

Algorithms for Solving Doubly Bordered Tridiagonal Linear Systems

Moawwad El-Mikkawy^{*1} and Faiz Atlan¹

¹Mathematics Department, Faculty of Science, Mansoura University, Mansoura 35516, Egypt

ABSTRACT

In this paper we focus on solving doubly bordered tridiagonal linear systems of equations via

transformation. It investigates numeric and symbolic algorithms for solving such systems. The computational cost of the algorithms are given. A MAPLE procedure based on these algorithms is listed. Some illustrative examples are introduced.

Keywords: Doubly bordered; Tridiagonal matrices; Partitioned matrices; Algorithm; LU factorization; MAPLE.

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1 INTRODUCTION AND BASIC DEFINITIONS

We begin this section by giving the following definitions:

Definition 1.1 [4]. An $n \times n$ matrix $A = (a_{ij})_{i,j=1}^n$ is said to have upper bandwidth p if p is the smallest integer such that:

$$a_{ij} = 0, \quad j - i \geq p,$$

and similarly, to have lower bandwidth q if q is the smallest integer such that:

$$a_{ij} = 0, \quad i - j \geq q.$$

The bandwidth, w for the matrix A is defined to be $w = p + q - 1$.

Definition 1.2 [6]. The symmetric matrix $A = (a_{ij})_{i,j=1}^n$ is called positive definite if and only if

$$\mathbf{x}^T A \mathbf{x} > 0, \quad \text{for all } \mathbf{x} \in \mathbb{R}^n, \quad \mathbf{x} \neq 0.$$

Definition 1.3 [6]. An $n \times n$ matrix A is called diagonally dominant if

$$|a_{ii}| \geq \sum_{\substack{j=1, \\ j \neq i}}^n |a_{ij}|, \quad \text{holds for each } i = 1, 2, \dots, n,$$

^{*}Corresponding author: E-mail: m-elmikkawy@yahoo.com

and strictly diagonally dominant if

$$|a_{ii}| > \sum_{\substack{j=1, \\ j \neq i}}^n |a_{ij}|, \quad \text{holds for each } i = 1, 2, \dots, n.$$

In many scientific and engineering applications, different special linear systems of equations arise. For such systems the coefficient matrix has special structure. Sparse matrices which contain a majority of zeros occur are often encountered. It is usually more efficient to solve these systems using tailor-made algorithms, much faster and with less storage than a full matrix. This can be achieved by taking advantage of the special structure of the coefficient matrix. Important examples are band matrices, and the most common cases are the matrices of tridiagonal type for which $p = q = w - 1 = 2$. Tridiagonal systems of linear equations take the form:

$$T \mathbf{x} = \mathbf{f}, \quad (1.1)$$

where

$$T = \begin{bmatrix} d_1 & a_1 & 0 & \dots & 0 \\ b_1 & d_2 & a_2 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & b_{n-2} & d_{n-1} & a_{n-1} \\ 0 & \dots & 0 & b_{n-1} & d_n \end{bmatrix}, \quad (1.2)$$

$\mathbf{x} = [x_1, x_2, \dots, x_n]^T$ and $\mathbf{f} = [f_1, f_2, \dots, f_n]^T$. The superscript T corresponds to the transpose operation. This type of matrices frequently appears in many applications. For example in parallel computing, telecommunication system analysis, solving differential equations using finite differences, heat conduction and fluid flow problems. A general $n \times n$ tridiagonal matrix of the form (1.2) can be stored in $3n - 2$ memory locations, rather than n^2 memory locations for a full matrix, by using three vectors $\mathbf{a} = [a_1, a_2, \dots, a_{n-1}]$, $\mathbf{b} = [b_1, b_2, \dots, b_{n-1}]$, and $\mathbf{d} = [d_1, d_2, \dots, d_n]$. This is always a good habit in computation in order to save memory space. To study tridiagonal matrices it is convenient to introduce vector $\mathbf{c}$ defined by [12, 14]:

$$\mathbf{c} = [c_1, c_2, \dots, c_n], \quad (1.3)$$

where

$$c_i = \begin{cases} d_1, & \text{for } i = 1 \\ d_i - \frac{a_{i-1} b_{i-1}}{c_{i-1}}, & \text{for } i = 2, 3, \dots, n. \end{cases} \quad (1.4)$$

In [10] it is shown that the tridiagonal matrix (1.2) is positive definite if and only if $c_i > 0$, for all $i = 1, 2, \dots, n$. This is an easy way to check whether a tridiagonal matrix is positive definite or not.

For some important results concerning tridiagonal matrix the reader may refer to [2, 5, 9–11, 13–17, 20, 21, 25–27, 29]. The motivation of the current paper is to derive algorithms for solving doubly bordered tridiagonal linear systems of two types of the forms:

$$\begin{bmatrix} d_1 & a_1 & 0 & \dots & \dots & 0 & p_1 \\ b_1 & d_2 & a_2 & \ddots & & \vdots & p_2 \\ 0 & b_2 & d_3 & a_3 & \ddots & \vdots & p_3 \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots \\ \vdots & & \ddots & b_{n-3} & d_{n-2} & a_{n-2} & p_{n-2} \\ 0 & \dots & \dots & 0 & b_{n-2} & d_{n-1} & a_{n-1} \\ q_1 & q_2 & q_3 & \dots & q_{n-2} & b_{n-1} & d_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-2} \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_{n-2} \\ f_{n-1} \\ f_n \end{bmatrix}, \quad (1.5)$$

and

$$\begin{bmatrix} d_1 & a_1 & p_{n-2} & \dots & p_3 & p_2 & p_1 \\ b_1 & d_2 & a_2 & 0 & \dots & \dots & 0 \\ q_{n-2} & b_2 & d_3 & a_3 & \ddots & & \vdots \\ \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ q_3 & \vdots & \ddots & b_{n-3} & d_{n-2} & a_{n-2} & 0 \\ q_2 & \vdots & & \ddots & b_{n-2} & d_{n-1} & a_{n-1} \\ q_1 & 0 & \dots & \dots & 0 & b_{n-1} & d_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-2} \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} g_1 \\ g_2 \\ g_3 \\ \vdots \\ g_{n-2} \\ g_{n-1} \\ g_n \end{bmatrix} \quad (1.6)$$

respectively. The linear systems (1.5) and (1.6) frequently occur in engineering computation and analysis, e.g. in computation of electric power system and in solution of partial differential equations, as referred in [1, 3, 7, 8, 18, 19, 22–24, 28, 30]. The linear systems defined by (1.5) and (1.6) are called doubly bordered tridiagonal linear systems of type (A) and of type (B), respectively.

Throughout this paper, the word 'simplify' means simplify the expression under consideration to its simplest rational form.

The organization of the paper is as follows. The main results are given in Section 2 and Section 3. In Section 4, a MAPLE procedure is introduced. Some illustrative examples are given in Section 5.

2 ALGORITHMS FOR SOLVING DOUBLY BORDERED TRIDIAGONAL LINEAR SYSTEMS OF TYPE (A)

In this section, we are going to consider the construction of new algorithms for solving doubly bordered tridiagonal linear systems of type (A) via transformation.

For this purpose it is convenient to introduce the vector $\mathbf{c} = [c_1, c_2, \dots, c_n]$ whose first $(n - 1)$ components, $c_1, c_2, \dots, c_{n-1}$ are given by (1.4).

