

On topological structure of the first non-abelian cohomology of topological groups

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**Original Research
Article**

Received: XX December 20XX

Accepted: XX December 20XX

Online Ready: XX December 20XX

Abstract

Let G , R , and A be topological groups. Suppose that G and R act continuously on A , and G acts continuously on R . In this paper, we define a partially crossed topological $G - R$ -bimodule (A, μ) , where $\mu : A \rightarrow R$ is a continuous homomorphism. Let $Der_c(G, (A, \mu))$ be the set of all (α, r) such that $\alpha : G \rightarrow A$ is a continuous crossed homomorphism and $\mu\alpha(g) = r^g r^{-1}$. We introduce a topology on $Der_c(G, (A, \mu))$. We show that $Der_c(G, (A, \mu))$ is a topological group, wherever G and R are locally compact. We define the first cohomology, $H^1(G, (A, \mu))$, of G with coefficients in (A, μ) as a quotient space of $Der_c(G, (A, \mu))$. Also, we state conditions under which $H^1(G, (A, \mu))$ is a topological group. Finally, we show that under what conditions $H^1(G, (A, \mu))$ is one of the following: k -space, discrete, locally compact and compact.

Keywords: Non-abelian cohomology of topological groups; Partially crossed topological bimodule; Evaluation map; Compactly generated group

2010 Mathematics Subject Classification: Primary 22A05; 20J06; Secondary 18G50

1 Introduction

The first non-abelian cohomology of groups (algebraically) was introduced by Guin [4]. The Guin's approach is extended by Inassaridze to any dimension with coefficients in (partially) crossed bimodules [8] and [9]. Hu [7] defined the cohomology of topological groups with coefficients in abelian topological modules. This paper is a part of an investigation about *non-abelian cohomology* of topological groups. We consider the first non-abelian cohomology in the topological context. The methods used here are motivated by [8] and [9].

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All topological groups are assumed to be Hausdorff (not necessarily abelian), unless otherwise specified. Let G and A be topological groups. It is said that A is a topological G -module, whenever G acts continuously on the left of A . For all $g \in G$ and $a \in A$ we denote the action of g on a by ${}^g a$. The center and the commutator of a topological group G is denoted by $Z(G)$ and $[G, G]$, respectively. If G and H are topological groups and $f : G \rightarrow H$ is a continuous homomorphism we denote by $\bar{f} : G \rightarrow f(G)$ the restricted map of f on its range and by $1 : G \rightarrow H$ the trivial homomorphism. The topological isomorphism and isomorphism are denoted respectively by " $\simeq$ " and " $\cong$ ". If the topological groups G and R act continuously on a topological group A , then the notation ${}^{gr}a$ means ${}^g({}^r a)$, $g \in G$, $r \in R$, $a \in A$. We assume that every topological group acts on itself by conjugation.

In section 2, we define precrossed, partially crossed and crossed topological R -module (A, μ) , where A is a topological R -module and $\mu : A \rightarrow R$ is a continuous homomorphism. Also, we generalize, these definitions to precrossed, partially crossed and crossed topological $G - R$ -bimodule (A, μ) , when G and R act continuously on A , and G acts continuously on R . We define the set $Der_c(G, (A, \mu))$, for a partially crossed topological $G - R$ -bimodule. We denote the set of all continuous maps from G into A , with compact-open topology, by $\mathcal{C}_k(G, A)$. Since $Der_c(G, (A, \mu)) \subset \mathcal{C}_k(G, A) \times R$, then we may consider $Der_c(G, (A, \mu))$ as a topological subspace of $\mathcal{C}_k(G, A) \times R$. We show that $Der_c(G, (A, \mu))$ is a topological group, whenever G and R are locally compact (Theorem 2.5). In addition, we prove that $Der_c(G, (A, \mu))$ is a topological G -module. Furthermore, we show that under what conditions, $Der_c(G, (A, \mu))$ is a precrossed topological $G - R$ -bimodule (Proposition 2.4).

In section 3, we define $H^1(G, (A, \mu))$ as a quotient of $Der_c(G, (A, \mu))$, where (A, μ) is a partially crossed topological $G - R$ -bimodule. We state conditions under which $H^1(G, (A, \mu))$ is a topological group (see Theorem 3.1). Moreover, since each partially crossed topological G -module can be naturally viewed as a partially crossed topological $G - G$ -bimodule, then we may define $H^1(G, (A, \mu))$, when (A, μ) is a partially crossed topological G -module. Finally, we find conditions under which $H^1(G, (A, \mu))$ is one of the following: k -space, discrete, locally compact and compact.

2 Partially Crossed topological $G - R$ -bimodule (A, μ)

In this section, we define a partially crossed topological $G - R$ -bimodule (A, μ) . We give some examples of precrossed, partially crossed and crossed topological $G - R$ -bimodules. Also, we define $Der_c(G, (A, \mu))$ and prove that if G and R are locally compact, then $Der_c(G, (A, \mu))$ is a topological group. Moreover, if the topological groups G and R act continuously on each other and on A compatibly, then $(Der_c(G, (A, \mu)), \gamma)$ is a precrossed topological $G - R$ -bimodule, where $\gamma : Der_c(G, (A, \mu)) \rightarrow R$, $(\alpha, r) \mapsto r$.

Definition 2.1. By a precrossed topological R -module we mean a pair (A, μ) where A is a topological R -module and $\mu : A \rightarrow R$ is a continuous homomorphism such that

$$\mu({}^r a) = {}^r \mu(a), \forall r \in R, a \in A.$$

If in addition we have the *Pieffer identity*

$$\mu({}^a b) = {}^a b, \forall a, b \in A,$$

then (A, μ) is called a crossed topological R -module.

Definition 2.2. A precrossed topological R -module (A, μ) is said to be a partially crossed topological R -module, whenever it satisfies the following equality

$$\mu({}^a b) = {}^a b,$$

for all $b \in A$ and for all $a \in A$ such that $\mu(a) \in [R, R]$.

It is clear that every crossed topological R -module is a partially crossed topological R -module.

Example 2.1. Suppose that A is a non-abelian topological group with nilpotency class of two (i.e., $[A, A] \subseteq Z(A)$). Take $R = A/[A, A]$. Let $\pi : A \rightarrow R$ be the canonical surjective map and suppose that R acts trivially on A . It is clear that ${}^{\pi(a)}b = {}^ab$, for all $b \in A$ if and only if $a \in Z(A)$. Hence, (A, π) is a partially crossed topological R -module which is not a crossed topological R -module.

