text{and} \quad \varphi(q) := \varphi_0(q) = \sum_{n=-\infty}^{\infty} q^{n^2}.$$

We also require the theta function

$$\psi(q) := \sum_{n=0}^{\infty} q^{n(n+1)/2}.$$

For example Cooper introduces the famous identity

$$q\psi^8(q) = \sum_{n=1}^{\infty} \frac{n^3 q^n}{1 - q^{2n}}$$

in ([3], Eq. (3.71), p. 136), therefore, by (1.1), we have

$$\xi_3(q) = q\psi^8(q). \quad (1.2)$$

Next we define the two-dimensional theta function $\Phi_{k,l,m}(q)$ by

$$\Phi_{k,l,m}(q) := \sum_{r,s=-\infty}^{\infty} \left(r\sqrt{l} + s\sqrt{-m} \right)^{2k} q^{lr^2 + ms^2}, \quad k, l, m \in \mathbb{N}, \quad (1.3)$$

and the quantities

$$\begin{aligned} A_{l,m}(q) &:= l\xi_1(-q^l) - m\xi_1(-q^m), \\ B_{l,m}(q) &:= l^2\xi_3(-q^l) + m^2\xi_3(-q^m). \end{aligned} \quad (1.4)$$

In this paper, we attempt to express two values

$$G(q) := \sum_{n=1}^{\infty} g(n)q^n = 2^4 q \prod_{n=1}^{\infty} \frac{(1+q^n)^{32} (1-q^n)^{16}}{(1+q^{2n})^{12}} \quad (1.5)$$

and

$$H(q) := \sum_{n=1}^{\infty} h(n)q^n = 2^{12} q^3 \prod_{n=1}^{\infty} (1+q^{2n})^4 (1-q^{4n})^{16} \quad (1.6)$$

into sums of squares based on K. S. Williams' paper [6]. Here $G(q)$ and $H(q)$ are first defined in the paper [4] to obtain some convolution sum formulae related to $\sum_{m < n/8} \sigma_3(m) \sigma_3(n-8m)$.

In Sections 2 and 3 we obtain as following results :

Theorem 1.1. *Let $n \in \mathbb{N}$. Then we have*

$$g(n) = 8 \sum_{\substack{(x_1, x_2, \dots, x_{11}, x_{12}) \in \mathbb{Z}^{12} \\ x_1^2 + x_2^2 + \dots + x_9^2 + x_{10}^2 + 2(x_{11}^2 + x_{12}^2) = n}} (x_{10}^2 - 2x_{12}^2).$$

Theorem 1.2. *Let $n \in \mathbb{N}$. Then we have*

$$h(n) = 1024 \sum_{\substack{(x_1, x_2, x_3, x_4) \in \mathbb{Z}^4 \\ 2(x_1^2 + x_2^2 + 2x_4^2) + x_3^2 = n}} (-1)^{x_1^2 + x_2^2 + 1} (x_1^4 - 3x_1^2 x_2^2) (x_3^2 - 4x_4^2).$$

Finally, in Section 4 we obtain some properties of $\varphi_1(q)$ according to $\xi_1(q)$, $\varphi(q)$, and $\psi(q)$:

Lemma 1.3. *For $q \in \mathbb{C}$ with $|q| < 1$ we obtain*

(a)

$$\varphi_1(q)\xi_1(q) + \varphi_1(-q)\xi_1(-q) = -4\xi_1(q)\xi_1(-q)\varphi(q^4),$$

(b)

$$\varphi_1(q)\xi_1(q) - \varphi_1(-q)\xi_1(-q) = -8q\xi_1(q)\xi_1(-q)\psi(q^8),$$

(c)

$$\varphi_1^2(q)\xi_1^2(q) - \varphi_1^2(-q)\xi_1^2(-q) = 32q\xi_1^2(q)\xi_1^2(-q)\psi^2(q^4),$$

(d)

$$\varphi_1(q)\varphi(-q) + \varphi_1(-q)\varphi(q) = 4\varphi(-q^2)\varphi_1(-q^2),$$

(e)

$$\varphi_1(q)\varphi(-q) - \varphi_1(-q)\varphi(q) = 4q\varphi^4(-q^4)\psi^2(q^4),$$

(f)

$$\varphi_1^2(q)\varphi^2(-q) - \varphi_1^2(-q)\varphi^2(q) = 16q\varphi(-q^2)\varphi^4(-q^4)\psi^2(q^4)\varphi_1(-q^2).$$

2 Proof of Theorem 1.1

We can find Proposition 2.1 and its proof as the main theorem in [6] :

