

DECOMPOSER TYPE FUNCTIONS ON GROUPS

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ABSTRACT. Decomposer functions in algebraic structures were introduced and studied in the paper "Decomposer and associative functional equations" in 2007. If $(G, \cdot)$ is a group and, $f : G \rightarrow G$ is a map, then f is called a right [resp. left] decomposer if and only if $f(f^*(x)f(y)) = f(y)$ [resp. $f(f(x)f_*(y)) = f(x)$] for all $x, y \in G$, where $f^*(x)f(x) = f(x)f_*(x) = x$. Also, f is called decomposer if it is left and right decomposer. There are many important connections between these functions and decomposition of groups by subsets. Now, we observe that if the structure is a group, then there are more important properties for them and also many connections among decomposer functions, multiplicative symmetric functions (introduced by J. G. Dhombres in 1973), separator functions and so on. For instance, every idempotent endomorphism in groups is (strong) decomposer. In the way, we introduce some other related types and generalizations of these functions such as semi-strong decomposer and weak decomposer which help us in the study of decomposer type functions in groups. Then, several important properties and relations for these functions will be proved. Finally, we completely characterize the (two-sided) decomposer functions in arbitrary groups, and so we give a general solution of the decomposer equations that were not solved in the paper 2007 (only left and right cases were characterized).

1. INTRODUCTION AND PRELIMINARIES

In [3], decomposer and strong decomposer functional equations on algebraic structures (with a single binary operation) were introduced and studied. They have close relations to associative and canceler functions (see [1, 2, 3]). If G is a group and $f : G \rightarrow G$ is an associative function, then f is decomposer, $f(G)$ is a factor of G , and $(G, \cdot_f)$ is a semigroup where $\cdot_f$ is f -multiplication ($x \cdot_f y = f(xy)$, see [7, 6]). In fact, $(G, \cdot_f)$ is a type of grouplikes that have recently been introduced in [4]. But, we find that for a perfect study and obtaining more properties of these functions, specially in groups, we need to introduce some other types of these functions and consider their relations.

Let $(G, \cdot)$ be a group with the identity element e and denote by $\iota = \iota_G$ the identity

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function on G . If f, g are functions from G to G , then define $f \cdot g$, f^- , f^* and f_* for every $x \in G$ by

$$f \cdot g(x) = f(x)g(x), \quad f^-(x) = f(x)^{-1}, \quad f^* = \iota_G \cdot f^-, \quad f_* = f^- \cdot \iota_G.$$

Hence, if $f = c$ is a constant function ($c \in G$), then $f \cdot g$ can be replaced by $c \cdot g$. The function f^* [resp. f_*] is called left [resp. right] $*$ -conjugation of f . They are also called $*$ -conjugations of f . Also, set $f_{**} := (f_*)^*$, $f^{**} := (f^*)^*$. Clearly $\iota^* = \iota_* = e$, $e^* = e_* = \iota$, $f = (f^*)_* = (f_*)^*$ and the identity $(fg)^- = f^-g$ implies $f^-f = (f^2)^-$, $(fg)^* = g^* \cdot f^*g$ and $(fg)_* = f_*g \cdot g_*$. Also, we have

$$f \text{ is idempotent} \Leftrightarrow f^*f = e \Leftrightarrow f_*f = e$$

$$f^{*2} = f^* \Leftrightarrow ff^* = e \Leftrightarrow f^{**}f^* = e, \quad f_*^2 = f_* \Leftrightarrow ff_* = e \Leftrightarrow f_{**}f_* = e.$$

Note that if f is an endomorphism, then $f^-f = ff^-$, $f^*f = ff^*$ and $f_*f = ff_*$ (i.e., the compositions of f and its $*$ -conjugations are commutative).

Additive notations. If $(G, +)$ is an additive group, then the notations e , f^- , $f \cdot g$, $f \cdot g^-$ are replaced by 0 , $-f$, $f + g$, $f - g$, and we have $f^* = f_* = \iota - f$.

