

Periodic travelling wave solutions of ZK(2,4,-2) equation *

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Abstract

In this paper, the qualitative analysis methods of a dynamical system are used to investigate the periodic travelling wave solutions of ZK(2, 4,-2) equation. The phase portrait bifurcation of the traveling wave system corresponding to the equation is given. The explicit expressions of the periodic travelling wave solutions are obtained by using the portraits. The graph of the solutions are given with the numerical simulation.

Key Words: Periodic travelling wave solutions, ZK(2,4,-2) equation, bifurcation method.

1 Introduction

The study of travelling wave solutions in particular, solitons, of Partial Differential Equations (what called PDEs), for various nonlinear evolution equations in mathematical physics plays an important role in soliton theory. To obtain the travelling wave solutions for PDEs, a lot of systematic methods have been developed for soliton equations, such as the inverse scattering method, the Bäcklund and the Darboux transformations, the

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tanh-function method, the homogeneous balance method, the extended tanh-function method and others [1-7].

Recently, the ZK equation given

$$u_t + auu_x + b(u_{xx} + u_{yy})_x = 0, \quad (1.1)$$

is investigated by BK. Shivamoggi[8], A. M. Wazwaz[9] and VE. Zakharov etc[10] with various distinct approaches.

More recently, by using the sine-cosine method and the tanh method, A. M. Wazwaz [11] investigated the ZK(n,-n,2n) equation,

$$u_t + a(u^n)_x + [bu^{-n}(u^{2n})_{xx} + k(u^n)_{yy}]_x = 0, \quad (1.2)$$

and the ZK(n,2n,-n) equation,

$$u_t + a(u^n)_x + [bu^{2n}(u^{-n})_{xx} + k(u^n)_{yy}]_x = 0, \quad (1.3)$$

where a , b , k are three non-zero real numbers, which obtained a family of solitary and compact patterns solutions:

$$u_1 = \begin{cases} \frac{2nc(b+k)}{a[b(3n-1)+k(n+1)]} \sin^2 \left[\frac{n-1}{2n} \sqrt{\frac{a}{b+k}} (x + y - ct) \right]^{\frac{1}{n-1}}, & |\mu\xi| < \pi, \\ 0, & \text{otherwise,} \end{cases}$$

$$u_2 = \begin{cases} \frac{2nc(b+k)}{a[b(3n-1)+k(n+1)]} \cos^2 \left[\frac{n-1}{2n} \sqrt{\frac{a}{b+k}} (x + y - ct) \right]^{\frac{1}{n-1}}, & |\mu\xi| < \frac{\pi}{2}, \\ 0, & \text{otherwise,} \end{cases}$$

other exact explicit solutions were listed in [11]. However, the bifurcation behavior of the travelling wave solutions for corresponding travelling wave equations haven't studied in its parameter space. It is very important to understand the dynamical behavior of solutions for the travelling wave equation of (1.2) and (1.3). In the present paper, we shall continue to study and obtain the periodic travelling wave solutions of Eq.(1.3) for $n = 2$ (simply called ZK(2,4,-2) equation), which employ bifurcation method of dynamical systems [12,13] to investigate the following equation:

$$u_t + a(u^2)_x + [bu^4(u^{-2})_{xx} + ku_{yy}^2]_x = 0 \quad (1.4)$$

Taking the transformation $u(x, t) = \phi(x + y - ct) = \phi(\xi)$, where c is the wave speed, then (1.4) becomes

$$-c\phi' + a(\phi^2)' + [b\phi^4(\phi^{-2})'' + k(\phi^2)'''] = 0, \quad (1.5)$$

where “ $\prime$ ” is the derivative with respect to ξ . Taking the integration once on both sides leads to

$$-c\phi + a\phi^2 + [b\phi^4(\phi^{-2})'' + k(\phi^2)''] = g, \quad (1.6)$$

where g is integration constant.