Definition 2.3. Let G , R and A be topological groups. A precrossed topological R -module (A, μ) is said to be a precrossed topological $G - R$ -bimodule, whenever

- (1) G acts continuously on R and A ;
- (2) $\mu : A \rightarrow R$ is a continuous G -homomorphism;
- (3) $({}^g r)_a = {}^{grg^{-1}}a$ (i.e., compatibility condition) for all $g \in G$, $r \in R$ and $a \in A$.

Definition 2.4. A precrossed topological $G - R$ -bimodule (A, μ) is said to be a crossed topological $G - R$ -bimodule, if (A, μ) is a crossed topological R -module.

Example 2.2. (1) Let A be an arbitrary topological G -module. Then $Z(A)$ is a topological G -module. Since A is Hausdorff, then $Z(A)$ is a closed subgroup of A . Thus, the quotient group $R = A/Z(A)$ is Hausdorff. Now, we define an action of R on A and an action of G on R by:

$${}^a Z(A)b = {}^ab, \forall a, b \in A, \quad {}^g(aZ(A)) = {}^ga, \forall g \in G, a \in A. \quad (2.1)$$

Let $\pi_A : A \rightarrow R$ be the canonical homomorphism. It is easy to see that under (2.1) the pair (A, π_A) is a crossed topological $G - R$ -bimodule.

- (2) By part (1), for any topological group G the pair (G, π_G) is a crossed topological $G - G/Z(G)$ -bimodule.

Definition 2.5. A precrossed topological $G - R$ -bimodule (A, μ) is said to be partially crossed topological $G - R$ -bimodule, if (A, μ) is a partially crossed topological R -module.

Let G be a locally compact group and $Aut(G)$ the group of all topological group automorphisms (i.e., continuous and open automorphisms) of G with the *Birkhoff topology* (see [2], [3] and [5]). This topology is known as the *generalized compact-open topology*. A neighborhood basis of the identity automorphism consists of sets $N(C, V) = \{\alpha \in Aut(G) : \alpha(x) \in Vx, \alpha^{-1}(x) \in Vx, \forall x \in C\}$, where C is a compact subset of G and V is a neighborhood of the identity of G . It is well-known that $Aut(G)$ is a Hausdorff topological group [5, p. 40]. The generalized compact-open topology is finer than the compact-open topology in $Aut(G)$ and if G is compact, then the generalized compact-open topology coincides with compact-open topology in $Aut(G)$ (see [3, p. 324]).

Lemma 2.3. Let A be a locally compact group and G a topological group. Suppose that A is a topological G -module. Then

- (i) the homomorphism $\iota_A : A \rightarrow Aut(A)$, $a \mapsto c_a$, is continuous, where $c_a(b) = aba^{-1}$, $\forall b \in A$;
- (ii) A is a topological $Aut(A)$ -module by the action ${}^\alpha x = \alpha(x)$, $\forall \alpha \in Aut(A)$, $x \in A$;
- (iii) $Aut(A)$ is a topological G -module by the action $({}^g \alpha)(x) = {}^g \alpha({}^{g^{-1}}x)$, $\forall g \in G$, $\alpha \in Aut(A)$, $x \in A$.

Proof. For (i) and (ii) see [3, p. 324] and [5, Proposition 3.1]. (iii): It is enough to prove that the map $\chi : G \times Aut(A) \rightarrow Aut(A)$, $(g, \alpha) \mapsto {}^g \alpha$ is continuous. By (ii), the maps $\phi : (G \times Aut(A)) \times A \rightarrow A$, $((g, \alpha), x) \mapsto {}^g \alpha({}^{g^{-1}}x)x^{-1}$ and $\psi : (G \times Aut(A)) \times A \rightarrow A$, $((g, \alpha), x) \mapsto {}^g \alpha^{-1}({}^{g^{-1}}x)x^{-1}$ are continuous. Let ${}^g \alpha \in N(C, V)$. Then, $\phi((g, \alpha), x) \in V$ and $\psi((g, \alpha), x) \in V$, for all $x \in C$. Thus,

$\phi(\{(g, \alpha)\} \times C) \subset V$ and $\psi(\{(g, \alpha)\} \times C) \subset V$. Now, $\phi^{-1}(V)$ and $\psi^{-1}(V)$ are open in $(G \times \text{Aut}(A)) \times A$ containing $\{(g, \alpha)\} \times C$. Hence, $\phi^{-1}(V) \cap \psi^{-1}(V) \cap (G \times \text{Aut}(A)) \times C$ is an open set in $(G \times \text{Aut}(A)) \times C$ containing the slice $\{(g, \alpha)\} \times C$ of $(G \times \text{Aut}(A)) \times C$. The tube lemma [13, Lemma 5.8] implies that there is an open neighbourhood U of (g, α) in $G \times \text{Aut}(A)$ such that the tube $U \times C$ lies in $\phi^{-1}(V) \cap \psi^{-1}(V)$. Then, for every $(h, \beta) \in U$, $x \in C$, we have $\phi((h, \beta), x) \in V$ and $\psi((h, \beta), x) \in V$, i.e., ${}^h\beta({}^{h^{-1}}x) \in Vx$ and ${}^h\beta^{-1}({}^{h^{-1}}x) \in Vx$. Therefore, ${}^h\beta \in N(C, V)$, for all $(h, \beta) \in U$. So χ is continuous. $\square$

Proposition 2.1. *Let A be a topological G -module and A a locally compact group. Then, (A, ι_A) is a crossed topological $G - \text{Aut}(A)$ -bimodule, where the homomorphism ι_A and the actions are defined as in Lemma 2.3.*

Proof. By Lemma 2.3, the homomorphism ι_A and the actions are continuous. Also,

1. For every $g \in G$ and $a, b \in A$, $\iota_A({}^ga)(b) = c_{{}^ga}(b) = {}^ga b {}^ga^{-1} = {}^gc_a(b)$. Hence, ι_A is a G -homomorphism.
2. For every $\alpha \in \text{Aut}(A)$ and $x, a \in A$, $\iota_A({}^\alpha x)(a) = \iota_A(\alpha(x))(a) = c_{\alpha(x)}(a) = \alpha(x)a\alpha(x)^{-1} = \alpha(x\alpha^{-1}(a)x^{-1}) = \alpha \circ c_x \circ \alpha^{-1}(a) = {}^\alpha c_x(a)$. So ι_A is a $\text{Aut}(A)$ -homomorphism.
3. For every $a, b \in A$, $\iota_A({}^ab) = c_a(b) = aba^{-1} = {}^ab$. Thus, the Pieffer identity is satisfied.
4. The compatibility condition is satisfied. Since for every $g \in G, \alpha \in \text{Aut}(A), x \in A$, then ${}^g\alpha x = ({}^g\alpha)(x) = {}^g\alpha({}^{g^{-1}}x) = {}^{g\alpha g^{-1}}x$.