Proposition 2.1. (See ([6], Theorem 1.1)) Let $k \in \mathbb{N}_0$ and $l \in \mathbb{N}$. Let $r, s, t, u \in \mathbb{N}_0$ be such that

$$r + s + t + u = k. \quad (2.1)$$

Let $v, w, x, y \in \mathbb{N}_0$ be such that

$$v + w + x + y = l. \quad (2.2)$$

Set

$$m = k + 2l \quad (2.3)$$

so that $m \in \mathbb{N}$ and $m \geq 2$. Let

$$\begin{aligned} P(x_1, \dots, x_m) = & \frac{1}{2^l} \prod_{g=r+1}^{r+v} (x_g^2 - 2x_{g+s+l+y}^2) \prod_{g=r+v+1}^{r+v+w} (x_g^2 - 3x_{g+s+t+l+y}^2) \\ & \times \prod_{g=r+v+w+1}^{r+v+w+x} (x_g^2 - 4x_{g+s+t+l+y+u}^2) \prod_{g=r+v+w+x+1}^{r+l} (x_g^4 - 3x_g^2 x_{g+y}^2) \end{aligned} \quad (2.4)$$

and

$$\begin{aligned} Q(x_1, \dots, x_m) = & x_1^2 + \dots + x_{r+l+y}^2 + 2x_{r+l+y+1}^2 \dots + 2x_{r+s+l+v+y}^2 \\ & + 3x_{r+s+l+v+y+1}^2 + \dots + 3x_{r+s+t+l+v+w+y}^2 \\ & + 4x_{r+s+t+l+v+w+y+1}^2 + \dots + 4x_m^2. \end{aligned} \quad (2.5)$$

Let

$$\begin{aligned} a_1 &= -2r + 2v + 4y, & a_6 &= 5t + 3w, \\ a_2 &= 5r - 2s + v + 3w + 2y, & a_8 &= -2s + 5u + 2v, \\ a_3 &= -2t, & a_{12} &= -2t, \\ a_4 &= -2r + 5s - 2u + v + 6x + 4y, & a_{16} &= -2u. \end{aligned} \quad (2.6)$$

Then, for $n \in \mathbb{N}$ with $n \geq l$, we have

$$\left[q^l E_1^{a_1} E_2^{a_2} E_3^{a_3} E_4^{a_4} E_6^{a_6} E_8^{a_8} E_{12}^{a_{12}} E_{16}^{a_{16}} \right]_n = \sum_{\substack{(x_1, \dots, x_m) \in \mathbb{Z}^m \\ Q(x_1, \dots, x_m) = n}} P(x_1, \dots, x_m) \quad (2.7)$$

and

$$a_1 + 2a_2 + 3a_3 + 4a_4 + 6a_6 + 8a_8 + 12a_{12} + 16a_{16} = 24l,$$

where P is a polynomial in $x_1, \dots, x_m$ with rational coefficients and Q is a positive-definite, diagonal, quadratic form in $x_1, \dots, x_m$ with integral coefficients. And K. S. Williams writes

$$[f(q)]_n = f_n, \quad n \in \mathbb{N}_0,$$

$$\text{if } f(q) = \sum_{n=0}^{\infty} f_n q^n.$$

Using Proposition 2.1 we can obtain Theorem 1.1 easily, but we need more skill for Theorem 1.2 so we calculate it in Section 3 independently.

Proof of Theorem 1.1. Let us change $G(q)$ in (1.5) as

$$\begin{aligned}
 G(q) &= 2^4 q \prod_{n=1}^{\infty} \frac{(1+q^n)^{32} (1-q^n)^{16}}{(1+q^{2n})^{12}} \\
 &= 2^4 q \prod_{n=1}^{\infty} \frac{(1+q^n)^{32} (1-q^n)^{16}}{(1+q^{2n})^{12}} \cdot \frac{(1-q^{2n})^{12}}{(1-q^{2n})^{12}} \cdot \frac{(1-q^n)^{16}}{(1-q^n)^{16}} \\
 &= 2^4 q \prod_{n=1}^{\infty} \frac{(1+q^n)^{16} (1-q^{2n})^{16} (1-q^{2n})^{12} (1-q^n)^{16}}{(1-q^{4n})^{12} (1-q^n)^{16}} \\
 &= 2^4 q \prod_{n=1}^{\infty} \frac{(1-q^{2n})^{44}}{(1-q^{4n})^{12} (1-q^n)^{16}} \\
 &= 2^4 q E_2^{44} E_4^{-12} E_1^{-16}.
 \end{aligned} \tag{2.8}$$