The system (1.5) can be completely described by the augmented matrix, G given by:

$$G := [T_n, \mathbf{f}] = \left[\begin{array}{cccccc|c} d_1 & a_1 & 0 & \dots & \dots & 0 & p_1 & f_1 \\ b_1 & d_2 & a_2 & \ddots & & \vdots & p_2 & f_2 \\ 0 & b_2 & d_3 & a_3 & \ddots & \vdots & p_3 & f_3 \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots & \vdots \\ \vdots & & \ddots & b_{n-3} & d_{n-2} & a_{n-2} & p_{n-2} & f_{n-2} \\ 0 & \dots & \dots & 0 & b_{n-2} & d_{n-1} & a_{n-1} & f_{n-1} \\ q_1 & q_2 & q_3 & \dots & q_{n-2} & b_{n-1} & d_n & f_n \end{array} \right]. \quad (2.1)$$

The matrix G contains all the information about the system (1.5) that is necessary to determine its solution, but in a compact form. Let $\mathbf{R}_i$ denotes the i -th row of the matrix G . Performing the following row operations on G , in the same order:

$$\mathbf{R}_1 \leftarrow \mathbf{R}_1 / c_1$$

For $i = 2, 3, \dots, n - 1$ do

$$\mathbf{R}_i \leftarrow \mathbf{R}_i - b_{i-1} \mathbf{R}_{i-1}$$

$$\mathbf{R}_i \leftarrow \mathbf{R}_i / c_i$$

End do.

For $j = 1, 2, \dots, n - 1$ do

$$\mathbf{R}_n \leftarrow \mathbf{R}_n - h_j \mathbf{R}_j$$

End do.

$$\mathbf{R}_n \leftarrow \mathbf{R}_n / c_n.$$

Then we have the resulting transformed linear system of the form:

$$\begin{bmatrix} 1 & y_1 & 0 & \dots & 0 & v_1 & z_1 \\ 0 & 1 & y_2 & \ddots & \vdots & v_2 & z_2 \\ \vdots & \ddots & \ddots & \ddots & 0 & \vdots & \vdots \\ \vdots & & \ddots & 1 & y_{n-2} & v_{n-2} & z_{n-2} \\ \vdots & & & \ddots & 1 & v_{n-1} & z_{n-1} \\ 0 & \dots & & & 0 & 1 & z_n \end{bmatrix}, \quad (2.2)$$

where

$$c_1 = d_1, \quad y_1 = \frac{a_1}{c_1}, \quad v_1 = \frac{p_1}{c_1}, \quad z_1 = \frac{f_1}{c_1}, \quad h_1 = q_1,$$

For $i = 2, 3, \dots, n-2$ do

$$\begin{aligned} c_i &= d_i - b_{i-1} y_{i-1}, \\ y_i &= \frac{a_i}{c_i}, \\ v_i &= \frac{1}{c_i} (p_i - b_{i-1} v_{i-1}), \\ z_i &= \frac{1}{c_i} (f_i - b_{i-1} z_{i-1}). \\ h_i &= q_i - h_{i-1} y_{i-1}, \end{aligned}$$

End do.

$$\begin{aligned} c_{n-1} &= d_{n-1} - b_{n-2} y_{n-2}, \\ v_{n-1} &= \frac{1}{c_{n-1}} (a_{n-1} - b_{n-2} v_{n-2}), \\ z_{n-1} &= \frac{1}{c_{n-1}} (f_{n-1} - b_{n-2} z_{n-2}), \\ h_{n-1} &= b_{n-1} - h_{n-2} y_{n-2}, \\ c_n &= d_n - \sum_{i=1}^{n-1} h_i v_i, \\ z_n &= \frac{1}{c_n} (f_n - \sum_{i=1}^{n-1} h_i z_i). \end{aligned} \quad (2.3)$$

It is well known that [4] the reduced system (2.2) has the same solution as the original linear system (1.5). At this stage, the determinant of the coefficient matrix in (1.5) can be computed using the following computational symbolic algorithm. It is an extension of the **DETGTRI** algorithm in [12] and the **PERTRI** algorithm in [13]. The parameter 's' in the algorithm is just a symbolic name. It is a dummy argument and its actual value is zero.

Algorithm 2.1. An algorithm for computing the determinant of doubly bordered

tridiagonal matrices of type (A).

To compute the determinant of the coefficient matrix in (1.5), we may proceed as follows:

INPUT: Order of the coefficient matrix n and the components, a_i, d_i, b_i, p_i, q_i .

OUTPUT: The determinant of the coefficient matrix in (1.5).

Step 1: Set $c_1 = d_1$. If $c_1 = 0$ then $c_1 = s$ end if.

Step 2: For $i = 2, 3, \dots, n - 1$ do

 Compute and simplify:

$$c_i = d_i - \frac{a_{i-1} b_{i-1}}{c_{i-1}}. \text{ If } c_i = 0 \text{ then } c_i = s \text{ end if.}$$

End do.

Step 3: Compute c_n using (2.3).

Step 4: Compute and simplify:

$$P(s) = \prod_{r=1}^n c_r.$$

Step 5: $\det(T_n) = P(0)$.

The Algorithm 2.1, will be referred to as **DETDTRI-A** algorithm.

The transformed system (2.2) is triangular and easy to solve by backward substitution. Consequently, the linear system (1.5) can be solved using the following algorithm:

Algorithm 2.2. Numeric algorithm for solving doubly bordered tridiagonal linear system of type (A).

To solve the linear system of the form (1.5), we may proceed as follows:

INPUT: Order of the coefficient matrix n and the components, $a_i, d_i, b_i, f_i, p_i, q_i$.

OUTPUT: The determinant of the coefficient matrix in (1.5) and the solution vector

$$\mathbf{x} = [x_1, x_2, \dots, x_n]^T.$$

Step 1: Set $c_1 = d_1, y_1 = \frac{a_1}{c_1}, v_1 = \frac{p_1}{c_1}, z_1 = \frac{f_1}{c_1}$, and $h_1 = q_1$.

Step 2: For $i = 2, 3, \dots, n - 2$ do

 Compute and simplify:

$$c_i = d_i - b_{i-1} y_{i-1},$$

$$y_i = \frac{a_i}{c_i},$$

$$v_i = \frac{1}{c_i}(p_i - b_{i-1} v_{i-1}),$$

$$z_i = \frac{1}{c_i}(f_i - b_{i-1} z_{i-1}),$$

$$h_i = q_i - h_{i-1} y_{i-1},$$

End do.

Step 3: Set $c_{n-1} = d_{n-1} - b_{n-2} y_{n-2}$,
 $v_{n-1} = \frac{1}{c_{n-1}}(a_{n-1} - b_{n-2} v_{n-2})$,
 $z_{n-1} = \frac{1}{c_{n-1}}(f_{n-1} - b_{n-2} z_{n-2})$,
 $h_{n-1} = b_{n-1} - h_{n-2} y_{n-2}$.

Step 4: Set $c_n = d_n - \sum_{i=1}^{n-1} h_i v_i$,
 $z_n = \frac{1}{c_n}(f_n - \sum_{i=1}^{n-1} h_i z_i)$.

Step 5: Use the **DETDBTRI-A** algorithm to check the non-singularity of the coefficient matrix of the system (1.5).

Step 6: If the determinant of the coefficient matrix in (1.5) equals zero, then Exiterror('No solutions') end if.

Step 7: Compute the solution vector $\mathbf{x} = [x_1, x_2, \dots, x_n]^T$ using

$$x_n = z_n$$

$$x_{n-1} = z_{n-1} - v_{n-1} x_n$$

Step 8: For $i = n-2, n-3, \dots, 1$ do

$$x_i = z_i - y_i x_{i+1} - v_i x_n$$

End do.