Example 1.1. Consider the additive group $\mathbb{R}$ and fix $b \in \mathbb{R} \setminus \{0\}$. For a real number a , denote by $[a]$ the largest integer not exceeding a and put $(a) = a - [a]$ (the fractional or decimal part of a). Now, set

$$[a]_b = b[\frac{a}{b}], \quad (a)_b = b(\frac{a}{b}).$$

We call $[a]_b$ b -integer part of a and $(a)_b$ b -decimal part of a . Also $[]_b$, $()_b$ are called b -decimal part function and b -integer part function, respectively. We have $()_b^* = []_b$ and $[]_b^* = ()_b$ and both are idempotent, so their compositions are zero. One can see an algebraic study of b -parts in [5]. We use b -parts functions for several examples and counterexamples in this paper.

2. DECOMPOSER TYPE FUNCTIONS ON GROUPS

Here, we introduce decomposer type functions in groups. Notice that decomposer (resp. strong decomposer) function is a solution of the decomposer (resp. strong decomposer) functional equation, introduced in [3].

Definition 2.1. We call a function f from G to G :

(a) *right [resp. left] strong decomposer* if

$$f(f^*(x)y) = f(y) \quad [\text{resp. } f(xf_*(y)) = f(x)] \quad : \forall x, y \in G.$$

(b) *right [resp. left] semi-strong decomposer* if

$$f(f^*(x)y) = f(f^*(e)y) \quad [\text{resp. } f(xf_*(y)) = f(xf_*(e))] \quad : \forall x, y \in G.$$

(c) *right [resp. left] decomposer* if

$$f(f^*(x)f(y)) = f(y) \quad [\text{resp. } f(f(x)f_*(y)) = f(x)] \quad : \quad \forall x, y \in G.$$

(d) *right [resp. left] weak decomposer* if

$$f(f^*(e)f(x)) = f(x), \quad f(f^*(x)f(e)) = f(e) \quad : \quad \forall x \in G.$$

$$[\text{resp. } f(f(x)f_*(e)) = f(x), \quad f(f(e)f_*(x)) = f(e) \quad : \quad \forall x \in G].$$

(e) *right [resp. left] separator* if $f^*(G) \cap f(G) = \{f(e)\}$ [resp. $f(G) \cap f_*(G) = \{f(e)\}$].

In each of the above parts, if $f(e) = e$, then we will add the word *standard* to the titles. For example "f is a standard right separator" means $f^*(G) \cap f(G) = \{e\}$. We call *f decomposer* or *two-sided decomposer* [resp. separator] if it is left and right decomposer [resp. separator].

Remark 2.2. Let $f : G \rightarrow G$ be a map. We always have

$$f(x) = f(y) \Rightarrow x = f^*(x)f(y) \quad [\text{resp. } f(x) = f(y) \Rightarrow x = f(y)f_*(x)]$$

for every $x, y \in G$. But its converse is valid if and only if *f is a* right [resp. left] decomposer. Also, we can state the property of a (two-sided) decomposer function *f* by the following doubled equation

$$(2.1) \quad f(f^*(x)f(y)) = f(f(y)f_*(x)) = f(y), \quad \forall x, y \in G.$$

If *G* is commutative, then left, right and two-sided cases are equivalent, so the words "left" and "right" can be omitted from the titles. Also, a function $f : G \rightarrow G$ is left weak decomposer [resp. decomposer, separator] if and only if f_* is right weak decomposer [resp. decomposer, separator], since considering $(f_*)^* = f$, we have

$$\begin{aligned} f_* \text{ is right decomposer} &\Leftrightarrow f_*(f(x)f_*(y)) = f_*(y) \Leftrightarrow \\ f^-(f(x)f_*(y))f(x)f_*(y) &= f_*(y) \Leftrightarrow f(f(x)f_*(y)) = f(x). \end{aligned}$$

The above property is not valid for strong and semi-strong decomposer functions, since the function $f = ()_b$ is (left) strong decomposer but $f_* = []_b$ is not (right) strong decomposer. Of course, we show that *f is a* left strong decomposer if and only if f_* *is a* right decomposer and $f_*(G) \leq G$ (i.e., $f_*(G)$ is a subgroup of *G*). Therefore, we focus on right (and two-sided) decomposer type and separator functions. The results for the left cases are similar.