Then we apply (2.8) to Proposition 2.1 and so, by (2.2), (2.6) and (2.7), we obtain

$$\begin{aligned}
 l &= v + w + x + y = 1, \\
 a_1 &= -2r + 2v + 4y = -16, \\
 a_2 &= 5r - 2s + v + 3w + 2y = 44, \\
 a_3 &= -2t = 0, \\
 a_4 &= -2r + 5s - 2u + v + 6x + 4y = -12, \\
 a_6 &= 5t + 3w = 0, \\
 a_8 &= -2s + 5u + 2v = 0, \\
 a_{12} &= -2t = 0, \\
 a_{16} &= -2u = 0.
 \end{aligned} \tag{2.9}$$

So from a_3, a_6 , and a_{16} we can deduce that

$$t = 0, \quad w = 0, \quad \text{and} \quad u = 0. \tag{2.10}$$

Since $u = 0$ we have

$$s = v$$

from a_8 and similarly, because $w = 0$ we obtain

$$v + x + y = 1$$

from l . Also a_1 leads

$$-r + v + 2y = -8 \tag{2.11}$$

and since $s = v$ and $w = 0$, the condition a_2 follows

$$5r - 2s + v + 3w + 2y = 5r - v + 2y = 44. \tag{2.12}$$

Again by using the facts $u = 0$, $s = v$, and $v + x + y = 1$ the condition a_4 shows that

$$\begin{aligned} & -2r + 5s - 2u + v + 6x + 4y \\ &= -2r + 5v + v + 6x + 4y \\ &= -2r + 6(v + x + y) - 2y \\ &= -2r + 6 - 2y \\ &= -12 \end{aligned}$$

so

$$r + y = 9. \quad (2.13)$$

Then, by (2.13), we can write (2.11) as

$$-r + v + 2y = -r + v + 2(9 - r) = -3r + v + 18 = -8$$

and so

$$v = 3r - 26 \quad (2.14)$$

which shows that

$$r \geq 9 \quad (2.15)$$

since Proposition 2.1 restricts $v, r \in \mathbb{N}_0$. Therefore, by combining (2.15) and $y \in \mathbb{N}_0$, (2.13) concludes that

$$r = 9 \quad \text{and} \quad y = 0. \quad (2.16)$$

Sequentially, by (2.16), Eq. (2.14) constructs

$$v = 1 \quad (2.17)$$

and it is obvious

$$s = 1 \quad (2.18)$$

by $s = v$. Lastly, by (2.16) and (2.17), the condition $v + x + y = 1$ leads

$$x = 0. \quad (2.19)$$

Now all values also confirm Eq. (2.12). Let us find k and m :

$$\begin{aligned} k &= r + s + t + u = 10, \\ m &= k + 2l = 12, \end{aligned}$$

where we used (2.1), (2.3), (2.9), (2.10), (2.16), and (2.18). So (2.4) becomes

$$P(x_1, \dots, x_{12}) = \frac{1}{2} (x_{10}^2 - 2x_{12}^2)$$

and (2.5) constructs

$$Q(x_1, \dots, x_{12}) = x_1^2 + \dots + x_{10}^2 + 2x_{11}^2 + 2x_{12}^2,$$

thus these polynomials deduce (2.7) as

$$[qE_1^{-16}E_2^{44}E_4^{-12}]_n = \sum_{\substack{(x_1, \dots, x_{12}) \in \mathbb{Z}^{12} \\ x_1^2 + \dots + x_{10}^2 + 2x_{11}^2 + 2x_{12}^2 = n}} \frac{1}{2} (x_{10}^2 - 2x_{12}^2).$$

Therefore we can write (2.8) as

$$\begin{aligned} G(q) &= \sum_{n=1}^{\infty} g(n)q^n = 2^4 \sum_{n=1}^{\infty} [qE_1^{-16}E_2^{44}E_4^{-12}]_n q^n \\ &= 2^3 \sum_{\substack{(x_1, x_2, \dots, x_{11}, x_{12}) \in \mathbb{Z}^{12} \\ x_1^2 + x_2^2 + \dots + x_9^2 + x_{10}^2 + 2(x_{11}^2 + x_{12}^2) = n}} (x_{10}^2 - 2x_{12}^2) q^n. \end{aligned}$$

□

3 Proof of Theorem 1.2

As mentioned in Section 2, we can not use Proposition 2.1 to express $H(q)$ as sums of squares :

Remark 3.1. From (1.6) we have

$$\begin{aligned} H(q) &= 2^{12} q^3 \prod_{n=1}^{\infty} (1 + q^{2n})^4 (1 - q^{4n})^{16} \\ &= 2^{12} q^3 \prod_{n=1}^{\infty} (1 + q^{2n})^4 \cdot \frac{(1 - q^{2n})^4}{(1 - q^{2n})^4} \cdot (1 - q^{4n})^{16} \\ &= 2^{12} q^3 \prod_{n=1}^{\infty} (1 - q^{4n})^{20} (1 - q^{2n})^{-4} \\ &= 2^{12} q^3 E_4^{20} E_2^{-4}. \end{aligned} \tag{3.1}$$