Clearly, (1.6) is equivalent to the following two-dimensional system

$$\frac{d\phi}{d\xi} = y, \quad \frac{dy}{d\xi} = \frac{g + c\phi - a\phi^2 - py^2}{q\phi}, \quad (1.7)$$

where $p = 6b + 2k$, $q = 2k - 2b$, which has the first integral

$$y^2 = \phi^{-\frac{2p}{q}} \left(\frac{g}{p} \phi^{\frac{2p}{q}} + \frac{2c}{2p+q} \phi^{\frac{2p}{q}+1} - \frac{a}{p+q} \phi^{\frac{2p}{q}+2} + h \right) \quad (1.8)$$

or

$$H(\phi, y) = \phi^{\frac{2p}{q}} y^2 - \phi^{\frac{2p}{q}} \left(\frac{g}{p} + \frac{2c}{2p+q} \phi - \frac{a}{p+q} \phi^2 \right) = h. \quad (1.9)$$

Obviously system (1.7) is a five-parameter planar dynamical system depending on the parameter group (g, c, a, p, q) . Since the phase orbits defined by the vector fields of Eq. (1.7) determine all traveling wave solutions of system (1.3), we should investigate the bifurcations of phase portraits of system (1.7) in the phase plane (ϕ, y) as the parameters g, c, a, p and q are changed.

The rest of this paper is organized as follows: in Section 2, we discuss the bifurcations of phase portraits of system (1.7), where explicit parametric conditions will be derived. In Section 3, we give exact explicit parametric representations for periodic solutions of Eq. (1.4) for $n = 2$. Section 4 contains the concluding remarks.

2 Bifurcations and phase portraits of system (1.7)

In this section, we discuss the existence of periodic solutions for (1.3) by the bifurcation method [13,14].

System (1.7) have the same phase orbits as the following system,

$$\frac{d\phi}{d\tau} = q\phi y, \quad \frac{dy}{d\tau} = g + c\phi - a\phi^2 - py^2, \quad (2.1)$$

except for the straight line $\phi = 0$, where $d\xi = q\phi d\tau$.

When $pg > 0, c^2 + 4ag > 0$, system (1.7) has four equilibrium points: $A_1(\phi_1, 0)$, $A_2(\phi_2, 0)$ and $S_{\pm}(0, \pm Y)$, where $\phi_1 = \frac{-c + \sqrt{c^2 + 4ag}}{-2a}$, $\phi_2 = \frac{c + \sqrt{c^2 + 4ag}}{2a}$ and $Y = \sqrt{\frac{g}{p}}$.

For the function defined by (1.9), we denote that

$$h_i = H(\phi_i, 0) = -(\phi_i)^{\frac{2p}{q}} \left(\frac{g}{p} + \frac{2c}{2p+q}\phi_i - \frac{a}{p+q}\phi_i^2 \right), (i = 1, 2), h_s = H(0, \pm Y) = 0.$$

If $\frac{g}{p} + \frac{2c}{2p+q}\phi_1 - \frac{a}{p+q}\phi_1^2 = 0$, i.e., $ag = \frac{2p}{p+q}c^2$, we have $H(\phi_1, 0) = H(0, \pm Y) = 0$.

Let $M(\phi_i, y_i)$ be the coefficient matrix of the linearized system of (2.1) at an equilibrium point (ϕ_i, y_i) , $J(\phi_i, y_i)$ is the corresponding Jacobi determinant of the $M(\phi_i, y_i)$. Then, if $p > 0, qg > 0, ag > 0, ag = \frac{2p}{p+q}c^2$, we have

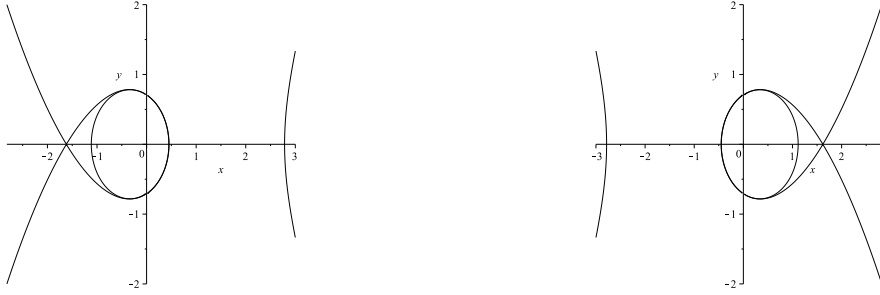
$$J(\phi_1, 0) = -\frac{q\sqrt{c^2 + 4ag}}{2a}(c + \sqrt{c^2 + 4ag}) < 0, \quad J(0, \pm Y) = 2pqY^2 > 0,$$

$$J(\phi_2, 0) = -\frac{q\sqrt{c^2 + 4ag}}{2a}(c - \sqrt{c^2 + 4ag}) > 0, \quad \text{Trace}(M(\phi_1, 0)) = 0,$$

By the theory of planar dynamical systems [12,13], we know that for an equilibrium point (ϕ_i, y_i) of a planar integrable system, if $J < 0$ then the equilibrium point is a saddle point; if $J > 0$ and $\text{Trace}(M(\phi_i, y_i)) = 0$ then it is a center point; if $J > 0$ and $(\text{Trace}(M(\phi_i, y_i)))^2 - 4J(\phi_i, y_i) > 0$ then it is a node; if $J = 0$ and the Poincaré index of the equilibrium point is zero then it is a cusp; if $J = 0$ and the index of the equilibrium point is not zero then it is a high order equilibrium point. Using the above qualitative analysis, we can obtain the bifurcation curves and phase portraits under various parameter conditions.