Therefore, (A, ι_A) is a crossed topological $G - \text{Aut}(A)$ -bimodule. $\square$

Remark 2.1. In a natural way any precrossed (crossed) topological R -module is a precrossed (crossed) topological $R - R$ -bimodule.

Remark 2.2. Let (A, μ) be a partially crossed (crossed) topological $G - R$ -bimodule. Then, $(A, \overline{\mu})$ is a partially crossed (crossed) topological $G - \mu(A)$ -bimodule. Thus, by Proposition 2.1, for any topological G -module A in which A is locally compact, we may associate the crossed topological $G - \text{Inn}(A)$ -bimodule $(A, \overline{\iota_A})$, where $\text{Inn}(A)$ is the topological group of all inner automorphisms of A .

Definition 2.6. Let (A, μ) be a partially crossed topological $G - R$ -bimodule. The map $\alpha : G \rightarrow A$ is called a crossed homomorphism whenever,

$$\alpha(gh) = \alpha(g){}^g\alpha(h), \forall g, h \in G.$$

Denote by $\text{Der}(G, (A, \mu))$ the set of all pairs (α, r) where $\alpha : G \rightarrow A$ is a crossed homomorphism and r is an element of R such that

$$\mu \circ \alpha(g) = r{}^gr^{-1}, \forall g \in G.$$

Let $\text{Der}_c(G, (A, \mu)) = \{(\alpha, r) | (\alpha, r) \in \text{Der}(G, (A, \mu)) \text{ and } \alpha \text{ is continuous}\}$. H. Inassaridze [9] introduced the product $\star$ in $\text{Der}(G, (A, \mu))$ by

$$(\alpha, r) \star (\beta, s) = (\alpha \star \beta, rs), \text{ where } \alpha \star \beta(g) = {}^r\beta(g)\alpha(g), \forall g \in G.$$

Definition 2.7. A family η of subsets of a topological space X is called a network on X if for each point $x \in X$ and each neighbourhood U of x there exists $P \in \eta$ such that $x \in P \subset U$. A network η is said to be compact (closed) if all its elements are compact (closed) subspaces of X . We say that a closed network η is hereditarily closed if for each $P \in \eta$ and any closed set B in P , $\blacksquare B \in \eta$.

Let X and Y be topological spaces. The set of all continuous functions $f : X \rightarrow Y$ is denoted by $\mathcal{C}(X, Y)$. Suppose that $U \subset X$ and $V \subset Y$. Take

$$[U, V] = \{f \in \mathcal{C}(X, Y) : f(U) \subset V\}.$$

Let X and Y be topological spaces, and η a network in X . The family $\{[P, V] : P \in \eta \text{ and } V \text{ is open in } Y\}$ is a subbase for a topology on $\mathcal{C}(X, Y)$, called the η -topology. We denote the set $\mathcal{C}(X, Y)$ with the η -topology by $\mathcal{C}_\eta(X, Y)$. If η is the family of all singleton subsets of X , then the η -topology is called the point-open topology; in this case $\mathcal{C}_\eta(X, Y)$ is denoted by $\mathcal{C}_p(X, Y)$. If η is the family of all compact subspaces of X , then the η -topology is called the compact-open topology and $\mathcal{C}_\eta(X, Y)$ is denoted by $\mathcal{C}_k(X, Y)$ (see [11]).

Now, suppose that A is a topological group, then $\mathcal{C}(X, A)$ is a group. For $f, g \in \mathcal{C}(X, A)$ the product, $f.g$, is defined by

$$(f.g)(x) = f(x).g(x), \forall x \in X. \quad (2.2)$$

Lemma 2.4. *Let X be a Tychonoff space and A a topological group. If η is a hereditarily closed, compact network on X , then under the product (2.2), $\mathcal{C}_\eta(X, A)$ is a topological group. In particular, $\mathcal{C}_p(X, A)$ and $\mathcal{C}_k(X, A)$ are topological groups.*

Proof. See [11, Theorem 1.1.7]. In particular, the set of all finite subsets of X and the set of all compact subsets of X are hereditarily closed, compact networks on X . $\square$

Suppose that X is a topological space and A a topological R -module. Then, $\mathcal{C}(X, A)$ is an R -module. If $r \in R, f \in \mathcal{C}(X, A)$, then the action ${}^r f$ is defined by

$$({}^r f)(x) = {}^r(f(x)), \forall x \in X. \quad (2.3)$$

Proposition 2.2. *Let X be a locally compact Hausdorff space, R a locally compact group and A a topological R -module. Then, by (2.3), $\mathcal{C}_k(X, A)$ is a topological R -module.*

Proof. Since X is a locally compact Hausdorff space, then by Lemma 2.4, $\mathcal{C}_k(X, A)$ is a topological group. By [13, Theorem 5.3], the evaluation map $e : X \times \mathcal{C}_k(X, A) \rightarrow A, (x, f) \mapsto f(x)$ is continuous. Thus, the map $F : R \times X \times \mathcal{C}_k(X, A) \rightarrow A, (r, x, f) \mapsto {}^r f(x)$ is continuous. By [13, Corollary 5.4], the induced map $\hat{F} : \mathcal{C}_k(X, A) \rightarrow \mathcal{C}_k(R \times X, A)$ is continuous, where $\hat{F}$ is defined by

$$\hat{F}(f)(r, x) = {}^r f(x).$$

On the other hand the exponential map $\Lambda : \mathcal{C}_k(R \times X, A) \rightarrow \mathcal{C}_k(R, \mathcal{C}_k(X, A)), u \mapsto \Lambda(u); \Lambda(u)(r)(x) = u(r, x)$, is a homeomorphism [11, Corollary 2.5.7]. Therefore, $\Lambda \circ \hat{F} : \mathcal{C}_k(X, A) \rightarrow \mathcal{C}_k(R, \mathcal{C}_k(X, A))$ is a continuous map. Since R is locally compact and Hausdorff then by [13, Corollary 5.4], $\Lambda \circ \hat{F}$ induces the continuous map $\chi : R \times \mathcal{C}_k(X, A) \rightarrow \mathcal{C}_k(X, A), \chi(r, f) = (\Lambda \circ \hat{F}(f))(r) = {}^r f$. Therefore, $\mathcal{C}_k(X, A)$ is a topological R -module. $\square$