Then, applying (3.1) to (2.7) in Proposition 2.1, we obtain (2.2) and (2.6) as

$$\begin{aligned} l &= v + w + x + y = 3, \\ a_1 &= -2r + 2v + 4y = 0, \\ a_2 &= 5r - 2s + v + 3w + 2y = -4, \\ a_3 &= -2t = 0, \\ a_4 &= -2r + 5s - 2u + v + 6x + 4y = 20, \\ a_6 &= 5t + 3w = 0, \\ a_8 &= -2s + 5u + 2v = 0, \\ a_{12} &= -2t = 0, \\ a_{16} &= -2u = 0. \end{aligned} \tag{3.2}$$

Thus from a_3, a_6, a_8 , and a_{16} in (3.2) we can observe that

$$t = 0, \quad w = 0, \quad u = 0, \quad \text{and} \quad s = v.$$

So since $w = 0$, l in (3.2) shows that

$$v + w + x + y = v + x + y = 3. \quad (3.3)$$

Also because $u = 0$ and $s = v$, a_4 leads that

$$\begin{aligned} & -2r + 5s - 2u + v + 6x + 4y \\ & = -2r + 5v + v + 6x + 4y \\ & = -2r + 6(v + x + y) - 2y \\ & = 20 \end{aligned}$$

so

$$-r + 3(v + x + y) - y = 10. \quad (3.4)$$

Combining (3.3) and (3.4), we have

$$r + y = -1,$$

which contradicts that $r, y \in \mathbb{N}_0$ as Proposition 2.1 suggests.

By the reason of Remark 3.1, we approach $H(q)$ in another way. Now, the infinite product representations of $\varphi(\pm q)$ and $\psi(\pm q)$ due to Jacobi are given by the following proposition :

Proposition 3.1. (See ([1], p. 851)) Let $n \in \mathbb{N}$. Then we have

$$\begin{aligned} \varphi(q) &= \prod_{n=1}^{\infty} \frac{(1 - q^{2n})^5}{(1 - q^n)^2 (1 - q^{4n})^2}, & \varphi(-q) &= \prod_{n=1}^{\infty} \frac{(1 - q^n)^2}{(1 - q^{2n})}, \\ \psi(q) &= \prod_{n=1}^{\infty} \frac{(1 - q^{2n})^2}{(1 - q^n)}, & \psi(-q) &= \prod_{n=1}^{\infty} \frac{(1 - q^n)(1 - q^{4n})}{(1 - q^{2n})}. \end{aligned}$$

And in ([1], Lemma 2.3) we can see that

$$q \prod_{n=1}^{\infty} (1 - q^{2n})^4 (1 - q^{4n})^4 = q\varphi^4(-q^2)\psi^4(q^2). \quad (3.5)$$

By using the definition of the two-dimensional theta function $\Phi_{k,l,m}(q)$, that is (1.3), K. S. Williams obtained very important results as :

Proposition 3.2. (See ([6], Theorem 2.4)) For $q \in \mathbb{C}$ with $|q| < 1$ we have

(a)

$$\Phi_{1,1,4}(q) = \sum_{r,s=-\infty}^{\infty} (r + s\sqrt{-4})^2 q^{r^2+4s^2} = -2A_{1,4}(q)\varphi(q)\varphi(q^4) = 2qE_4^6,$$

(b)

$$\Phi_{2,1,1}(q) = \sum_{r,s=-\infty}^{\infty} (r + s\sqrt{-1})^4 q^{r^2+s^2} = -2B_{1,1}(q)\varphi^2(q) = 4qE_1^4 E_2^2 E_4^4.$$

Proof of Theorem 1.2. We can rewrite $H(q)$ in (1.6) as

$$\begin{aligned} H(q) &= 2^{12} q^3 \prod_{n=1}^{\infty} (1 + q^{2n})^4 (1 - q^{4n})^{16} \\ &= 2^{12} q^3 \prod_{n=1}^{\infty} (1 + q^{2n})^4 \cdot \frac{(1 - q^{2n})^4}{(1 - q^{2n})^4} \cdot (1 - q^{4n})^{16} \\ &= 2^{12} q^3 \prod_{n=1}^{\infty} \frac{(1 - q^{4n})^{16}}{(1 - q^{2n})^4} \cdot (1 - q^{4n})^4 \\ &= 2^{12} q^3 \prod_{n=1}^{\infty} \frac{(1 - q^{4n})^{16}}{(1 - q^{2n})^8} \cdot (1 - q^{2n})^4 (1 - q^{4n})^4 \\ &= 2^{12} q^2 \left(\prod_{n=1}^{\infty} \frac{(1 - q^{4n})^2}{(1 - q^{2n})} \right)^8 \left(q \prod_{n=1}^{\infty} (1 - q^{2n})^4 (1 - q^{4n})^4 \right) \\ &= 2^{12} q^2 \psi^8(q^2) \cdot q\varphi^4(-q^2)\psi^4(q^2), \end{aligned} \tag{3.6}$$