The Algorithm 2.2, will be referred to as **TRANSDBTRI-AI** algorithm. The cost of the algorithm is $(11n-16)$ multiplications/divisions and $(8n-13)$ additions/subtractions. Note that the algorithm **TRANSDBTRI-AI** works properly only if $c_i \neq 0$ for all $i = 1, 2, \dots, n$.

The following symbolic version algorithm is developed in order to remove the cases where the numeric algorithm **TRANSDBTRI-AI** fails.

Algorithm 2.3. Symbolic version algorithm for **TRANSDBTRI-AI** algorithm.

To solve the linear system of the form (1.5), we may proceed as follows:

INPUT: Order of the matrix n and the components, $a_i, d_i, b_i, f_i, p_i, q_i$.

OUTPUT: The determinant of the coefficient matrix in (1.5) and the solution vector

$$\mathbf{x} = [x_1, x_2, \dots, x_n]^T.$$

Step 1: Set $c_1 = d_1$. If $c_1 = 0$ then $c_1 = s$ end if. $y_1 = \frac{a_1}{c_1}$, $v_1 = \frac{p_1}{c_1}$, $z_1 = \frac{f_1}{c_1}$, and $h_1 = q_1$.

Step 2: For $i = 2, 3, \dots, n-2$ do

Compute and simplify:

$$c_i = d_i - b_{i-1} y_{i-1}. \text{ If } c_i = 0 \text{ then } c_i = s \text{ end if.}$$

$$\begin{aligned}
y_i &= \frac{a_i}{c_i}, \\
v_i &= \frac{1}{c_i}(p_i - b_{i-1} v_{i-1}), \\
z_i &= \frac{1}{c_i}(f_i - b_{i-1} z_{i-1}), \\
h_i &= q_i - h_{i-1} y_{i-1},
\end{aligned}$$

End do.

Step 3: Set $c_{n-1} = d_{n-1} - b_{n-2} y_{n-2}$. If $c_{n-1} = 0$ then $c_{n-1} = s$ end if.

$$\begin{aligned}
v_{n-1} &= \frac{1}{c_{n-1}}(a_{n-1} - b_{n-2} v_{n-2}), \\
z_{n-1} &= \frac{1}{c_{n-1}}(f_{n-1} - b_{n-2} z_{n-2}), \\
h_{n-1} &= b_{n-1} - h_{n-2} y_{n-2}.
\end{aligned}$$

Step 4: Set $c_n = d_n - \sum_{i=1}^{n-1} h_i v_i$. If $c_n = 0$ then $c_n = s$ end if.

$$z_n = \frac{1}{c_n}(f_n - \sum_{i=1}^{n-1} h_i z_i).$$

Step 5: Use the **DETDBTRI-A** algorithm to check the non-singularity of the coefficient matrix of the system (1.5).

Step 6: If the determinant of the coefficient matrix in (1.5) equals zero, then Exiterror('No solutions') end if.

Step 7: Compute the solution vector $\mathbf{x} = [x_1, x_2, \dots, x_n]^T$ using

$$\begin{aligned}
x_n &= z_n \\
x_{n-1} &= z_{n-1} - v_{n-1} x_n
\end{aligned}$$

Step 8: For $i = n-2, n-3, \dots, 1$ do

$$x_i = z_i - y_i x_{i+1} - v_i x_n$$

End do.

Step 9: Substitute $s = 0$ in all expressions of the solution vector $x_i, i = 1, 2, \dots, n$.

The Algorithm 2.3, will be referred to as **TRANSDBTTRI-All** algorithm.

At this point it is worth mentioned that:

- The values $z_i, i = 1, 2, \dots, n$ in (2.3) satisfy:

$$\begin{bmatrix} c_1 & 0 & \dots & \dots & \dots & 0 \\ b_1 & c_2 & \ddots & & & \vdots \\ 0 & b_2 & c_3 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & b_{n-3} & c_{n-2} & \ddots \\ 0 & \dots & \dots & 0 & b_{n-2} & c_{n-1} & 0 \\ h_1 & h_2 & h_3 & \dots & h_{n-2} & h_{n-1} & c_n \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ \vdots \\ z_{n-2} \\ z_{n-1} \\ z_n \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_{n-2} \\ f_{n-1} \\ f_n \end{bmatrix}. \quad (2.4)$$

- The solution vector, $\mathbf{x} = [x_1, x_2, \dots, x_n]^T$ of the system (1.5) satisfies:

$$\begin{bmatrix} 1 & y_1 & 0 & \dots & 0 & v_1 \\ 0 & 1 & y_2 & \ddots & \vdots & v_2 \\ \vdots & \ddots & \ddots & \ddots & 0 & \vdots \\ \vdots & & \ddots & 1 & y_{n-2} & v_{n-2} \\ \vdots & & & \ddots & 1 & v_{n-1} \\ 0 & \dots & & & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_{n-2} \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ \vdots \\ z_{n-2} \\ z_{n-1} \\ z_n \end{bmatrix}. \quad (2.5)$$

Hence using (2.4) and (2.5), we get

$$\begin{bmatrix} c_1 & 0 & \dots & \dots & \dots & 0 \\ b_1 & c_2 & \ddots & & & \vdots \\ 0 & b_2 & c_3 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & b_{n-3} & c_{n-2} & \ddots \\ 0 & \dots & \dots & 0 & b_{n-2} & c_{n-1} \\ h_1 & h_2 & h_3 & \dots & h_{n-2} & h_{n-1} & c_n \end{bmatrix} \begin{bmatrix} 1 & y_1 & 0 & \dots & 0 & v_1 \\ 0 & 1 & y_2 & \ddots & \vdots & v_2 \\ \vdots & \ddots & \ddots & \ddots & 0 & \vdots \\ \vdots & & \ddots & 1 & y_{n-2} & v_{n-2} \\ \vdots & & & \ddots & 1 & v_{n-1} \\ 0 & \dots & & & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_{n-2} \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ \vdots \\ f_{n-2} \\ f_{n-1} \\ f_n \end{bmatrix}. \quad (2.6)$$

From (1.5) and (2.6), we obtain the Crout LU factorization [4] of the matrix in (1.5) in the form:

$$T_n = L_1 U_1$$

where

$$L_1 = \begin{bmatrix} c_1 & 0 & \dots & \dots & \dots & 0 \\ b_1 & c_2 & \ddots & & & \vdots \\ 0 & b_2 & c_3 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & b_{n-3} & c_{n-2} & \ddots \\ 0 & \dots & \dots & 0 & b_{n-2} & c_{n-1} \\ h_1 & h_2 & h_3 & \dots & h_{n-2} & h_{n-1} & c_n \end{bmatrix} \quad \text{and} \quad U_1 = \begin{bmatrix} 1 & y_1 & 0 & \dots & 0 & v_1 \\ 0 & 1 & y_2 & \ddots & \vdots & v_2 \\ \vdots & \ddots & \ddots & \ddots & 0 & \vdots \\ \vdots & & \ddots & 1 & y_{n-2} & v_{n-2} \\ \vdots & & & \ddots & 1 & v_{n-1} \\ 0 & \dots & & & 0 & 1 \end{bmatrix}. \quad (2.7)$$