Thus, the equilibrium point A_1 is a saddle point, the equilibrium points $S_{\pm}$ and A_2 are center points.

For our purpose, in the parameter region: $pg > 0, c^2 + 4ag > 0, ag = \frac{2p}{p+q}c^2$, we show the phase portraits of system (1.7) in Fig. 1.



(1-1) $a < 0, q > 0, p < 0, g < 0, c > 0$. (1-2) $a < 0, q > 0, p < 0, g < 0, c < 0$.

Fig. 1 The phase portraits of system (1.7) for $pg > 0, c^2 + 4ag > 0, ag = \frac{2p}{p+q}c^2, p \geq -2q$.

3 Exact explicit periodic solutions of (1.4)

Lemma 3.1. When $h \rightarrow h_1$, the periodic orbits of the periodic annulus surrounding $(\phi_1, 0)$ approach to the boundary curves. Let $(\phi, y = \phi')$ be a point in the periodic orbits of system (1.6). Then, the periodic travelling wave solution is defined by $h = h_1$.

In the following, we give exact explicit parametric representations of periodic solutions.

Case (I). $a < 0, q > 0, p < 0, g < 0, c > 0, ag = \frac{2p}{p+q}c^2$.

In this case, we have $\phi_1 = \frac{c}{2a}\sqrt{1 + \frac{3p+q}{p+q}}$ and the phase portrait of system (1.7) shown in Fig. 1(1-1). Notice that $H(A_1) = H(S_{\pm}) = 0$, periodic orbit surrounding the center point $A_1\left(\frac{c}{2a}\sqrt{1 + \frac{3p+q}{p+q}}, 0\right)$ are

$$y^2 = \frac{a}{p+q} \left[-\left(\phi - \frac{c(p+q)}{a(2p+q)} \right)^2 + \frac{c^2}{a^2} \left(\frac{(p+q)^2}{(2p+q)^2} + 2 \right) \right]. \quad (3.1)$$

Substituting (3.1) into the first equation of (1.7) and integrating it, we get

$$d\xi = \frac{d\phi}{\sqrt{\frac{a}{p+q} \left[D^2 - \left(\phi - \frac{c(p+q)}{a(2p+q)} \right)^2 \right]}}$$

that is

$$\xi = \sqrt{\frac{p+q}{a}} \int_{\phi_1}^{\phi} \frac{d\phi}{\sqrt{D^2 - (\phi + B)^2}}, \quad (3.2)$$

where $D^2 = \frac{c^2}{a^2} \left(\frac{(p+q)^2}{(2p+q)^2} + 2 \right)$, $B = -\frac{c(p+q)}{a(2p+q)}$, which implies the following parametric

representations:

$$\phi(\xi) = -\frac{c}{2a} \left(1 + \sqrt{\frac{3p+q}{p+q}} \right) = D \sin \left(\sqrt{\frac{a}{p+q}} \xi \right). \quad (3.3)$$

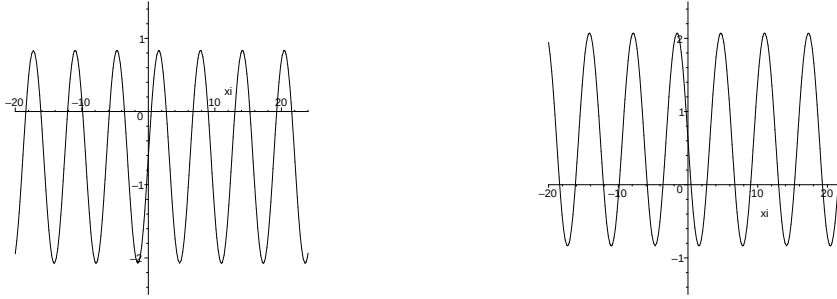
(3.3) gives a **periodic** solution and the profile is shown in Fig. 2(2-1).