Note that $\text{Der}_c(G, (A, \mu)) \subset \text{Der}_c(G, A) \times R \subset \mathcal{C}(G, A) \times R$, where $\text{Der}_c(G, A) = \{\alpha | \alpha \text{ is a continuous crossed homomorphism from } G \text{ into } A\}$. Thus, $\mathcal{C}_k(G, A) \times R$ induces the subspace topology on $\text{Der}_c(G, (A, \mu))$. Here, the induced subspace topology on $\text{Der}_c(G, (A, \mu))$ is called the induced topology by compact-open topology. From now on, we consider $\text{Der}_c(G, (A, \mu))$ with this topology.

In the following theorem $\star$ is as in Definition 2.6.

Theorem 2.5. *Let G and R be locally compact groups and (A, μ) a partially crossed topological $G - R$ -bimodule. Then, $(\text{Der}_c(G, (A, \mu)), \star)$ is a topological group.*

Proof. By [9, Proposition 3], $Der(G, (A, \mu))$ is a group. If $(\alpha, r), (\beta, s) \in Der_c(G, (A, \mu)) \subset Der(G, (A, \mu))$, then $(\alpha, r) \star (\beta, s) \in Der_c(G, (A, \mu))$ and $(\alpha, r)^{-1} = (\bar{\alpha}, r^{-1}) \in Der_c(G, (A, \mu))$, where $\bar{\alpha}(g) = r^{-1}\alpha(g)^{-1}, \forall g \in G$. It is clear that $\alpha \star \beta$ and $\bar{\alpha}$ are continuous. Thus, $Der_c(G, (A, \mu))$ is a subgroup of $Der(G, (A, \mu))$.

By Proposition 2.2, $C_k(G, A)$ is a topological R -module. Thus, it is clear that

$$\begin{aligned} \phi : (C_k(G, A) \times R) \times (C_k(G, A) \times R) &\rightarrow C_k(G, A) \times R \\ ((f, r), (g, s)) &\mapsto ({}^r g f, r s) \end{aligned}$$

and

$$\begin{aligned} \psi : C_k(G, A) \times R &\rightarrow C_k(G, A) \times R \\ (f, r) &\mapsto \bar{f} = ({}^{r^{-1}} f^{-1}, r^{-1}) \end{aligned}$$

are continuous. Obviously, the restrictions of ϕ and ψ to $Der_c(G, (A, \mu)) \times Der_c(G, (A, \mu))$ and $Der_c(G, (A, \mu))$ are continuous, respectively. Consequently, $(Der_c(G, (A, \mu)), \star)$ is a topological group. $\square$

Proposition 2.3. (i) Let (A, μ) be a partially crossed topological G - R -bimodule. Then, $Der_c(G, (A, \mu))$ is a closed subspace of $Der_c(G, A) \times R$;

(ii) Let A be a topological G -module. Then, $Der_c(G, A)$ is a closed subspace of $C_k(G, A)$.

Proof. (i). Consider the map

$$\phi_g : C_k(G, A) \times R \rightarrow R, (\alpha, r) \mapsto r^{-1}\mu\alpha(g)^g r,$$

for $g \in G$. By [15, 9.6 Lemma], ϕ_g is continuous, for all $g \in G$. Hence, $\phi_g^{-1}(1)$ is closed in $C_k(G, A) \times R$, for all $g \in G$. It is easy to see that

$$Der_c(G, (A, \mu)) = \bigcap_{g \in G} \phi_g^{-1}(1) \cap (Der_c(G, A) \times R).$$

Therefore, $Der_c(G, (A, \mu))$ is closed in $Der_c(G, A) \times R$.

(ii). By a similar argument as in (i), we consider the continuous map

$$\chi_{(g, h)} : C_k(G, A) \rightarrow A, \alpha \mapsto \alpha(gh)^{-1}\alpha(g)^g\alpha(h),$$

for $(g, h) \in G \times G$. Since

$$Der_c(G, A) = \bigcap_{(g, h) \in G \times G} \chi_{(g, h)}^{-1}(1),$$

then $Der_c(G, A)$ is closed in $C_k(G, A)$. $\square$

We immediately obtain the following two corollaries.

Corollary 2.6. Let (A, μ) be a partially crossed topological G - R -bimodule. Then, $Der_c(G, (A, \mu))$ is a closed subspace of $C_k(G, A) \times R$.

Corollary 2.7. Let G be a topological group and A an abelian topological group. Then, $Hom_c(G, A)$ is a closed subgroup of $C_k(G, A)$.

Suppose that (A, μ) is a partially crossed topological G - R -bimodule. There is an action of G on $Der(G, (A, \mu))$ defined by

$${}^g(\alpha, r) = (\tilde{\alpha}, {}^g r), g \in G, r \in R \quad (2.4)$$

with $\tilde{\alpha}(h) = {}^g\alpha(g^{-1}h), h \in G$ [9].

Note that if $(\alpha, r) \in Der_c(G, (A, \mu))$, then ${}^g(\alpha, r) \in Der_c(G, (A, \mu)), \forall g \in G$, since $\tilde{\alpha}$ is continuous. This shows that $Der_c(G, (A, \mu))$ is a G -submodule of $Der(G, (A, \mu))$.

Lemma 2.8. Let G and R be locally compact groups and (A, μ) a partially crossed topological module. Then by (2.4), $Der_c(G, (A, \mu))$ is a topological G -module.