Example 2.3. (a) Every idempotent endomorphism in groups is strong decomposer.

(b) Consider $G = \{1, a, a^2, a^3, b, ba, ba^2, ba^3\} \cong D_4$ ($a^4 = b^2 = 1, bab = a^{-1} = a^3$). Put $\Omega = \{1, ba, ba^2, ba^3\}$ and

$$f(x) = \begin{cases} x, & x \in \Omega \\ bx, & x \notin \Omega \end{cases}$$

Considering the relation $x \notin \Omega \Leftrightarrow bx \in \Omega$, it can be seen that f is a (standard) right strong decomposer.

(c) The b -parts functions are decomposer functions in the additive real number group. Moreover, the b -decimal part function $(\)_b$ is strong decomposer. Also, for every constant real number c , the function $f(x) = (\)_b + c$ is semi-strong decomposer.

Now, we want to show that every multiplicative symmetric function introduced by J. G. Dhombres in [2] on groups is decomposer, so it is a type of (two-sided) decomposer functions. For if $f : G \rightarrow G$ is multiplicative symmetric ($f(xf(y)) = f(f(x)y)$ for every $x, y \in G$), then

$$\begin{aligned} f(f^*(x)f(y)) &= f(ff^*(x)y) = f(ff^*(x)f(x)f^-(x)y) = f(f^*(x)f(f(x)f^-(x)y)) \\ &= f(f^*(x)f(xf(f^-(x)y))) = f(ff^*(x)xf(f^-(x)y)) = f(f(ff^*(x)x)f^-(x)y) \\ &= f(f(f^*(x)f(x))f^-(x)y) = f(f(x)f^-(x)y) = f(y). \end{aligned}$$

So, f is a right decomposer. Similarly, it is also a left decomposer. But, the converse is not true. Consider the additive integer group $(\mathbb{Z}, +)$ and define a map $f : \mathbb{Z} \rightarrow \mathbb{Z}$ by

$$f(x) := x + (x)_2 - 1 = \begin{cases} x, & x \in \mathbb{Z}_o \\ x - 1, & x \in \mathbb{Z}_e \end{cases}$$

$f : \mathbb{Z} \rightarrow \mathbb{Z}$ is decomposer, since

$$\begin{aligned} f(f^*(x) + f(y)) &= f(1 - (x)_2 + y + (y)_2 - 1) = y + (y)_2 - (x)_2 + (y + (y)_2 - (x)_2)_2 \\ &= y + (y)_2 - (x)_2 + (y + y - x)_2 = y + (y)_2 - (x)_2 + (-x)_2 = y + (y)_2 = f(y), \end{aligned}$$

but it is not multiplicative symmetric ($f(f(1) + 2) = 3 \neq f(1 + f(2)) = 1$).

The following lemma shows that every decomposer type function in groups gives us the standard decomposer case.

Lemma 2.4. *Let $f : G \rightarrow G$ be a map.*

- (a) *If f is a right semi-strong decomposer, right decomposer or right weak decomposer, then $c \cdot f$ is the same type of decomposer for all $c \in G$.*
- (b) *If f is a right strong decomposer, then $c \cdot f$ is right semi-strong decomposer for all $c \in G$. Moreover, $c \cdot f$ is right strong decomposer if and only if $c \in f^*(G)$.*
- (c) *If f is a type of right decomposer (weak, ordinary, semi-strong or strong), then the function $f_e = f(e)^{-1} \cdot f$ is the standard form of the same type of decomposer.*