The Doolittle LU factorization of the coefficient matrix in (1.5) is given by:

$$T_n = L_2 U_2$$

where

$$L_2 = \begin{bmatrix} 1 & 0 & \dots & & \dots & \dots & 0 \\ \frac{b_1}{c_1} & 1 & \ddots & & & & \vdots \\ 0 & \frac{b_2}{c_2} & 1 & \ddots & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \frac{b_{n-3}}{c_{n-3}} & 1 & \ddots & \vdots \\ 0 & \dots & \dots & 0 & \frac{b_{n-2}}{c_{n-2}} & 1 & 0 \\ \frac{h_1}{c_1} & \frac{h_2}{c_2} & \frac{h_3}{c_3} & \dots & \frac{h_{n-2}}{c_{n-2}} & \frac{h_{n-1}}{c_{n-1}} & 1 \end{bmatrix} \text{ and } U_2 = \begin{bmatrix} c_1 & a_1 & 0 & \dots & 0 & c_1 v_1 \\ 0 & c_2 & a_2 & \ddots & \vdots & c_2 v_2 \\ \vdots & \ddots & \ddots & \ddots & 0 & \vdots \\ \vdots & & \ddots & c_{n-2} & a_{n-2} & c_{n-2} v_{n-2} \\ \vdots & & & \ddots & c_{n-1} & c_{n-1} v_{n-1} \\ 0 & \dots & & \dots & 0 & c_n \end{bmatrix}. \quad (2.8)$$

- The values v_i , $i = 1, 2, \dots, n-1$ in (2.3) satisfy:

$$\begin{bmatrix} c_1 & 0 & \dots & & \dots & 0 \\ b_1 & c_2 & \ddots & & & \vdots \\ 0 & b_2 & c_3 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & b_{n-3} & c_{n-2} & 0 \\ 0 & \dots & \dots & 0 & b_{n-2} & c_{n-1} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_{n-2} \\ v_{n-1} \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ \vdots \\ p_{n-2} \\ a_{n-1} \end{bmatrix} \quad (2.9)$$

- The values h_i , $i = 1, 2, \dots, n-1$ in (2.3) satisfy:

$$\begin{bmatrix} 1 & 0 & \dots & & \dots & 0 \\ y_1 & 1 & \ddots & & & \vdots \\ 0 & y_2 & 1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & y_{n-3} & 1 & 0 \\ 0 & \dots & \dots & 0 & y_{n-2} & 1 \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \\ h_3 \\ \vdots \\ h_{n-2} \\ h_{n-1} \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ \vdots \\ q_{n-2} \\ b_{n-1} \end{bmatrix}. \quad (2.10)$$

Armed with (2.7)-(2.10) we may write the partitioned forms:

$$\left[\begin{array}{cccccc|c} d_1 & a_1 & 0 & \dots & \dots & 0 & p_1 \\ b_1 & d_2 & a_2 & \ddots & & \vdots & p_2 \\ 0 & b_2 & d_3 & a_3 & \ddots & \vdots & p_3 \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots \\ \vdots & & \ddots & b_{n-3} & d_{n-2} & a_{n-2} & p_{n-2} \\ 0 & \dots & \dots & 0 & b_{n-2} & d_{n-1} & a_{n-1} \\ \hline q_1 & q_2 & q_3 & \dots & q_{n-2} & b_{n-1} & d_n \end{array} \right]$$

$$= \left[\begin{array}{cccccc|c} c_1 & 0 & \dots & & \dots & 0 & 0 \\ b_1 & c_2 & \ddots & & & \vdots & \vdots \\ 0 & b_2 & c_3 & \ddots & & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & & \vdots & \vdots \\ \vdots & & \ddots & b_{n-3} & c_{n-2} & 0 & \vdots \\ 0 & \dots & \dots & 0 & b_{n-2} & c_{n-1} & 0 \\ \hline h_1 & h_2 & h_3 & \dots & h_{n-2} & h_{n-1} & c_n \end{array} \right] \left[\begin{array}{cccccc|c} 1 & y_1 & 0 & \dots & \dots & 0 & v_1 \\ 0 & 1 & y_2 & \ddots & & \vdots & v_2 \\ \vdots & \ddots & 1 & y_3 & \ddots & \vdots & v_3 \\ \vdots & & \ddots & \ddots & \ddots & 0 & \vdots \\ \vdots & & & \ddots & 1 & y_{n-2} & v_{n-2} \\ 0 & \dots & \dots & \dots & 0 & 1 & v_{n-1} \\ \hline 0 & \dots & \dots & \dots & \dots & 0 & 1 \end{array} \right], \quad (2.11)$$

and

$$\left[\begin{array}{cccccc|c} d_1 & a_1 & 0 & \dots & \dots & 0 & p_1 \\ b_1 & d_2 & a_2 & \ddots & & \vdots & p_2 \\ 0 & b_2 & d_3 & a_3 & \ddots & \vdots & p_3 \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 & \vdots \\ \vdots & & \ddots & b_{n-3} & d_{n-2} & a_{n-2} & p_{n-2} \\ 0 & \dots & \dots & 0 & b_{n-2} & d_{n-1} & a_{n-1} \\ \hline q_1 & q_2 & q_3 & \dots & q_{n-2} & b_{n-1} & d_n \end{array} \right] =$$

$$= \left[\begin{array}{cccccc|c} 1 & 0 & \dots & \dots & \dots & 0 & 0 \\ \frac{b_1}{c_1} & 1 & \ddots & & & \vdots & \vdots \\ 0 & \frac{b_2}{c_2} & 1 & \ddots & & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & & \vdots & \vdots \\ \vdots & & \ddots & \frac{b_{n-3}}{c_{n-3}} & 1 & 0 & \vdots \\ 0 & \dots & \dots & 0 & \frac{b_{n-2}}{c_{n-2}} & 1 & 0 \\ \hline h_1 & h_2 & h_3 & \dots & h_{n-2} & h_{n-1} & 1 \end{array} \right] \left[\begin{array}{cccccc|c} c_1 & a_1 & 0 & \dots & \dots & 0 & v_1 \\ 0 & c_2 & a_2 & \ddots & & \vdots & v_2 \\ \vdots & \ddots & c_3 & a_3 & \ddots & \vdots & v_3 \\ \vdots & & \ddots & \ddots & \ddots & 0 & \vdots \\ \vdots & & & \ddots & c_{n-2} & a_{n-2} & v_{n-2} \\ 0 & \dots & \dots & \dots & 0 & c_{n-1} & v_{n-1} \\ \hline 0 & \dots & \dots & \dots & \dots & 0 & c_n \end{array} \right]. \quad (2.12)$$

3 ALGORITHMS FOR SOLVING DOUBLY BORDERED TRIDIAGONAL LINEAR SYSTEMS OF TYPE (B)

In this section, we are going to consider the construction of new algorithms for solving doubly bordered tridiagonal linear systems of type (B) via transformation. For this purpose it is convenient to introduce the vector $\mathbf{e} = [e_1, e_2, \dots, e_n]$ whose components $e_n, e_{n-1}, \dots, e_2$ are given by:

$$e_i = \begin{cases} d_n, & \text{for } i = n \\ d_i - \frac{a_i b_i}{e_{i+1}}, & \text{for } i = n-1, n-2, \dots, 2. \end{cases} \quad (3.1)$$

The system (1.6) can be completely described by the augmented matrix, H given by:

$$H := [\hat{T}_n, \mathbf{g}] = \left[\begin{array}{ccccccc|c} d_1 & a_1 & p_{n-2} & \dots & p_3 & p_2 & p_1 & g_1 \\ b_1 & d_2 & a_2 & 0 & \dots & \dots & 0 & g_2 \\ q_{n-2} & b_2 & d_3 & a_3 & \ddots & & \vdots & g_3 \\ \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ q_3 & \vdots & \ddots & b_{n-3} & d_{n-2} & a_{n-2} & 0 & g_{n-2} \\ q_2 & \vdots & & \ddots & b_{n-2} & d_{n-1} & a_{n-1} & g_{n-1} \\ q_1 & 0 & \dots & \dots & 0 & b_{n-1} & d_n & g_n \end{array} \right]. \quad (3.2)$$

The matrix H contains all the information about the system (1.6) that is necessary to determine its solution, but in a compact form. Let $\mathbf{R}_i$ denotes the i -th row of the matrix H . Performing the following row operations on H , in the same order:

$\mathbf{R}_n \leftarrow \mathbf{R}_n / e_n$

For $i = n-1, n-2, \dots, 2$ do

$\mathbf{R}_i \leftarrow \mathbf{R}_i - a_i \mathbf{R}_{i+1}$

$\mathbf{R}_i \leftarrow \mathbf{R}_i / e_i$

End do.