Proof. Since G is locally compact and Hausdorff, then the evaluation map $e : G \times \mathcal{C}_k(G, A) \rightarrow A$, $(g, \alpha) \mapsto \alpha(g)$ is continuous. Thus, the map

$$\Phi : G \times G \times \mathcal{C}_k(G, A) \rightarrow A, (g, h, \alpha) \mapsto {}^g\alpha({}^{g^{-1}}h)$$

is continuous. By a similar argument as in the proof of Proposition 2.2, the map $G \times \mathcal{C}_k(G, A) \rightarrow \mathcal{C}_k(G, A)$, $(g, \alpha) \mapsto \tilde{\alpha}$ is continuous, where $\tilde{\alpha}(h) = {}^g\alpha({}^{g^{-1}}h)$, $h \in G$. Hence,

$$(G \times \mathcal{C}_k(G, A)) \times R \rightarrow \mathcal{C}_k(G, A) \times R$$

$$((g, \alpha), r) \mapsto (\tilde{\alpha}, {}^g r)$$

is continuous. Therefore, by restriction of this map to $G \times Der_c(G, (A, \mu))$ we get the continuous map

$$G \times Der_c(G, (A, \mu)) \rightarrow Der_c(G, (A, \mu))$$

$$((g, \alpha), r) \mapsto (\tilde{\alpha}, {}^g r)$$

and this completes the proof. $\square$

Let (A, μ) be a partially crossed topological $G - R$ -bimodule. If G is a topological R -module, and the compatibility condition

$$({}^r g)a = {}^{rg}r^{-1}a \text{ and } ({}^r g)s = {}^{rg}r^{-1}s; \forall r, s \in R, g \in G, a \in A,$$

holds, then $Der(G, (A, \mu))$ is an R -module via

$${}^r(\alpha, s) = (\tilde{\alpha}, {}^r s) \quad (2.5)$$

where $\tilde{\alpha}(g) = {}^r\alpha({}^{r^{-1}}g)$, $g \in G$ [9].

It is easy to see that $Der_c(G, (A, \mu))$ is an R -submodule of $Der(G, (A, \mu))$.

Lemma 2.9. Let G and R be locally compact groups and (A, μ) a partially crossed topological $G - R$ -bimodule. Then by (2.5), $Der_c(G, (A, \mu))$ is a topological R -module.

Proof. This can be proved by a similar argument as in Lemma 2.8. $\square$

Definition 2.8. Let G and R be topological groups acting continuously on each other. These actions are said to be compatible if

$$({}^r g)s = {}^{rg}r^{-1}s \text{ and } ({}^g r)h = {}^{gr}g^{-1}h; \forall r, s \in R, g, h \in G.$$

Also, it is said that the topological groups G and R act (continuously) on a topological group A compatibly if

$$({}^r g)a = {}^{rg}r^{-1}a \text{ and } ({}^g r)a = {}^{gr}g^{-1}a; \forall r \in R, g \in G, a \in A.$$

Proposition 2.4. Let G and R be locally compact groups and (A, μ) a partially crossed topological $G - R$ -bimodule. Let the topological groups G and R act continuously on each other and on A compatibly. Then, $(Der_c(G, (A, \mu)), \gamma)$ is a precrossed topological $G - R$ -bimodule, where $\gamma : Der_c(G, (A, \mu)) \rightarrow R$, $(\alpha, r) \mapsto r$.

Proof. Since G and R are locally compact groups, then by Lemma 2.8 and Lemma 2.9, G and R act continuously on $Der_c(G, (A, \mu))$. The map γ is continuous, since $\pi_2 : \mathcal{C}_k(G, A) \times R \rightarrow R$, $(\alpha, r) \mapsto r$ is continuous. Also, γ is a G -homomorphism and an R -homomorphism. Since ${}^g r(\alpha, s) = {}^{gr}g^{-1}(\alpha, s)$ for all $g \in G, r \in R, (\alpha, s) \in Der_c(G, (A, \mu))$ [9, Proposition 5], we conclude that $(Der_c(G, (A, \mu)), \gamma)$ is a precrossed topological $G - R$ -bimodule. $\square$

3 The first non-abelian cohomology of a topological group as a topological space

In this section we define the first non-abelian cohomology $H^1(G, (A, \mu))$ of G with coefficients in a partially crossed topological $G - R$ -bimodule (A, μ) . We will introduce a topological structure on $H^1(G, (A, \mu))$. It will be shown that under what conditions $H^1(G, (A, \mu))$ is a topological group. As a result, $H^1(G, (A, \mu))$ is a topological group for every partially crossed topological G -module. In addition, we verify some topological properties of $H^1(G, (A, \mu))$.

Let R be a topological G -module, then we define

$$H^0(G, R) = \{r \mid {}^g r = r, \forall g \in G\}.$$

Let (A, μ) be a partially crossed topological $G - R$ -bimodule. H. Inassaridze [8] introduced an equivalence relation on the group $Der(G, (A, \mu))$ as follows:

$$\begin{aligned} (\alpha, r) \sim (\beta, s) &\Leftrightarrow (\exists a \in A \wedge (\forall g \in G \Rightarrow \beta(g) = a^{-1}\alpha(g)^g a)) \\ &\wedge (s = \mu(a)^{-1}r \bmod H^0(G, R)) \end{aligned}$$

Let $\sim'$ be the restriction of $\sim$ to $Der_c(G, (A, \mu))$. Therefore, $\sim'$ is an equivalence relation. In other word, $(\alpha, r) \sim' (\beta, s)$ if and only if $(\alpha, r) \sim (\beta, s)$, whenever $(\alpha, r), (\beta, s) \in Der_c(G, (A, \mu))$.

Definition 3.1. Let (A, μ) be a partially crossed topological $G - R$ -bimodule. The quotient set $Der_c(G, (A, \mu)) / \sim'$ will be called the first cohomology of G with the coefficients in (A, μ) and is denoted by $H^1(G, (A, \mu))$. (In this definition, the groups G, R and A are not necessarily Hausdorff.)

Theorem 3.1. Let G and R be locally compact groups and (A, μ) a partially crossed topological $G - R$ -bimodule satisfying the following conditions

- (i) $H^0(G, R)$ is a normal subgroup of R ;
- (ii) for every $c \in H^0(G, R)$ and $(\alpha, r) \in Der_c(G, (A, \mu))$, there exists $a \in A$ such that $\mu(a) = 1$ and ${}^c \alpha(g) = a^{-1}\alpha(g)^g a, \forall g \in G$.