Proof. Put $g = c \cdot f$. Since $g^* = f^* \cdot c^{-1}$ then

$$g(g^*(x)g(y)) = cf(f^*(x)f(y)), \quad g(g^*(x)y) = cf(f^*(x)c^{-1}y).$$

Thus, (a) is proved. Now, if f is a right strong decomposer and $c \in f^*(G)$, then $f^*(x)c^{-1} \in f^*(G)$, since $f^*(G) \leq G$ (by the next theorem), and thus, $g(g^*(x)y) = g(y)$. Conversely if $g(g^*(x)y) = g(y)$, then $f^*(x)c^{-1}f^*(y) = f^*(f^*(x)c^{-1}y)$, so $c^{-1}, c \in f^*(G)$. Putting $c = f(e)^{-1}$ in (a), (b) and considering $f_e(e) = e$, $c = f^*(e) \in f^*(G)$, we arrive at (c). $\square$

Note. We do not have the property of the above lemma for (right) separator functions. For if $G = \mathbb{R}$ and $f(x) = |x|$, then f is a separator. But if $c \neq 0$ and $g = c + f$, then

$$g^*(G) \cap g(G) = \begin{cases} \emptyset, & c > 0 \\ [c, -c], & c < 0 \end{cases}$$

So g is not (right) separator.

Now we prove some important relations between decomposer type functions.

Theorem 2.5. *Let $f : G \rightarrow G$ be a map.*

(a) *f is a right strong decomposer $\Rightarrow f$ is a right semi-strong decomposer $\Rightarrow f$ is a right decomposer $\Rightarrow f$ is a right weak decomposer.*

(b) *f is a standard right strong decomposer $\Leftrightarrow f$ is standard right semi-strong decomposer $\Rightarrow f$ is a standard right decomposer $\Rightarrow f$ is a standard right weak decomposer $\Rightarrow f$ is a standard right separator.*

(c) *If f is a right strong decomposer, then f is a right separator, idempotent, $ff^* = f(e)$ and*

$$f^*(e).ff^* = f^*f = e, \quad \langle f(e) \rangle \leq f^*(G) \leq G.$$

(Note that $\langle f(e) \rangle \leq f^*(G)$ means that the cyclic group generated by $f(e)$ is a subgroup of $f^*(G)$.)

(d) *f is a standard right [resp. left] weak decomposer if and only if f and f^* [resp. f and f_*] are idempotents. Therefore, f is standard weak decomposer if and only if $ff^* = f^*f = ff_* = f_*f = e$.*

Proof. Let f be a right strong decomposer. Then

$$f(x) = f(f^*(x)f(x)) = f(f(x)) : \quad \forall x \in G,$$

$$ff^*(x) = f(f^*(x)e) = f(e) : \quad \forall x \in G.$$

So $f^2 = f$, $ff^* = f(e)$, hence

$$f^*(e).ff^* = f(e)^{-1}.ff^* = e = f^*f.$$

Now we have

$$f(f^*(x)f(y)) = f(f(y)) = f(y) : \forall x, y \in G,$$

this means f is a decomposer.

On the other hand, a simple calculation shows that f is a right strong decomposer if and only if f satisfies the following equation.

$$(2.2) \quad f^*(f^*(x)y) = f^*(x)f^*(y) : \forall x, y \in G.$$

Putting $y = [f^*(x)]^{-1}$ in (2.2) implies

$$(2.3) \quad f^*(x)^{-1} = f^*(f^*(x)^{-1})f(e) : \forall x \in G.$$

Also putting $x = e$, $y = f(e)^2$ in the right strong decomposer equation implies $f(f(e)^2) = f(e)$ thus

$$(2.4) \quad f(e) = f^*(f(e)^2) = f(f(e)^2)$$

The relations (2.2), (2.3) and (2.4) imply $f^*(G) \leq G$. Also (2.4) implies $f(e) \in f^*(G) \cap f(G) \neq \emptyset$. But if $x \in f^*(G) \cap f(G)$, then $x = f(y) = f^*(z)$ for some $y, z \in G$, and thus

$$x = f(y) = f^2(y) = ff^*(z) = f(e),$$

so f is a right separator.