For $j = 1, 2, \dots, n-1$ do

$\mathbf{R}_1 \leftarrow \mathbf{R}_1 - v_j \mathbf{R}_{n-j+1}$

End do.

$\mathbf{R}_1 \leftarrow \mathbf{R}_1 / e_1$.

Then we have the resulting transformed linear system of the form:

$$\left[\begin{array}{ccccccc|c} 1 & 0 & \dots & & \dots & \dots & 0 & Z_1 \\ h_{n-1} & 1 & \ddots & & & & \vdots & Z_2 \\ h_{n-2} & Y_2 & 1 & \ddots & & & \vdots & Z_3 \\ \vdots & 0 & \ddots & \ddots & \ddots & & \vdots & \vdots \\ h_3 & \vdots & \ddots & Y_{n-3} & 1 & \ddots & \vdots & Z_{n-2} \\ h_2 & \vdots & & \ddots & Y_{n-2} & 1 & 0 & Z_{n-1} \\ h_1 & 0 & \dots & \dots & 0 & Y_{n-1} & 1 & Z_n \end{array} \right], \quad (3.3)$$

where

$$e_n = d_n, \quad h_1 = \frac{q_1}{e_n}, \quad Z_n = \frac{g_n}{e_n}, \quad v_1 = p_1,$$

For $i = n - 1, n - 2, \dots, 2$ do

$$\begin{aligned} Y_i &= \frac{b_i}{e_{i+1}}, \\ e_i &= d_i - a_i Y_i, \\ Z_i &= \frac{1}{e_i} (g_i - a_i Z_{i+1}). \end{aligned}$$

End do.

For $i = 2, 3, \dots, n - 2$ do

$$\begin{aligned} h_i &= \frac{1}{e_{n-i+1}} (q_i - h_{i-1} a_{n-i+1}), \\ v_i &= p_i - Y_{n-i+1} v_{i-1}, \end{aligned} \tag{3.4}$$

End do.

$$h_{n-1} = \frac{1}{e_2} (b_1 - h_{n-2} a_2),$$

$$v_{n-1} = a_1 - Y_2 v_{n-2},$$

$$e_1 = d_1 - \sum_{r=1}^{n-1} h_r v_r,$$

$$Z_1 = \frac{1}{e_1} (g_1 - \sum_{r=1}^{n-1} v_r Z_{n-r+1}).$$

At this stage, the determinant of the coefficient matrix in (1.6) can be computed using the following computational symbolic algorithm.

Algorithm 3.1. An algorithm for computing the determinant of doubly bordered tridiagonal matrices of type (B).

To compute the determinant of the coefficient matrix in (1.6), we may proceed as follows:

INPUT: Order of the coefficient matrix n and the components, a_i, d_i, b_i, p_i, q_i .

OUTPUT: The determinant of the coefficient matrix in (1.6).

Step 1: Set $e_n = d_n$. If $e_n = 0$ then $e_n = s$ end if.

Step 2: For $i = n - 1, n - 2, \dots, 2$ do

 Compute and simplify:

$$e_i = d_i - \frac{a_i b_i}{e_{i+1}}. \text{ If } e_i = 0 \text{ then } e_i = s \text{ end if.}$$

End do.

Step 3: Compute e_1 from (3.4).

Step 4: Compute and simplify:

$$P(s) = \prod_{r=1}^n e_r.$$

Step 5: $\det(\hat{T}_n) = P(0)$.

The Algorithm 3.1, will be referred to as **DETDBTRI-B** algorithm.

The reduced transformed system (3.3) is triangular and easy to solve by forward substitution. Consequently, the linear system (1.6) can be solved using the following symbolic algorithm:

Algorithm 3.2. Symbolic version algorithm for solving doubly bordered tridiagonal linear

system of type (B).

To solve the linear system of the form (1.6), we may proceed as follows:

INPUT: Order of the matrix n and the components, $a_i, d_i, b_i, g_i, p_i, q_i$.

OUTPUT: The determinant of the coefficient matrix in (1.6) and the solution vector

$$\mathbf{x} = [x_1, x_2, \dots, x_n]^T.$$

Step 1: Set $e_n = d_n$. If $e_n = 0$ then $e_n = s$ end if. $h_1 = \frac{q_1}{e_n}$, $Z_1 = \frac{g_n}{e_n}$, and $v_1 = p_1$.

Step 2: For $i = n - 1, n - 2, \dots, 2$ do

 Compute and simplify:

$$Y_i = \frac{b_i}{e_{i+1}},$$

$e_i = d_i - a_i Y_i$. If $e_i = 0$ then $e_i = s$ end if.

$$Z_i = \frac{1}{e_i}(g_i - a_i Z_{i+1}).$$

 End do.

Step 3: For $i = 2, 3, \dots, n - 2$ do

 Compute and simplify:

$$h_i = \frac{1}{e_{n-i+1}}(q_i - a_{n-i+1} h_{i-1}),$$

$$v_i = p_i - Y_{n-i+1} v_{i-1},$$

 End do.

Step 4: Set $h_{n-1} = \frac{1}{e_2}(b_1 - a_2 h_{n-2})$,

$$v_{n-1} = a_1 - Y_2 v_{n-2}.$$

Step 5: Set $e_1 = d_1 - \sum_{r=1}^{n-1} h_r v_r$. If $e_1 = 0$ then $e_1 = s$ end if.

$$Z_1 = \frac{1}{e_1}(g_1 - \sum_{r=1}^{n-1} v_r Z_{n-r+1}).$$

Step 6: Use the **DETDBTRI-B** algorithm to check the non-singularity of the coefficient

matrix of the system (1.6).

Step 7: If the determinant of the coefficient matrix in (1.6) equals zero, then Exiterror('No solutions') end if.

Step 8: Compute the solution vector $\mathbf{x} = [x_1, x_2, \dots, x_n]^T$ using

$$x_1 = Z_1$$

$$x_2 = Z_2 - h_{n-1} x_1$$

Step 9: For $i = 3, 4, \dots, n$ do

$$x_i = Z_i - Y_{i-1} x_{i-1} - h_{n-i+1} x_1$$

End do.

Step 9: Substitute $s = 0$ in all expressions of the solution vector $x_i, i = 1, 2, \dots, n$.

The Algorithm 3.2, will be referred to as **TRANSDBTRI-BI** algorithm.

The cost of this algorithm is exactly the same as the cost of the **TRANSDBTRI-AI** algorithm.