where we used Proposition 3.1 and (3.5) for the last line. Now, by (1.4) and Proposition 3.2 (b), we note that

$$B_{1,1}(q) = 2\xi_3(-q)$$

and so

$$\Phi_{2,1,1}(q) = -2B_{1,1}(q)\varphi^2(q) = -4\xi_3(-q)\varphi^2(q). \tag{3.7}$$

Therefore, by (1.2) and (3.7), Eq. (3.6) becomes

$$\begin{aligned} H(q) &= 2^{12} \xi_3(q^2) \cdot q\varphi^4(-q^2)\psi^4(q^2) \\ &= 2^{12} \left(-\frac{\Phi_{2,1,1}(-q^2)}{4\varphi^2(-q^2)} \right) \cdot q\varphi^4(-q^2)\psi^4(q^2) \\ &= -2^{10} \Phi_{2,1,1}(-q^2) \cdot q\varphi^2(-q^2)\psi^4(q^2). \end{aligned} \tag{3.8}$$

Similarly, let us investigate the last part $q\varphi^2(-q^2)\psi^4(q^2)$ in (3.8) :

$$\begin{aligned} q\varphi^2(-q^2)\psi^4(q^2) &= q \left(\prod_{n=1}^{\infty} \frac{(1 - q^{2n})^2}{(1 - q^{4n})} \right)^2 \left(\prod_{n=1}^{\infty} \frac{(1 - q^{4n})^2}{(1 - q^{2n})} \right)^4 \\ &= q \prod_{n=1}^{\infty} \frac{(1 - q^{2n})^4 (1 - q^{4n})^8}{(1 - q^{4n})^2 (1 - q^{2n})^4} \\ &= q \prod_{n=1}^{\infty} (1 - q^{4n})^6 = qE_4^6 = \frac{1}{2} \Phi_{1,1,4}(q), \end{aligned} \tag{3.9}$$

where we used Proposition 3.2 (a) for the last line. So, by (3.9), Eq. (3.8) becomes

$$H(q) = -2^{10}\Phi_{2,1,1}(-q^2) \cdot \frac{1}{2}\Phi_{1,1,4}(q) = -2^9\Phi_{2,1,1}(-q^2)\Phi_{1,1,4}(q). \quad (3.10)$$

Then by using Proposition 3.2 again, we have

$$\begin{aligned} \Phi_{1,1,4}(q) &= \sum_{r,s=-\infty}^{\infty} (r + s\sqrt{-4})^2 q^{r^2+4s^2} \\ &= \sum_{r,s=-\infty}^{\infty} (r^2 + 2\sqrt{-4}rs - 4s^2) q^{r^2+4s^2} \\ &= \sum_{r,s=-\infty}^{\infty} (r^2 - 4s^2) q^{r^2+4s^2}, \end{aligned} \quad (3.11)$$

since $\sum_{r=-\infty}^{\infty} r^{2k-1}q^{r^2} = 0$ for $k \in \mathbb{N}$, and also,

$$\begin{aligned} \Phi_{2,1,1}(-q^2) &= \sum_{r,s=-\infty}^{\infty} (r + s\sqrt{-1})^4 (-q^2)^{r^2+s^2} \\ &= \sum_{r,s=-\infty}^{\infty} (r + s\sqrt{-1})^4 (-1)^{r^2+s^2} q^{2(r^2+s^2)} \\ &= \sum_{r,s=-\infty}^{\infty} (r^2 + 2\sqrt{-1}rs - s^2)^2 (-1)^{r^2+s^2} q^{2(r^2+s^2)} \\ &= \sum_{r,s=-\infty}^{\infty} (r^4 + 4\sqrt{-1}r^3s - 4\sqrt{-1}rs^3 - 6r^2s^2 + s^4) (-1)^{r^2+s^2} q^{2(r^2+s^2)} \\ &= \sum_{r,s=-\infty}^{\infty} (r^4 - 6r^2s^2 + s^4) (-1)^{r^2+s^2} q^{2(r^2+s^2)} \\ &= \sum_{r,s=-\infty}^{\infty} \left\{ (r^4 - 3r^2s^2) (-1)^{r^2+s^2} q^{2(r^2+s^2)} + (s^4 - 3s^2r^2) (-1)^{s^2+r^2} q^{2(s^2+r^2)} \right\} \\ &= 2 \sum_{r,s=-\infty}^{\infty} (r^4 - 3r^2s^2) (-1)^{r^2+s^2} q^{2(r^2+s^2)}. \end{aligned} \quad (3.12)$$