Then, $Der_c(G, (A, \mu))$ induces a topological group structure on $H^1(G, (A, \mu))$.

Proof. By [8, Theorem 2.1] the group $Der(G, (A, \mu))$ induces the following action on $Der(G, (A, \mu)) / \sim$

$$[(\alpha, r)][(\beta, s)] = [({}^r \beta \alpha, rs)].$$

Thus, $N = \{(\alpha, r) \mid (\alpha, r) \in Der(G, (A, \mu)), (\alpha, r) \sim (1, 1)\}$ is a normal subgroup of $Der(G, (A, \mu))$. Therefore, $N' = \{(\alpha, r) \mid (\alpha, r) \in Der_c(G, (A, \mu)), (\alpha, r) \sim (1, 1)\}$ is a normal subgroup of $Der_c(G, (A, \mu))$. By Theorem 2.5, $Der_c(G, (A, \mu))$ is a topological group. Obviously, $H^1(G, (A, \mu)) = Der_c(G, (A, \mu)) / N'$. Therefore, $H^1(G, (A, \mu))$ is a topological group. $\square$

Notice 3.2. (i) Note that Hausdorffness of A is not needed in Theorem 3.1.

- (ii) Let A be a topological G -module. The first cohomology, $H^1(G, A)$, of G with coefficients in A is defined as in [14]. Thus, the compact-open topology on $Der_c(G, A)$ induces a quotient topology on $H^1(G, A)$. From now on, we consider $H^1(G, A)$ with this topology. Define $Inn(G, A) = \{Inn(a) \mid a \in A\}$, where for all $a \in A, g \in G$, $Inn(a)(g) = a^g a^{-1}$. If A is abelian, then by [14, Remark 2.4. (i)], $Inn(G, A)$ is a normal subgroup of $Der_c(G, A)$ and $H^1(G, A) = Der_c(G, A) / Inn(G, A)$; moreover, $H^1(G, A)$ is a topological group, and it is Hausdorff if and only if $Inn(G, A)$ is closed in $Der_c(G, A)$.
- (iii) Define $Inn(G, (A, \mu)) = \{(Inn(a), \mu(a)z) \mid a \in A, z \in H^0(G, R)\}$. Note that $H^1(G, (A, \mu))$ is a topological group if and only if $Inn(G, (A, \mu))$ is a normal subgroup of $Der_c(G, (A, \mu))$. Thus, by hypotheses of Theorem 3.1, $Inn(G, (A, \mu))$ is a normal subgroup of $Der_c(G, (A, \mu))$ and $H^1(G, (A, \mu)) = Der_c(G, (A, \mu)) / Inn(G, (A, \mu))$.

In the following, we give an example for this fact that: in general, $H^1(G, A)$ and $H^1(G, (A, \mu))$ are not necessarily Hausdorff.

Example 3.3. Let G be an abelian discrete group; let $(\mathbb{Z}, +)$ be the integer numbers group with the indiscrete topology τ , (i.e., $\tau = \{\mathbb{Z}, \emptyset\}$) such that $\chi : G \rightarrow \text{Aut}(\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ is a nontrivial homomorphism. Equip $\text{Aut}(\mathbb{Z})$ with the compact-open topology. Then, χ induces a nontrivial continuous action of G on $\mathbb{Z}$ given by ${}^g z = \chi(g)(z)$, $\forall g \in G, z \in \mathbb{Z}$. For all $g \in G$, we have $[\{g\}, \mathbb{Z}] \cap \text{Der}_c(G, \mathbb{Z}) = \text{Der}_c(G, \mathbb{Z})$. Hence, the compact-open topology on $\text{Der}_c(G, \mathbb{Z})$ is the indiscrete topology. Thus, $H^1(G, \mathbb{Z}) = \text{Der}_c(G, \mathbb{Z})/\text{Inn}(G, \mathbb{Z})$ has the indiscrete topology. On the other hand, discreteness of G implies that $\text{Der}_c(G, \mathbb{Z}) = \text{Der}(G, \mathbb{Z})$. Hence by [1, Theorem 3.2], $H^1(G, \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} \neq 1$. Hence, $H^1(G, \mathbb{Z})$ is not Hausdorff. Consequently, $\text{Inn}(G, \mathbb{Z})$ is not closed in $\text{Der}_c(G, \mathbb{Z})$. Now, note that $(\mathbb{Z}, 1 : \mathbb{Z} \rightarrow G)$ is a crossed $G - G$ -bimodule. It is easy to see that $\text{Inn}(G, (\mathbb{Z}, 1)) = \text{Inn}(G, \mathbb{Z}) \times G$. Hence $\text{Inn}(G, (\mathbb{Z}, 1))$ is not closed in $\text{Der}_c(G, (\mathbb{Z}, 1))$ and so $H^1(G, (\mathbb{Z}, 1))$ is not Hausdorff.

Remark 3.1. Let A be an abelian topological G -module and A be compact Hausdorff. Then, $H^1(G, A)$ is a Hausdorff topological group.

Let (A, μ) be a partially crossed G -module. Naturally (A, μ) is a crossed $G - G$ -bimodule. Thus, we define the first cohomology of G with coefficients in (A, μ) as the set $H^1(G, (A, \mu))$.

Theorem 3.4. Let G be a locally compact group and (A, μ) a partially crossed topological G -module. Then, $H^1(G, (A, \mu))$ is a topological group. In addition, if any of the following conditions is satisfied, then $H^1(G, (A, \mu))$ is Hausdorff.

- (i) A is compact and G has trivial center;
- (ii) A is a trivial G -module;
- (iii) A and $Z(G)$ are compact, in particular if both topological groups A and G are compact.

Proof. Note that $H^0(G, G) = Z(G)$. For any $c \in Z(G)$ and $(\alpha, g) \in \text{Der}_c(G, (A, \mu))$, $\alpha(cx) = \alpha(xc)$ for all $x \in G$. Thus, ${}^c \alpha(x) = \alpha(c)^{-1} \alpha(x) \alpha(c)$, $\forall x \in G$ and $\mu(\alpha(c)) = {}^g c c^{-1} = 1$. Since G is locally compact, then by Theorem 3.1, $H^1(G, (A, \mu))$ is a topological group.