Similar to the situation with the doubly bordered tridiagonal linear systems of type (A), the following points are worth to mention:

- The solution vector, $\mathbf{x} = [x_1, x_2, \dots, x_n]^T$ of the system (1.6) satisfies:

$$\begin{bmatrix} 1 & 0 & \dots & \dots & \dots & 0 \\ h_{n-1} & 1 & \ddots & & & \vdots \\ h_{n-2} & Y_2 & 1 & \ddots & & \vdots \\ \vdots & 0 & \ddots & \ddots & \ddots & \vdots \\ h_3 & \vdots & \ddots & Y_{n-3} & 1 & \ddots \\ h_2 & \vdots & & \ddots & Y_{n-2} & 1 & 0 \\ h_1 & 0 & \dots & \dots & 0 & Y_{n-1} & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-2} \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \\ \vdots \\ Z_{n-2} \\ Z_{n-1} \\ Z_n \end{bmatrix}. \quad (3.5)$$

- The values $Z_i, i = 1, 2, \dots, n$ in (3.4) satisfy:

$$\begin{bmatrix} e_1 & v_{n-1} & \dots & \dots & v_2 & v_1 \\ 0 & e_2 & a_2 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & e_{n-2} & a_{n-2} & 0 \\ \vdots & & & \ddots & e_{n-1} & a_{n-1} \\ 0 & \dots & \dots & 0 & e_n & \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \\ \vdots \\ Z_{n-2} \\ Z_{n-1} \\ Z_n \end{bmatrix} = \begin{bmatrix} g_1 \\ g_2 \\ g_3 \\ \vdots \\ g_{n-2} \\ g_{n-1} \\ g_n \end{bmatrix}. \quad (3.6)$$

Hence using (3.5) and (3.6), we get

$$\begin{bmatrix} e_1 & v_{n-1} & \cdots & \cdots & v_2 & v_1 \\ 0 & e_2 & a_2 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & e_{n-2} & a_{n-2} & 0 \\ \vdots & & & \ddots & e_{n-1} & a_{n-1} \\ 0 & \cdots & \cdots & 0 & e_n & \end{bmatrix} \begin{bmatrix} 1 & 0 & \cdots & \cdots & \cdots & 0 \\ h_{n-1} & 1 & \ddots & & & \vdots \\ h_{n-2} & Y_2 & 1 & \ddots & & \vdots \\ \vdots & 0 & \ddots & \ddots & \ddots & \vdots \\ h_3 & \vdots & \ddots & Y_{n-3} & 1 & \ddots \\ h_2 & \vdots & & \ddots & Y_{n-2} & 1 \\ h_1 & 0 & \cdots & \cdots & 0 & Y_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-2} \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} g_1 \\ g_2 \\ g_3 \\ \vdots \\ g_{n-2} \\ g_{n-1} \\ g_n \end{bmatrix} \quad (3.7)$$

From (1.6) and (3.7), we obtain the Doolittle UL factorization [4] of the matrix in (1.6) in the form:

$$\hat{T}_n = U_1 L_1$$

where

$$U_1 = \begin{bmatrix} e_1 & v_{n-1} & \cdots & \cdots & v_2 & v_1 \\ 0 & e_2 & a_2 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & e_{n-2} & a_{n-2} & 0 \\ \vdots & & & \ddots & e_{n-1} & a_{n-1} \\ 0 & \cdots & \cdots & 0 & e_n & \end{bmatrix} \text{ and } L_1 = \begin{bmatrix} 1 & 0 & \cdots & \cdots & \cdots & 0 \\ h_{n-1} & 1 & \ddots & & & \vdots \\ h_{n-2} & Y_2 & 1 & \ddots & & \vdots \\ \vdots & 0 & \ddots & \ddots & \ddots & \vdots \\ h_3 & \vdots & \ddots & Y_{n-3} & 1 & \ddots \\ h_2 & \vdots & & \ddots & Y_{n-2} & 1 \\ h_1 & 0 & \cdots & \cdots & 0 & Y_{n-1} \end{bmatrix}. \quad (3.8)$$

The Crout UL factorization of the coefficient matrix in (1.6) is given by:

$$\hat{T}_n = U_2 L_2$$

where

$$U_2 = \begin{bmatrix} 1 & \frac{v_{n-1}}{e_2} & \cdots & \cdots & \frac{v_2}{e_{n-1}} & \frac{v_1}{e_n} \\ 0 & 1 & \frac{a_2}{e_3} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 & \frac{a_{n-2}}{e_{n-1}} & 0 \\ \vdots & & & \ddots & 1 & \frac{a_{n-1}}{e_n} \\ 0 & \cdots & \cdots & 0 & 1 & \end{bmatrix} \text{ and } L_2 = \begin{bmatrix} e_1 & 0 & \cdots & \cdots & \cdots & 0 \\ e_2 h_{n-1} & e_2 & \ddots & & & \vdots \\ e_3 h_{n-2} & b_2 & e_3 & \ddots & & \vdots \\ \vdots & 0 & \ddots & \ddots & \ddots & \vdots \\ e_{n-2} h_3 & \vdots & \ddots & b_{n-3} & e_{n-2} & \ddots \\ e_{n-1} h_2 & \vdots & & \ddots & b_{n-2} & e_{n-1} \\ e_n h_1 & 0 & \cdots & \cdots & 0 & b_{n-1} \end{bmatrix}. \quad (3.9)$$

- The values v_i , $i = 1, 2, \dots, n-1$ in (3.4) satisfy:

$$\begin{bmatrix} 1 & 0 & \dots & \dots & 0 \\ Y_{n-1} & 1 & \ddots & & \vdots \\ 0 & Y_{n-2} & 1 & \ddots & \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & Y_3 & 1 & 0 \\ 0 & \dots & \dots & 0 & Y_2 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_{n-2} \\ v_{n-1} \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ \vdots \\ p_{n-2} \\ a_1 \end{bmatrix} \quad (3.10)$$

- The values h_i , $i = 1, 2, \dots, n-1$ in (3.4) satisfy:

$$\begin{bmatrix} e_n & 0 & \dots & \dots & 0 \\ a_{n-1} & e_{n-1} & \ddots & & \vdots \\ 0 & a_{n-2} & 1 & \ddots & \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & a_3 & e_3 & 0 \\ 0 & \dots & \dots & 0 & a_2 & e_2 \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \\ h_3 \\ \vdots \\ h_{n-2} \\ h_{n-1} \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ \vdots \\ q_{n-2} \\ b_1 \end{bmatrix}. \quad (3.11)$$

Armed with (3.8)-(3.11) we have the partitioned form:

$$= \begin{bmatrix} d_1 & a_1 & p_{n-2} & \dots & p_3 & p_2 & p_1 \\ b_1 & d_2 & a_2 & 0 & \dots & \dots & 0 \\ q_{n-2} & b_2 & d_3 & a_3 & \ddots & & \vdots \\ \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ q_3 & \vdots & \ddots & b_{n-3} & d_{n-2} & a_{n-2} & 0 \\ q_2 & \vdots & & \ddots & b_{n-2} & d_{n-1} & a_{n-1} \\ q_1 & 0 & \dots & \dots & 0 & b_{n-1} & d_n \end{bmatrix} \begin{bmatrix} e_1 & v_{n-1} & v_{n-2} & v_{n-3} & \dots & v_2 & v_1 \\ 0 & e_2 & a_2 & 0 & \dots & \dots & 0 \\ \vdots & 0 & e_3 & a_3 & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & e_{n-2} & a_{n-2} & 0 \\ \vdots & \vdots & & & \ddots & e_{n-1} & a_{n-1} \\ 0 & 0 & & \dots & \dots & 0 & e_n \end{bmatrix} \begin{bmatrix} 1 & 0 & \dots & \dots & \dots & 0 \\ h_{n-1} & 1 & 0 & \dots & \dots & 0 \\ h_{n-2} & Y_2 & 1 & \ddots & & \vdots \\ \vdots & 0 & \ddots & \ddots & \ddots & \vdots \\ h_2 & \vdots & \ddots & Y_{n-2} & 1 & 0 \\ h_1 & 0 & \dots & 0 & Y_{n-1} & 1 \end{bmatrix}, \quad (3.12)$$