So, by (3.11) and (3.12), we can write (3.10) as

$$\begin{aligned}
 H(q) &= -2^9 \left(2 \sum_{r,s=-\infty}^{\infty} (r^4 - 3r^2 s^2) (-1)^{r^2+s^2} q^{2(r^2+s^2)} \right) \left(\sum_{r,s=-\infty}^{\infty} (r^2 - 4s^2) q^{r^2+4s^2} \right) \\
 &= -2^{10} \left(\sum_{(x_1, x_2) \in \mathbb{Z}^2} (x_1^4 - 3x_1^2 x_2^2) (-1)^{x_1^2+x_2^2} q^{2(x_1^2+x_2^2)} \right) \\
 &\quad \times \left(\sum_{(x_3, x_4) \in \mathbb{Z}^2} (x_3^2 - 4x_4^2) q^{x_3^2+4x_4^2} \right) \\
 &= -2^{10} \sum_{(x_1, x_2, x_3, x_4) \in \mathbb{Z}^4} (-1)^{x_1^2+x_2^2} (x_1^4 - 3x_1^2 x_2^2) (x_3^2 - 4x_4^2) q^{2(x_1^2+x_2^2+2x_4^2)+x_3^2} \\
 &= 2^{10} \sum_{\substack{(x_1, x_2, x_3, x_4) \in \mathbb{Z}^4 \\ 2(x_1^2+x_2^2+2x_4^2)+x_3^2=n}} (-1)^{x_1^2+x_2^2+1} (x_1^4 - 3x_1^2 x_2^2) (x_3^2 - 4x_4^2) q^n.
 \end{aligned} \tag{3.13}$$

Thus

$$\begin{aligned}
 H(q) &= \sum_{n=1}^{\infty} h(n) q^n \\
 &= 2^{10} \sum_{\substack{(x_1, x_2, x_3, x_4) \in \mathbb{Z}^4 \\ 2(x_1^2+x_2^2+2x_4^2)+x_3^2=n}} (-1)^{x_1^2+x_2^2+1} (x_1^4 - 3x_1^2 x_2^2) (x_3^2 - 4x_4^2) q^n.
 \end{aligned}$$

□

4 Some properties of $\varphi_1(q)$

First let us search the basic identities satisfied by $\varphi(q)$ and $\psi(q)$:

Proposition 4.1. (See ([2], p. 15, 71, 72)) For $q \in \mathbb{C}$ with $|q| < 1$ we have

(a)

$$\varphi(q) + \varphi(-q) = 2\varphi(q^4),$$

(b)

$$\varphi(q) - \varphi(-q) = 4q\psi(q^8),$$

(c)

$$\varphi(q)\psi(q^2) = \psi^2(q),$$

(d)

$$\varphi(q)\varphi(-q) = \varphi^2(-q^2),$$

(e)

$$\varphi^2(q) + \varphi^2(-q) = 2\varphi^2(q^2),$$

(f)

$$\varphi^2(q) - \varphi^2(-q) = 8q\psi^2(q^4).$$

Some results of Toh [5] construct $\varphi_1(q)$ with $\xi_1(q)$, $\xi_3(q)$, and $\varphi(q)$.

Proposition 4.2. (See ([6], Theorem 2.1), ([5], p. 187)) For $q \in \mathbb{C}$ with $|q| < 1$ we have

(a)

$$\varphi_1(q) = -2\varphi(q)\xi_1(-q),$$

(b)

$$\varphi_2(q) = 2\varphi(q) (6\xi_1^2(-q) - \xi_3(-q)).$$

Using Proposition 4.2 we can obtain some properties as follows :

Lemma 4.1. For $q \in \mathbb{C}$ with $|q| < 1$ we obtain

(a)

$$\xi_1(q) + \xi_1(-q) = 4\xi_1(q^2),$$

(b)

$$\xi_1(q) - \xi_1(-q) = 2q\psi^4(q^2),$$

(c)

$$\xi_1^2(q) - \xi_1^2(-q) = 8q\psi^4(q^2)\xi_1(q^2).$$

Proof. (a) By (1.1) we can observe that

$$\xi_1(q) = \sum_{n=1}^{\infty} \frac{nq^n}{1-q^{2n}} = \sum_{\substack{n=1 \\ 2|n}}^{\infty} \frac{nq^n}{1-q^{2n}} + \sum_{\substack{n=1 \\ 2 \nmid n}}^{\infty} \frac{nq^n}{1-q^{2n}} \quad (4.1)$$

and

$$\xi_1(-q) = \sum_{n=1}^{\infty} \frac{n(-q)^n}{1-(-q)^{2n}} = \sum_{\substack{n=1 \\ 2|n}}^{\infty} \frac{nq^n}{1-q^{2n}} - \sum_{\substack{n=1 \\ 2 \nmid n}}^{\infty} \frac{nq^n}{1-q^{2n}}. \quad (4.2)$$