On the other hand, if f is a standard right weak decomposer, then f and f^* are idempotents (clearly), and so $ff^* = f^*f = e$ thus $e \in f^*(G) \cap f(G)$ and $f^*(G) \cap f(G) = \{e\}$.

Now if f is a right semi-strong decomposer, then

$$f(f^*(x)f(y)) = f(f^*(e)f(y)) = f(f^*(y)f(y)) = f(y).$$

So, f is a right decomposer. The proof of the remaining parts is similar, employing.

□

The followings are some examples of decomposer type functions on groups. Also, these examples show that, in general, a separator (resp. weak decomposer) function may be not weak decomposer (resp. decomposer) and so on.

(i) Right weak decomposer $\nRightarrow$ right separator.

The real function

$$f(x) = \begin{cases} x - 1, & x \geq 0 \\ -1, & x < 0 \end{cases}$$

is (right) weak decomposer, but it is not (right) separator.

Of course "(right) separator $\nRightarrow$ (right) weak decomposer". Consider

$$f(x) = \begin{cases} x - 1, & x \geq 2 \\ 1, & x < 2 \end{cases}$$

(ii) Right decomposer $\nRightarrow$ right separator.

Put $G = \{1, a, a^2, b, ab, ba\} \cong S_3$ ($a^3 = b^2 = 1$, $bab = a^{-1} = a^2$). Define a map $f : G \rightarrow G$ by $f(1) = f(ab) = f(b) = a^2$, $f(a) = f(ba) = f(a^2) = ab$. Then f is a right decomposer but it is not a right separator (since $f^*(G) \cap f(G) = \emptyset$).

(iii) Right weak decomposer $\nRightarrow$ right decomposer.

Consider, e.g., the first real function in part (i).

(iv) Right semi-strong decomposer $\nRightarrow$ right strong decomposer.

Consider the real function $f = () + \frac{1}{2}$.

(v) Right decomposer $\nRightarrow$ right semi-strong decomposer.

Consider the function f in part (ii) ($f(f^*(ab)1) = a^2 \neq f(f^*(1)1) = ab$).

All the above examples are non-standard. Now, we consider some standard examples.

(vi) In the additive real number group, $|x|$ is standard separator (and idempotent) but it is not (standard) weak decomposer. The function $f(x) = \max\{x, 0\}$ is standard weak decomposer, but it is not (standard) decomposer. Finally, $[]_b$ is standard decomposer, but it is not (standard) strong decomposer.

3. CHARACTERIZATION OF TWO-SIDED DECOMPOSER FUNCTIONS IN ARBITRARY GROUPS

If Δ and Ω are subsets of G , then one can define a product of group subsets in a natural way ($\Delta\Omega = \{\delta\omega : \delta \in \Delta \text{ and } \omega \in \Omega\}$). (Note that Δ and Ω need not be subgroups. The associativity of this product follows from that of the group product. The product $\Delta\Omega$ is called *direct*, and denoted by $\Delta \cdot \Omega$, if $\delta_1\omega_1 = \delta_2\omega_2$, where $\delta_1, \delta_2 \in \Delta$, $\omega_1, \omega_2 \in \Omega$, then $\delta_1 = \delta_2$ and $\omega_1 = \omega_2$. If $A = \Delta \cdot \Omega$, then we say that A is a *direct product of (subsets) Δ and Ω* . By the notation $A = \Delta : \Omega$, we mean $A = \Delta \cdot \Omega$ and $\Delta \cap \Omega = \{e\}$ and say that A is a *standard direct product of Δ and Ω* . If $A = \Delta \cdot \Omega$, then Δ [resp. Ω] is called *left [resp. right] factor* of A . Note that additive notations are $\Delta \dot{+} \Omega$ (direct sum of subsets) and $\Delta \ddot{+} \Omega$ (standard direct sum of subsets).