(i). If A is compact and G has trivial center then by the assumption $Z(G) = 1$. So $\text{Inn}(G, (A, \mu)) = \{(\text{Inn}(a), \mu(a)) | a \in A\}$. It is easy to see that the map $\text{Inn} : A \rightarrow \text{Der}_c(G, A), a \mapsto \text{Inn}(a)$ is continuous. Thus, compactness of A implies that $\text{Inn}(G, (A, \mu))$ is a compact subset of $\text{Der}_c(G, (A, \mu))$. Hence, $\text{Inn}(G, (A, \mu))$ is closed in $\text{Der}_c(G, (A, \mu))$. So $H^1(G, (A, \mu))$ is Hausdorff.

(ii). If G acts trivially on A , then ${}^g \mu(a) = \mu(a)$, for every $g \in G$ and $a \in A$. Thus, $\text{Inn}(G, (A, \mu)) = \{1\} \times Z(G)$. Hence, $\text{Inn}(G, (A, \mu))$ is closed in $\text{Der}_c(G, (A, \mu))$.

(iii). Consider the continuous map $A \times Z(G) \rightarrow \text{Der}_c(G, (A, \mu)), (a, z) \mapsto (\text{Inn}(a), \mu(a)z)$. Consequently, the part (iii) is proved. $\square$

Lemma 3.5. Let G be a locally compact group and A an abelian topological group. Then, there is a natural topological isomorphism

$$\text{Hom}_c(G, A) \simeq \text{Hom}_c(G/\overline{[G, G]}, A).$$

Proof. Since G is locally compact, then $G/\overline{[G, G]}$ is a locally compact group. Let $\pi : G \rightarrow G/\overline{[G, G]}$ be the natural epimorphism. Then, obviously $\chi : \text{Hom}_c(G/\overline{[G, G]}, A) \rightarrow \text{Hom}_c(G, A), f \mapsto \pi f$ is a one to one and onto continuous homomorphism. We show that χ is an open map. It suffices to show that for every neighborhood Γ of 1 in $\text{Hom}_c(G, A)$, $\chi(\Gamma)$ is a neighborhood of 1 in $\text{Hom}_c(G/\overline{[G, G]}, A)$. Since $\text{Hom}_c(G, A)$ is a topological group, so it is a homogeneous space. It is clear that the network of all compact subset of G is closed under finite unions. Now, by a similar argument as in [11,

p. 7], there is an open neighborhood of 1 of the form $S(C, U)$ in Γ . Note that $S(C, U) = \{f | f \in \text{Hom}_c(G/[G, G], A), f(C) \subset U\}$, where C is compact in $G/[G, G]$ and U is open in A . Since G is locally compact, then by [6, 5.24.b], there is a compact subset D of G such that $\pi(D) = C$. It is easy to see that $\chi(S(C, U)) = S(D, U) \subset \chi(\Gamma)$. Therefore, χ is a topological isomorphism. $\square$

Recall that a topological group G has no small subgroups (or is without small subgroups) if there is a neighborhood of the identity that contains no nontrivial subgroup of G . For example if n is a positive integer number, then the n -dimensional vector group, the n -dimensional torus, and general linear groups over the complex numbers are without small subgroups. It is well-known that the property of having no small subgroups is an extension property [13, 6.15 Theorem]. A topological group G is called compactly generated if there exists a compact subset K so that it generates G , that is $G = \langle K \rangle$.

Proposition 3.1. (1) If G is a locally compact group and A is a compact abelian group without small subgroups, then $\text{Hom}_c(G, A)$ is a locally compact group.

- (2) If G is a locally compact compactly generated group and A is a locally compact abelian group without small subgroups, then $\text{Hom}_c(G, A)$ is a locally compact group.
- (3) If G is a compact group and A is an abelian group without small subgroups, then $\text{Hom}_c(G, A)$ is a discrete group.
- (4) If G is a discrete group and A is a compact group, then $\text{Hom}_c(G, A)$ is a compact group.
- (5) If G is a finite discrete group and A is a compact abelian group without small subgroups, then $\text{Hom}_c(G, A)$ is a finite discrete group.
- (6) Let A be a topological G -module. If G is discrete and A is compact, then $\text{Der}_c(G, A)$ is a compact group.

Proof. Since A is abelian, by Lemma 3.5, $\text{Hom}_c(G, A) \simeq \text{Hom}_c(G/[G, G], A)$. Therefore, (1) and (2) follow from two corollaries of [11, p. 377]. Also (3) is obtained by [11, Theorem 4.1].

(4) Since G is discrete, then $\mathcal{C}_k(G, A) = \mathcal{C}_p(G, A)$. By Corollary 2.7, $\text{Hom}_c(G, A)$ is closed in $\mathcal{C}_k(G, A)$. Let $B = \prod_{g \in G} A_g$, where $A_g = A, \forall g \in G$. It is clear that the map $\Phi : \mathcal{C}_p(G, A) \rightarrow B, f \mapsto \{f(g)\}_{g \in G}$ is continuous. In addition, since G is discrete, then the map $G \times B \rightarrow A, (h, \{a_g\}_{g \in G}) \mapsto a_h$ is continuous. Hence, this map induces the continuous map $\Psi : B \rightarrow \mathcal{C}_p(G, A), \{a_g\}_{g \in G} \mapsto f$, where $f(g) = a_g$. Obviously, $\Phi\Psi = \text{Id}$ and $\Psi\Phi = \text{Id}$. Consequently, $\mathcal{C}_p(G, A)$ is homeomorphic to B . Thus, $\mathcal{C}_p(G, A)$ is compact. So $\text{Hom}_c(G, A)$ is compact.

(5) This is an immediate result from (3) and (4).

(6) By Proposition 2.3, $\text{Der}_c(G, A)$ is closed in $\mathcal{C}_k(G, A)$. We have seen in the proof of (4) that $\mathcal{C}_k(G, A)$ is compact. Consequently, $\text{Der}_c(G, A)$ is compact. $\square$

Recall that a topological space X is called a k -space if every subset of X , whose intersection with every compact $K \subset X$ is relatively open in K , is open in X . A topological space X is a k -space if and only if X is the quotient image of a locally compact space [16, Characterization (1)]. For example, locally compact spaces and first-countable spaces are k -spaces. It is well-known that the k -space property is preserved by the closed subsets and the quotients. Also, the product of a locally compact space with a k -space is a k -space (see [16, Result (1)]). We call a topological group to be a k -group if it is a k -space as a topological space.