Thus combining (4.1) and (4.2) we have

$$\xi_1(q) + \xi_1(-q) = 2 \sum_{\substack{n=1 \\ 2|n}}^{\infty} \frac{nq^n}{1-q^{2n}} = 2 \sum_{n=1}^{\infty} \frac{2nq^{2n}}{1-q^{4n}} = 4 \sum_{n=1}^{\infty} \frac{nq^{2n}}{1-q^{4n}} = 4\xi_1(q^2).$$

(b) In ([6], p. 85) we can see that

$$A_{1,2}(q) = -q\varphi^2(-q)\psi^2(q^4). \quad (4.3)$$

And by the definition of $A_{l,m}(q)$ in (1.4) we have $A_{1,2}(q) = \xi_1(-q) - 2\xi_1(-q^2)$ so combining (4.3) we obtain

$$\begin{aligned}\xi_1(q) - \xi_1(-q) &= A_{1,2}(-q) - A_{1,2}(q) \\ &= q\varphi^2(q)\psi^2(q^4) - (-q\varphi^2(-q)\psi^2(q^4)) \\ &= q\psi^2(q^4) (\varphi^2(q) + \varphi^2(-q)).\end{aligned}$$

This leads that

$$\xi_1(q) - \xi_1(-q) = q\psi^2(q^4) \cdot 2\varphi^2(q^2) = 2q (\varphi(q^2)\psi(q^4))^2 = 2q\psi^4(q^2), \quad (4.4)$$

where we used Proposition 4.1 (c) and (e).

(c) Multiplying Lemma 4.1 (a) and (b) we have

$$\begin{aligned}\xi_1^2(q) - \xi_1^2(-q) &= (\xi_1(q) + \xi_1(-q)) (\xi_1(q) - \xi_1(-q)) \\ &= 4\xi_1(q^2) \cdot 2q\psi^4(q^2) = 8q\psi^4(q^2)\xi_1(q^2).\end{aligned}$$

□

Proof of Lemma 1.3. (a) By Proposition 4.1 (a) and Proposition 4.2 (a) we obtain

$$\begin{aligned}\varphi_1(q)\xi_1(q) + \varphi_1(-q)\xi_1(-q) \\ &= -2\varphi(q)\xi_1(-q)\xi_1(q) + (-2\varphi(-q)\xi_1(q)\xi_1(-q)) \\ &= -2\xi_1(q)\xi_1(-q) (\varphi(q) + \varphi(-q)) \\ &= -2\xi_1(q)\xi_1(-q) \cdot 2\varphi(q^4) \\ &= -4\xi_1(q)\xi_1(-q)\varphi(q^4).\end{aligned}$$

(b) From Proposition 4.1 (b) and Proposition 4.2 (a) we have

$$\begin{aligned}\varphi_1(q)\xi_1(q) - \varphi_1(-q)\xi_1(-q) \\ &= -2\varphi(q)\xi_1(-q)\xi_1(q) - (-2\varphi(-q)\xi_1(q)\xi_1(-q)) \\ &= -2\xi_1(q)\xi_1(-q) (\varphi(q) - \varphi(-q)) \\ &= -2\xi_1(q)\xi_1(-q) \cdot 4q\psi(q^8) \\ &= -8q\xi_1(q)\xi_1(-q)\psi(q^8).\end{aligned}$$

(c) Multiply Lemma 1.3 (a) and (b), that is,

$$\begin{aligned}\varphi_1^2(q)\xi_1^2(q) - \varphi_1^2(-q)\xi_1^2(-q) \\ &= (\varphi_1(q)\xi_1(q) + \varphi_1(-q)\xi_1(-q)) (\varphi_1(q)\xi_1(q) - \varphi_1(-q)\xi_1(-q)) \\ &= -4\xi_1(q)\xi_1(-q)\varphi(q^4) (-8q\xi_1(q)\xi_1(-q)\psi(q^8)) \\ &= 32q\xi_1^2(q)\xi_1^2(-q) \cdot \varphi(q^4)\psi(q^8) \\ &= 32q\xi_1^2(q)\xi_1^2(-q)\psi^2(q^4),\end{aligned}$$

where we used Proposition 4.1 (c) for the last line.