Clearly if $\Delta\Omega = \Delta \cdot \Omega$, then $|\Delta\Omega| = |\Delta||\Omega| = |\Omega\Delta|$. Also if Δ and Ω are non-empty subsets of G , then $\Delta\Omega = \Delta \cdot \Omega$ if and only if $(\Delta^{-1}\Delta) \cap (\Omega\Omega^{-1}) = \{e\}$ (in additive notation $(\Delta - \Delta) \cap (\Omega - \Omega) = \{0\}$). Moreover, if Δ and Ω are finite, then $\Delta\Omega = \Delta \cdot \Omega$ if and only if $|\Delta\Omega| = |\Delta||\Omega|$. If $\Delta\Omega = \Delta \cdot \Omega$ and $\Delta \cap \Omega$ has an element that commutes with every element of $\Delta \cap \Omega$, then $|\Delta \cap \Omega| = 1$. Especially

if $\Delta\Omega = \Delta \cdot \Omega$ and $e \in \Delta \cap \Omega$, then $\Delta\Omega = \Delta : \Omega$. If $G = \Delta \cdot \Omega$, then $G = \Delta_e : \Omega_e$, where $\Delta_e = \Delta\delta_0^{-1}$, $\Omega_e = \omega_0^{-1}\Omega$ and $e = \delta_0\omega_0$. For more properties of factorization of groups by subsets see [8].

Example 3.1. Consider the additive real number group $\mathbb{R}$ and put $\mathbb{R}_b = b[0, 1) = \{bd | 0 \leq d < 1\}$, $\langle b \rangle = b\mathbb{Z}$. We have $\mathbb{R} = \langle b \rangle \dot{+} \mathbb{R}_b$ (see [3]). The natural numbers set is not a factor (subset) of $\mathbb{R}$. Also, $S_3 = \langle \sigma \rangle : \langle \tau \rangle$, where σ and τ are the elements of order two and three, respectively, but S_3 is not decomposable by its non-trivial (normal) subgroups (note that $\langle \sigma \rangle$ is not normal in S_3).

Projections. Let $G = \Delta \cdot \Omega$. Define the functions P_Δ, P_Ω from G to G , by $P_\Delta(x) = \delta$, $P_\Omega(x) = \omega$, where $x = \delta\omega, \delta \in \Delta, \omega \in \Omega$. Clearly, they are well-defined and $P_\Delta(G) = \Delta$, $P_\Omega(G) = \Omega$, $P_\Omega^* = P_\Delta$. We call P_Ω , (resp. P_Δ) *right (resp. left) projection*. For example, the b -parts functions are projections of $\mathbb{R} = \langle b \rangle \dot{+} \mathbb{R}_b$. Now we are ready to solve (two-sided) decomposer equations (introduced in [3]) in arbitrary groups.

Theorem 3.2. *In every group G , general solution of the (two-sided) decomposer equation is*

$$f = P_\Omega : \text{ for all factorizations } G = \Delta \cdot \Omega = \Omega \cdot \Lambda \text{ with } \Delta\omega = \omega\Lambda \text{ for all } \omega \in \Omega.$$

Also, a general solution of the standard decomposer equation in G is of the form

$$f = P_\Omega : \text{ for all factorizations } G = \Delta : \Omega \text{ (or } G = \Omega : \Delta) \text{ with } \Omega \subseteq N_G(\Delta),$$

where $N_G(\Delta)$ denotes the normalizer of the subset Δ in G .

Proof. Putting $\Omega = f(G)$, $\Delta = f^*(G)$ and $\Lambda = f_*(G)$, we have $G = \Delta \cdot \Omega = \Omega \cdot \Lambda$, by Corollary 2.2 of [3]. If $\delta \in \Delta$ and $\omega \in \Omega$, then there exist $x, y \in G$ such that $\delta = f^*(x)$, $\omega = f(y)$ and

$$\begin{aligned} \delta\omega &= f^*(x)f(y) = f(f^*(x)f(y))f_*(f^*(x)f(y)) \\ &= f(y)f_*(f^*(x)f(y)) = \omega f_*(f^*(x)f(y)) \in \omega\Lambda. \end{aligned}$$