and

$$\begin{aligned}
 & \left[\begin{array}{c|cccccc} d_1 & a_1 & p_{n-2} & \dots & p_3 & p_2 & p_1 \\ \hline b_1 & d_2 & a_2 & 0 & \dots & \dots & 0 \\ q_{n-2} & b_2 & d_3 & a_3 & \ddots & & \vdots \\ \vdots & 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ q_3 & \vdots & \ddots & b_{n-3} & d_{n-2} & a_{n-2} & 0 \\ q_2 & \vdots & & \ddots & b_{n-2} & d_{n-1} & a_{n-1} \\ q_1 & 0 & \dots & \dots & 0 & b_{n-1} & d_n \end{array} \right] \\
 &= \left[\begin{array}{c|cccccc} 1 & \frac{v_{n-1}}{e_2} & \frac{v_{n-2}}{e_3} & \frac{v_{n-3}}{e_4} & \dots & \frac{v_2}{e_{n-1}} & \frac{v_1}{e_n} \\ \hline 0 & 1 & \frac{a_2}{e_3} & 0 & \dots & \dots & 0 \\ \vdots & 0 & 1 & \frac{a_3}{e_4} & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & 1 & \frac{a_{n-2}}{e_{n-1}} & 0 \\ \vdots & \vdots & & & \ddots & 1 & \frac{a_{n-1}}{e_n} \\ 0 & 0 & & \dots & \dots & 0 & 1 \end{array} \right] \left[\begin{array}{c|cccccc} e_1 & 0 & \dots & \dots & \dots & 0 \\ \hline e_2 h_{n-1} & e_2 & 0 & \dots & \dots & 0 \\ e_3 h_{n-2} & b_2 & e_3 & \ddots & & \vdots \\ \vdots & 0 & \ddots & \ddots & \ddots & \vdots \\ e_{n-1} h_2 & \vdots & \ddots & b_{n-2} & e_{n-1} & 0 \\ e_n h_1 & 0 & \dots & 0 & b_{n-1} & e_n \end{array} \right], \quad (3.13)
 \end{aligned}$$

4 COMPUTER PROGRAM

In this section, we are going to introduce a MAPLE procedure for solving linear systems of doubly bordered tridiagonal types (A) and (B) in (1.5) and (1.6), respectively. The procedure is based on the algorithms **DETDBTRI-A**, **TRANSDBTRI-AII**, **DETDBTRI-B** and **TRANSDBTRI-BI**.

```
#####
restart:
doublytritrans:= proc(d::vector, a::vector, b::vector,q::vector,p::vector,f::vector,
n::posint,indic::nonnegint)
local i,r:
global x,T,detA:
x:= vector(n):
#####
#Apply TRANDBTRI-AII algorithm if indic =1#
if indic=1 then
```

```

if d[1] = 0 then d[1]:=t fi;
a[1]:=simplify(a[1]/d[1]); f[1]:=simplify(f[1]/d[1]); p[1]:=p[1]/d[1];
for i from 2 to n-2 do
  d[i] := simplify(d[i]-a[i-1]*b[i-1]);if d[i] = 0 then d[i] := t; fi;
  a[i] := simplify(a[i]/d[i]);
  p[i] := simplify((p[i] - p[i-1]*b[i-1])/d[i]);
  f[i] := simplify((f[i]-f[i-1]*b[i-1])/d[i]);
  q[i] := simplify(q[i] - q[i-1]*a[i-1]);
od;
d[n-1] := simplify(d[n-1]-a[n-2]*b[n-2]);
if d[n-1] = 0 then d[n-1] := t; fi;
p[n-1] := simplify((a[n-1] - p[n-2]*b[n-2])/d[n-1]);
f[n-1] := simplify((f[n-1] - f[n-2]*b[n-2])/d[n-1]);
q[n-1] := simplify((b[n-1] - q[n-2]*a[n-2]));
d[n] := simplify(d[n]-sum(q[r]*p[r],r=1..n-1));
f[n] := simplify((f[n]-sum(q[r]*f[r],r=1..n-1))/d[n]);
if d[n] = 0 then d[n] := t; fi;
#To compute the determinant of the doubly bordered tridiagonal matrix#
T := simplify(subs(t =0,simplify(product(d[r],r= 1..n)))):
detA:= eval(T);
if T = 0 then error("Singular Matrix"); fi;
# To compute the Solution of the system X. #
x[n]:=simplify(f[n]);
x[n-1]:=simplify((f[n-1]-p[n-1]*x[n]));
for i from n-2 by -1 to 1 do
  x[i]:=simplify((f[i]-a[i]*x[i+1]-p[i]*x[n]));
od;
eval(x);
#####
#Apply TRANDBTRI-BI algorithm if indic = 2#
elif indic=2 then
  if d[n] = 0 then d[n]:=t fi;
  g[n]:=simplify(g[n]/d[n]); q[1]:=simplify(q[1]/d[n]);
  for i from n-1 by -1 to 2 do
    b[i] := simplify(b[i]/d[i+1]);
    d[i] := simplify(d[i]-b[i]*a[i]);if d[i] = 0 then d[i] := t; fi;
    g[i] := simplify((g[i]-g[i+1]*a[i])/d[i]);
  od;
  for i from 2 to n-2 do
    q[i] := simplify((q[i] - q[i-1]*a[n-i+1])/d[n-i+1]);
    p[i] := simplify(p[i] - p[i-1]*b[n-i+1]);
  od;

```

```

q[n-1] := simplify((b[1] - q[n-2]*a[2])/d[2]):
p[n-1] := simplify(a[1] - p[n-2]*b[2]):
d[1] := simplify(d[1]-sum(q[r]*p[r],r=1..n-1)):
if d[1] = 0 then d[1] := t; fi:
g[1] := simplify((g[1]-sum(p[r]*f[n-r+1],r=1..n-1))/d[1]):
#To compute the determinant of the doubly bordered tridiagonal matrix#
T := simplify(subs(t =0,simplify(product(d[r],r= 1..n)))):
detA:= eval(T);
if T = 0 then error("Singular Matrix");fi;
# To compute the Solution of the system X. #
x[1]:=simplify(g[1]):
x[2]:=simplify((g[2]-q[n-1]*x[1])):
for i from 3 to n do
x[i]:=simplify((g[i]-b[i-1]*x[i-1]-q[n-i+1]*x[1]));
od:
eval(x);
elif (indic<>1 and indic<>2) then
print("Error. The value of indic is out of range"):
fi:
end proc:

```

5 ILLUSTRATIVE EXAMPLES

Example 5.1. [26]. Consider the singly bordered tridiagonal matrix

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p_1 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p_2 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & p_3 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & p_4 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & p_5 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 & p_6 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 & 0 & p_7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 & p_8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 2 \end{bmatrix},$$

find $\det(A)$.

Solution: We have

$n = 10$, $indic = 1$, $\mathbf{a} = [1, 1, 1, 1, 1, 1, 1, 1, 1, 1]$, $\mathbf{d} = [2, 2, 2, 2, 2, 2, 2, 2, 2, 2]$,
 $\mathbf{b} = [1, 1, 1, 1, 1, 1, 1, 1, 1, 1]$, $\mathbf{p} = [p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, 0]^T$ and $\mathbf{q} = [0, 0, 0, 0, 0, 0, 0, 0, 0, 0]$.