(d) By Proposition 4.1 (d), Proposition 4.2 (a) and Lemma 4.1 (a) we note that

$$\begin{aligned}
 & \varphi_1(q)\varphi(-q) + \varphi_1(-q)\varphi(q) \\
 &= -2\varphi(q)\xi_1(-q)\varphi(-q) + (-2\varphi(-q)\xi_1(q)\varphi(q)) \\
 &= -2\varphi(q)\varphi(-q) (\xi_1(q) + \xi_1(-q)) \\
 &= -2 \cdot \varphi(q)\varphi(-q) \cdot 4\xi_1(q^2) \\
 &= -2 \cdot \varphi^2(-q^2) \cdot 4\xi_1(q^2) \\
 &= 4\varphi(-q^2) (-2\varphi(-q^2)\xi_1(q^2)) \\
 &= 4\varphi(-q^2)\varphi_1(-q^2).
 \end{aligned}$$

(e) By Proposition 4.1 (d), Proposition 4.2 (a), and (4.4) we can know that

$$\begin{aligned}
 & \varphi_1(q)\varphi(-q) - \varphi_1(-q)\varphi(q) \\
 &= -2\varphi(q)\xi_1(-q)\varphi(-q) - (-2\varphi(-q)\xi_1(q)\varphi(q)) \\
 &= 2\varphi(q)\varphi(-q) (\xi_1(q) - \xi_1(-q)) \\
 &= 2 \cdot \varphi(q)\varphi(-q) \cdot 2q\varphi^2(q^2)\psi^2(q^4) \\
 &= 2 \cdot \varphi^2(-q^2) \cdot 2q\varphi^2(q^2)\psi^2(q^4) \\
 &= 4q \cdot \varphi^2(-q^2)\varphi^2(q^2) \cdot \psi^2(q^4) \\
 &= 4q\varphi^4(-q^4)\psi^2(q^4).
 \end{aligned}$$

(f) Multiplying Lemma 1.3 (d) and (e) we have

$$\begin{aligned}
 & \varphi_1^2(q)\varphi^2(-q) - \varphi_1^2(-q)\varphi^2(q) \\
 &= (\varphi_1(q)\varphi(-q) + \varphi_1(-q)\varphi(q)) (\varphi_1(q)\varphi(-q) - \varphi_1(-q)\varphi(q)) \\
 &= 4\varphi(-q^2)\varphi_1(-q^2) \cdot 4q\varphi^4(-q^4)\psi^2(q^4) \\
 &= 16q\varphi(-q^2)\varphi^4(-q^4)\psi^2(q^4)\varphi_1(-q^2).
 \end{aligned}$$

□

5 Appendix

We list first eighteen values of $g(n)$ in the Table 1

n	$g(n)$	n	$g(n)$	n	$g(n)$
1	16	7	-25728	13	233056
2	256	8	-49152	14	260096
3	1728	9	-44976	15	398976
4	6144	10	-53760	16	393216
5	10976	11	-55744	17	-301280
6	3072	12	73728	18	-523008

TABLE 1. $g(n)$ for n ($1 \leq n \leq 18$)

and $h(n)$ in the Table 2.

n	$h(n)$	n	$h(n)$	n	$h(n)$
1	0	7	−24576	13	540672
2	0	8	0	14	0
3	4096	9	−163840	15	385024
4	0	10	0	16	0
5	16384	11	−20480	17	−163840
6	0	12	0	18	0

TABLE 2. $h(n)$ for n ($1 \leq n \leq 18$)

6 Conclusions

The purpose of this paper is to express two values

$$G(q) := \sum_{n=1}^{\infty} g(n)q^n = 2^4 q \prod_{n=1}^{\infty} \frac{(1+q^n)^{32} (1-q^n)^{16}}{(1+q^{2n})^{12}},$$

$$H(q) := \sum_{n=1}^{\infty} h(n)q^n = 2^{12} q^3 \prod_{n=1}^{\infty} (1+q^{2n})^4 (1-q^{4n})^{16}$$

into sums of squares as

$$g(n) = 8 \sum_{\substack{(x_1, x_2, \dots, x_{11}, x_{12}) \in \mathbb{Z}^{12} \\ x_1^2 + x_2^2 + \dots + x_9^2 + x_{10}^2 + 2(x_{11}^2 + x_{12}^2) = n}} (x_{10}^2 - 2x_{12}^2),$$

$$h(n) = 1024 \sum_{\substack{(x_1, x_2, x_3, x_4) \in \mathbb{Z}^4 \\ 2(x_1^2 + x_2^2 + 2x_4^2) + x_3^2 = n}} (-1)^{x_1^2 + x_2^2 + 1} (x_1^4 - 3x_1^2 x_2^2) (x_3^2 - 4x_4^2).$$

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