Therefore, $\Delta\omega \subseteq \omega\Lambda$. Also, the identity

$$f(y)f_*(x) = f^*(f(y)f_*(x))f(y),$$

implies $\omega\Lambda \subseteq \Delta\omega$. Denote the projections of Ω with respect to the direct factorization $G = \Delta \cdot \Omega$ [resp. $G = \Omega \cdot \Lambda$] by P_Ω^r [resp. P_Ω^ℓ]. We claim that $P_\Omega^\ell = P_\Omega^r = f$. For if $x \in G$, then $x = \delta\omega_1 = \omega_2\lambda$ for some $\omega_1, \omega_2 \in \Omega$, $\delta \in \Delta$ and $\lambda \in \Lambda$, thus there exists $\lambda' \in \Lambda$ such that $\omega_2\lambda = x = \delta\omega_1 = \omega_1\lambda'$ so $P_\Omega^\ell(x) = \omega_2 = \omega_1 = P_\Omega^r(x)$ and we may denote the common value by $P_\Omega(x)$ that is equal to $f(x)$.

For the converse, first note that the mentioned form implies $P_\Omega^\ell = P_\Omega^r = P_\Omega(x)$ so, there is no any risk of confusion in $f = P_\Omega$. Then, we have $f^* = P_\Delta$ and $f_* = P_\Lambda$

and it is not difficult to see that f is a decomposer.

If $f(1) = 1$, then we conclude that $\Delta = \Lambda$ in the first part, and we arrive at the second part. $\square$

4. CONCLUSION

A main result of the study is that general solution of every right [resp. left, two-sided] decomposer equation (in a group) is all right [resp. left, two-sided] projections with respect to a direct decomposition by two subsets. For the two-sided case, Theorem 3.2 states that a decomposer function f is right and also left projection with respect to a direct decompositions $G = \Delta \cdot \Omega$ and $\Omega \cdot \Lambda$ (respectively). For the standard case, we have $\Delta = \Omega$. But, in general, the following open problem is still reminded.

Unsolved problem. Let $f : G \rightarrow G$ be a non-standard (two-sided) decomposer function and Δ, Ω be as mentioned in Theorem 3.2. Does $\Delta = \Omega$?

For further developments of the topic, one may consider the mentioned functional equations on semigroups, monoids, loops, etc. Many related results can be seen in [3,2].

Finally, we emphasize that the methods of study (in [3] and this paper) is usable for studying such equations on more general algebraic structures than groups. Hence, we can use them for further developments of the topic, e.g., every multiplicative symmetric equation is two-sided decomposer (see the question before Lemma 2.4) and so its general solution is a sub-class of subset-projections (as an application of the topic for future studies of such equations).

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REFERENCES

- [1] N. Brillouët-Belluot, *Multiplicative symmetry and related functional equations*, Aequationes Math., **51**(1996), 21-47.
- [2] J. G. Dhombres, *Solution gnrale sur un groupe ablien de l'equation fonctionnelle $f(x * f(y)) = f(f(x) * y)$* , Aequationes Math., **15**(1977), 173-193.
- [3] M. H. Hooshmand and H. K. Haili, *Decomposer and associative functional equations*, Indag. Math., New Ser. **18**(2007), no. 4, 539-554.
- [4] M. H. Hooshmand, *Grouplikes*, Bull. Iran. Math. Soc., **39**(2013), no. 1, 65-86.
- [5] M. H. Hooshmand and H. K. Haili, *Some algebraic properties of b-Parts of real numbers*, Šiauliai Math. Semin., **3**(2008), no. 11, 115-121.
- [6] M. H. Hooshmand, *f-Grouplikes*, submitted.
- [7] M. H. Hooshmand and H. K. Haili, *f-Multiplication of binary operations and decompositional groups*, in: Proceedings of the 20th Seminar on Algebra 2009, Karaj, Iran, 95-98.

- [8] S. Szabó and A.D. Sands, *Factoring Groups into Subsets*, CRC Press, Taylor and Francis Group, Boca Raton, 2009.

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