By applying the **DETDBTRI-A** algorithm , we obtain

- $\mathbf{c} = [2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \frac{6}{5}, \frac{7}{6}, \frac{8}{7}, \frac{9}{8}, \frac{10}{9}, \frac{11}{10} + \frac{4}{5}p_8 - \frac{7}{10}p_7 + \frac{3}{5}p_6 - \frac{1}{2}p_5 + \frac{2}{5}p_4 - \frac{3}{10}p_3 + \frac{1}{5}p_2 - \frac{1}{10}p_1]$.
- $\det A = \prod_{i=1}^{10} c_i = 11 + 8p_8 - 7p_7 + 6p_6 - 5p_5 + 4p_4 - 3p_3 + 2p_2 - p_1 = 11 + \sum_{i=1}^8 (-1)^i i p_i$.

This result is in complete agreement with the result in [26].

Example 5.2. Solve the doubly bordered tridiagonal linear system

$$\begin{bmatrix} 5 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \\ 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 12 \\ 0 & -2 & 5 & 2 & 0 & 0 & 0 & 0 & 0 & 7 \\ 0 & 0 & 1 & 2 & 7 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 3 & 10 & 2 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 15 & 3 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 9 & 2 & 5 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 7 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 4 & 2 \\ 3 & 2 & -2 & 7 & -6 & 1 & 4 & 5 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \end{bmatrix} = \begin{bmatrix} 5 \\ -5 \\ 8 \\ 12 \\ 13 \\ 22 \\ 19 \\ 24 \\ 16 \\ 34 \end{bmatrix}$$

Solution: We have a doubly bordered tridiagonal linear systems of type (A) for which:

$n = 10$, $indic = 1$, $\mathbf{a} = [2, 1, 2, 7, 2, 3, 5, 7, 2]$, $\mathbf{d} = [5, 1, 5, 2, 10, 15, 2, 1, 4, 1]$, $\mathbf{b} = [2, -2, 1, 3, 1, 9, 1, 3, 1]$,

$\mathbf{p} = [4, 12, 7, 2, 5, 3, 6, 2, 0]^T$,

$\mathbf{q} = [3, 2, -2, 7, -6, 1, 4, 5, 0]$ and

$\mathbf{f} = [5, -5, 8, 12, 13, 22, 19, 24, 16, 34]^T$.

By applying the **TRANSDBTRI-AII**, we get

- $\mathbf{c} = [5, \frac{1}{5}, 15, \frac{28}{15}, \frac{-5}{4}, \frac{83}{5}, \frac{31}{83}, \frac{-384}{31}, \frac{729}{128}, \frac{-161620}{567}]$.
- $\det A = \prod_{i=1}^{10} c_i = -4363740$.
- The solution vector is $\mathbf{x} = [1, 2, 3, 2, 1, 1, 3, 2, 3, -1]^T$.

By using **TRANSDBTRI-AI**, we obtained the same results.

Example 5.3. Solve the doubly bordered tridiagonal linear system

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 \\ 1 & 1 & 12 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 9 & 2 & 5 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 3 & 15 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 & 3 & 10 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 7 & 1 & 2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & -5 & 2 & 2 & 0 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 1 & 12 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 2 & 4 \\ 3 & 2 & 1 & 7 & 5 & -2 & 4 & 2 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \end{bmatrix} = \begin{bmatrix} 6 \\ 16 \\ 14 \\ 35 \\ 2 \\ 8 \\ 12 \\ 15 \\ 10 \\ 33 \end{bmatrix}$$

Solution: We have a doubly bordered tridiagonal linear systems of type (A) for which:

$n = 10$, $indic = 1$, $\mathbf{a} = [1, 12, 5, 1, 10, 2, 2, 1, 4]$, $\mathbf{d} = [1, 1, 2, 15, 3, 1, 2, 1, 2, 5]$, $\mathbf{b} = [1, 9, 3, 2, 7, -5, 2, 5, 1]$, $\mathbf{p} = [5, 3, 2, 1, 5, 2, 7, 12, 0]^T$, $\mathbf{q} = [3, 2, 1, 7, 5, -2, 4, 2, 0]$ and $\mathbf{f} = [6, 16, 14, 35, 2, 8, 12, 15, 10, 33]^T$.

By applying the **TRANSDBTRI-All**, we have

- $\mathbf{c} = [1, s, 2 \frac{s-54}{s}, \frac{15}{2} \frac{s-108}{s-54}, \frac{1}{15} \frac{41s-4644}{s-108}, -\frac{1009s-108756}{41s-4644}, 24 \frac{67s-7128}{1009s-108756}, -\frac{1}{6} \frac{607s-65988}{67s-7128}, 8 \frac{403s-43227}{607s-65988}, -\frac{1}{8} \frac{80797s-8813606}{403s-43227}]$.
- $\det A = (\prod_{i=1}^{10} c_i)_{s=0} = (-323188 * s + 35254424)_{s=0} = 35254424$.
- The solution vector is $\mathbf{x} = [1, 0, 1, 2, 1, -1, 0, 0, 3, 1]^T$.

Example 5.4. Solve the doubly bordered tridiagonal linear system

$$\begin{bmatrix} 5 & 2 & 2 & 6 & 3 & 5 & 2 & 7 & 12 & 4 \\ 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 5 & -2 & 5 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 1 & 2 & 7 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 3 & 10 & 2 & 0 & 0 & 0 & 0 \\ -6 & 0 & 0 & 0 & 1 & 15 & 3 & 0 & 0 & 0 \\ 7 & 0 & 0 & 0 & 0 & 9 & 2 & 5 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 7 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 1 & 1 \\ 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \end{bmatrix} = \begin{bmatrix} 34 \\ 5 \\ 4 \\ 3 \\ 0 \\ 18 \\ 32 \\ 3 \\ 9 \\ 4 \end{bmatrix}$$

Solution: We have a doubly bordered tridiagonal linear systems of type (B) for which:

$n = 10$, $indic = 2$, $\mathbf{a} = [2, 1, 2, 7, 2, 3, 5, 7, 1]$, $\mathbf{d} = [5, 1, 5, 2, 10, 15, 2, 1, 1, 1]$, $\mathbf{b} = [2, -2, 1, 3, 1, 9, 1, 3, 1]$,

$$\mathbf{p} = [4, 12, 7, 2, 5, 3, 6, 2, 0]^T,$$

$$\mathbf{q} = [3, 2, -2, 7, -6, 1, 4, 5, 0] \text{ and}$$

$$\mathbf{g} = [34, 5, 4, 3, 0, 18, 32, 3, 9, 4]^T.$$

By applying the **TRANSDBTRI-BI**, we have

- $\mathbf{e} = \left[\frac{11}{3} * \frac{(104*s-1637)}{(32*s+43)}, 3 * \frac{(32*s+43)}{(88*s+107)}, \frac{(88*s+107)}{(4*s+11)}, \frac{-1}{2} * \frac{(4*s+11)}{(17*s+13)}, \frac{14}{3} * \frac{(17*s+13)}{(8*s+7)}, 3 * \frac{(8*s+7)}{(s+14)}, -3 * \frac{(s+14)}{(s-21)}, \frac{(s-21)}{s}, s, 1 \right].$
- $\det T = \left(\prod_{i=1}^{10} e_i \right)_{s=0} = (24024 * s - 378147)_{s=0} = -378147.$
- The solution vector is
 $\mathbf{x} = [1, 2, 1, -1, 0, 1, 3, 2, 0, 1]^T.$

6 CONCLUSIONS

The current article focuses on solving doubly bordered tridiagonal linear systems of equations via transformation. It investigates numeric and symbolic algorithms for solving such systems. MAPLE was used to simulate the problem considered. If n is large ($n > 500$), it may be preferable to use iterative methods for solving such systems. Consequently, the efficiency of the new methods in this paper should be compared with existing methods such as static and dynamic tuning method or the GPU accelerated tridiagonal solver. This will be the subject of the next paper.

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COMPETING INTERESTS

The authors declare that they have no competing interests.

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