Theorem 3.6. Let G be a locally compact group; let (A, μ) be a partially crossed topological $G - R$ -bimodule such that G acts trivially on A and R .

- (1) If R is a k -group and A is compact without small subgroups, then $H^1(G, (A, \mu))$ is a k -space.
- (2) If G is compactly generated, R is a k -group and A is locally compact without small subgroups, then $H^1(G, (A, \mu))$ is a k -space.

- (3) If G is compact, A has no small subgroups and R is discrete, then $H^1(G, (A, \mu))$ is discrete.
- (4) If G and R are finite discrete and A is compact without small subgroups, then $H^1(G, (A, \mu))$ is a finite discrete space.

Proof. Since G acts trivially on A and R , then it is easy to see that $Der_c(G, (A, \mu))$ is homeomorphic to $Hom_c(G, Ker\mu) \times R$. Note that $Ker\mu$ is closed in $Z(A)$. Now by Proposition 3.1, the assertions (1) to (4) hold. $\square$

Theorem 3.7. Let G be a locally compact abelian topological group; let (A, μ) be a partially crossed topological G -module and A a trivial G -module.

- (1) If A is compact without small subgroups, then $H^1(G, (A, \mu))$ is a locally compact abelian group.
- (2) If G is compactly generated and A is locally compact without small subgroups, then $H^1(G, (A, \mu))$ is a locally compact abelian group.
- (3) If G is finite discrete and A is compact without small subgroups, then $H^1(G, (A, \mu))$ is a finite discrete abelian group.

Proof. Since G is a locally compact abelian group and acts trivially on A , one can see $Der_c(G, (A, \mu)) \simeq Hom_c(G, Ker\mu) \times G$. Therefore, by Proposition 3.1, the proof is completed. $\square$

Let G and A be topological groups; let K be an abelian subgroup of A . We denote the set of all continuous homomorphisms $f : G \rightarrow A$ with $f(G) \subset K$ by $Hom_c(G, A|K)$. Obviously, if G is locally compact, then $Hom_c(G, A|K)$ with compact-open topology is an abelian topological group.

Remark 3.2. (1) Let (A, μ) be a partially crossed topological G -module. Suppose that G is a locally compact abelian group which acts trivially on A . Then, $H^1(G, (A, \mu)) \simeq Hom_c(G, A|Ker\mu)$.

- (2) Let A be an abelian topological G -module. Then, $(A, 1)$ is a crossed topological $G - R$ -bimodule for every topological group R , and $H^1(G, (A, 1))$ is homeomorphic to $H^1(G, A)$.
- (3) Let G be a locally compact group and A an abelian topological G -module. Then, $(A, 1)$ is a crossed topological G -module, and $H^1(G, (A, 1)) \simeq H^1(G, A)$. In particular if G acts trivially on A , then $H^1(G, (A, 1)) \simeq Hom_c(G/[G, G], A)$.
- (4) Let G be a locally compact group and A an abelian topological G -module. Then, $H^1(G, (A, \pi_A)) = H^1(G, (A, 1)) \simeq H^1(G, A)$.

Theorem 3.8. Let (A, μ) be a partially crossed topological $G - R$ -bimodule. Suppose that G is a discrete group, A and R are compact. Then, $H^1(G, (A, \mu))$ is compact.

Proof. By Proposition 2.3, $Der_c(G, (A, \mu))$ is closed in $Der_c(G, A) \times R$. Obviously, if R is compact, then $H^1(G, (A, \mu))$ is compact. $\square$

As an immediate result of Theorem 3.8, we have the following corollary:

Corollary 3.9. Let (A, μ) be a partially crossed topological G -module, G be finite discrete and A be compact. Then $H^1(G, (A, \mu))$ is a compact group.

Definition 3.2. A topological group A is radical-based, if it has a countable base $\{U_n\}_{n \in \mathbb{N}}$ at 1, such that each U_n is symmetric and for all $n \in \mathbb{N}$:

- (1) $(U_n)^n \subset U_1$;
- (2) $a, a^2, \dots, a^n \in U_1$ implies $a \in U_n$.

For example, if n is a positive integer, then the n -dimensional vector group, the n -dimensional torus and the rational numbers are radical-based groups. For another example see [10].

Theorem 3.10. *Let (A, μ) be a partially crossed topological $G - R$ -bimodule, and G a first countable group. Let R be locally compact and A a compact radical-based group with $H^0(G, A) = A$. Then, $H^1(G, (A, \mu))$ is a k -space.*

Proof. Since $H^0(G, A) = A$, then it follows from Proposition 2.3 that $Der_c(G, (A, \mu))$ is closed in $Hom_c(G, A) \times R$. By [10, Theorem 1], $Hom_c(G, A)$ is a k -space. Thus, $Hom_c(G, A) \times R$ is a k -space. Consequently, $H^1(G, (A, \mu))$ is a k -space. $\square$

By Theorem 3.10, the next corollary is immediate.

Corollary 3.11. *Let (A, μ) be a partially crossed topological G -module, let G be locally compact first countable and A a compact radical-based group with $H^0(G, A) = A$. Then, $H^1(G, (A, \mu))$ is a k -group.*

4 CONCLUSIONS

- a We have proved that if G and R are locally compact groups and (A, μ) a partially crossed topological $G - R$ -bimodule, then $Der_c(G, (A, \mu))$ is a topological group (Theorem 2.5). Moreover, if the locally compact groups G and R act continuously on each other and on A compatibly then $(Der_c(G, (A, \mu)), \gamma)$ is a precrossed topological $G - R$ -bimodule, where $\gamma : Der_c(G, (A, \mu)) \rightarrow R, (\alpha, r) \mapsto r$ (Proposition 2.4).
- b We have showed that under what conditions $H^1(G, (A, \mu))$ is a topological group (Theorem 3.1). In particular, if G is a locally compact group, then $H^1(G, (A, \mu))$ is a topological group for every partially crossed topological G -module (A, μ) . Furthermore, we have found conditions under which $H^1(G, (A, \mu))$ is one of the following: k -space, discrete, locally compact and compact.

Competing interests

The authors declare that they have no competing interests.

Acknowledgment

The authors gratefully acknowledge the financial support for this work that was provided by University of Guilan.

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