Remark. To the best of our knowledge, the solution (3.3) of Eq. (1.4) has not been reported in literature.

Case (II). $a < 0$, $q > 0$, $p < 0$, $g < 0$, $c < 0$, $ag = \frac{2p}{p+q}c^2$.

In this case, we have the phase portrait of system (1.7) shown in Fig. 1(1-2). **Paralleled** to the Cases (I), system (1.7) has a parametric representation of the **periodic** orbit as (3.3).

It gives another **periodic** solution and the profile is shown in Fig. 2(2-2).



(2-1) $a = g = -1$, $c = q = 1$, $p = -2$. (2-2) $a = g = c = -1$, $q = 1$, $p = -2$.

Fig. 2 **Periodic** solutions of Eq.(1.1) for $pg > 0$, $c^2 + 4ag > 0$, $ag = \frac{2pc^2}{p+q}$, $p \geq -2q$.

4 Discussion

In this paper, we used the qualitative analysis methods of a dynamical system to investigate the **periodic** solutions of ZK(2, 4,-2,) equation. As a result, we obtained two of new exact **periodic** solutions. The phase portrait bifurcation of the **travelling** wave system corresponding to the equation is given. The graph of the solutions are given with the numerical simulation. **However, what shall to solve the solitary wave solutions of ZK(2, 4,-2,) equation by the paralleled method, it is our further study.**

References

- [1] M.J. Ablowitz, H. Segur, Solitons and the Inverse Scattering Transform, SIAM, Philadelphia, 1981.
- [2] V.B. Matveev, M.A. Salle, Darboux Transformations and Solitons, Springer, Berlin, 1991.
- [3] C.H. Gu, Soliton Theory and its Application, Springer, Berlin, 1995.
- [4] E.G. Fan, Extended tanh-function method and its applications to nonlinear equations, Phys. Lett. A, **277** (2000) 212-218.
- [5] M.L. Wang, Exact solutions for a compound KdV-Burgers equation, Phys. Lett. A, **213** (1996) 279-287.
- [6] C.W. Cao, Y.T. Wu, X.G. Geng, Relation between the Kadomtsev-Petviashvili equation and the Confocal involutive system, J. Math. Phys., **40** (1999) 3948-3970.
- [7] M.L. Wang, Y.B. Zhou, Z.B. Li, Application of a homogeneous balance method to exact solutions of nonlinear equations in mathematical physics, Phys. Lett. A, **216** (1996) 67-75.
- [8] BK. Shivamoggi. The Painlevé analysis of the Zakharov-Kuznetsov equation. Phys. Scripta., **42** (1990), 641-642.
- [9] A.M. Wazwaz Exact solutions with solitons and periodic structures for the Zakharov-Kuznetsov (ZK) equation and its modified form. Commun Nonlinear Sci. Numer. Simul., **10** (2005), 597-606.
- [10] VE. Zakharov and EA. Kuznetsov, On three-dimensional solitons, Sov. Phys., **39** (1974), 285-288.
- [11] A.M.Wazwaz, Explicit travelling wave solutions of variants of the K(n,n) and ZK(n,n) equations with compact and noncompact structures, Appl. Math. Comput., **173** (2006), 213-230.

- [12] S. Tang, W. Huang. Bifurcations of travelling wave solutions for the $K(n, -n, 2n)$ equations. *Appl. Math. Comput.*, **203** (2008), 39-49.
- [13] L. Qiao, H. Fu, S. Tang. PEAKON SOLITON SOLUTIONS OF $K(2, -2, 4)$ EQUATION. *J. Appl. Anal. Comput.*, **4** (2013), 399-403.
- [14] J. Li, Exact Explicit Peakon and Periodic Cusp Wave Solutions for Several Nonlinear Wave Equations. *J. Diff. Eqns.*, **20**(2008), 909-922.
- [15] J. Li, H.H. Dai. On the Study of Singular Nonlinear Traveling Wave Equations: Dynamical System Approach. Beijing: Science Press, 2007 (in English).
- [16] M.J. Ablowitz, P.A. Clarkson, Solitons, Nonlinear Evolution Equations and Inverse Scattering, Cambridge University Press, Cambridge, 1991.
- [17] G. Wang, T. Xu etc., Solitons and Lie group analysis to an extended quantum Zakharov-Kuznetsov equation, *Astrophysics and Space Science*, **349**(2014), 317-327.