

Co- $\ast^n$ -tuple of Contravariant Functors

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Abstract: In this work we generalize the concept of Co- $\ast^n$ -modules to the concept of Co- $\ast^n$ -tuple of Contravariant Functors between abelian categories.

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1 Introduction

For a unital associative ring A , a fixed right A -module M , and $D = \text{End}_A(M)$, let $\text{fgd-tl}(M_A)$ denote the class of all torsionless right A -modules whose M -dual are finitely generated over D and $\text{fg-tl}({}_D M)$ denote the class of all finitely generated torsionless left D -modules. M is called costar module if

$$\text{Hom}_A(-, M) : \text{fgd-tl}(M_A) \rightleftarrows \text{fg-tl}({}_D M) : \text{Hom}_D(-, M)$$

is a duality. Costar modules were introduced by Colby and Fuller in [5]. M is said to be an r-costar module provided that any exact sequence

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

such that X and Y are M -reflexive, remains exact after applying the functor $\text{Hom}_A(-, M)$ if and only if Z is M -reflexive. The notion of r-costar module was introduced by Liu and Zhang in [6]. We say that a right A -module X is n -finitely M -copresented if there exists a long exact sequence

$$0 \longrightarrow X \longrightarrow M^{k_0} \longrightarrow M^{k_1} \longrightarrow \dots \longrightarrow M^{k_{n-1}}$$

such that n is a positive integer and k_i are positive integers for $0 \leq i \leq n-1$. The class of all n -finitely M -copresented modules is denoted by $n\text{-cop}(M)$. We say that a right A -module M is a finitistic n -self-cotilting module provided that $n\text{-cop}(M) = (n+1)\text{-cop}(M)$ and for any exact sequence

$$0 \longrightarrow X \longrightarrow M^n \longrightarrow Z \longrightarrow 0$$

such that $Z \in n\text{-cop}(M)$ and n is a positive integer, remains exact after applying the functor $\text{Hom}_A(-, M)$. Finitistic n -self-cotilting modules were introduced by Breaz in [2]. In [8], L. Yao and J. Chen introduced the concept of $\text{co-}*\text{-}^n$ -modules, which is a generalization of finitistic n -self-cotilting modules to the infinite case, i.e. we say that a right A -module M is a $\text{co-}*\text{-}^n$ -module provided that $n\text{-cop}(M) = (n+1)\text{-cop}(M)$ and for any exact sequence

$$0 \longrightarrow X \longrightarrow M^I \longrightarrow Z \longrightarrow 0$$

such that $Z \in n\text{-cop}(M)$ and I is a set, remains exact after applying the functor $\text{Hom}_A(-, M)$.

In [4] Castaño-Iglesias generalizes the notion of costar module to Grothendieck categories. Pop in [7] generalizes the notion of finitistic n -self-cotilting module to finitistic n - F -cotilting object in abelian categories and he describes a family of dualities between some special abelian categories. Breaz and Pop in [3] generalize a duality exhibited in [2, Theorem 2.8] to abelian categories.

In [1], the author generalizes the notion of r -costar module to r -costar pair of contravariant functors between abelian categories, by generalizing the work in [6]. In this paper we generalize the work in [8] by generalizing the notion of $\text{Co-}*\text{-}^n$ -modules to a $\text{Co-}*\text{-}^n$ -tuple of contravariant functors between abelian categories. We use the same technique of proofs as in [8].

2 Preliminaries

Let $F : \mathfrak{C} \longrightarrow \mathfrak{D}$ and $G : \mathfrak{D} \longrightarrow \mathfrak{C}$ be additive and contravariant functors between two abelian categories $\mathfrak{C}$ and $\mathfrak{D}$. It is said that they are adjoint on the right if there are natural isomorphisms

$$\eta_{X,Y} : \text{Hom}_{\mathfrak{C}}(X, G(Y)) \longrightarrow \text{Hom}_{\mathfrak{D}}(Y, F(X))$$

for all $X \in \mathfrak{C}$ and all $Y \in \mathfrak{D}$. Then they induce two natural transformations $\delta : 1_{\mathfrak{C}} \longrightarrow GF$ and $\delta' : 1_{\mathfrak{D}} \longrightarrow FG$ defined by $\delta_X = \eta_{X, F(X)}^{-1}(1_{F(X)})$ and $\delta'_Y = \eta_{G(Y), Y}(1_{G(Y)})$. Moreover the following identities are satisfied for each $X \in \mathfrak{C}$ and $Y \in \mathfrak{D}$.

$$F(\delta_X) \circ \delta'_{F(X)} = 1_{F(X)} \text{ and } G(\delta'_Y) \circ \delta_{G(Y)} = 1_{G(Y)}.$$

F and G are **left** exact, since they are adjoint on the right. The pair (F, G) is called a duality if there are functorial isomorphisms $GF \simeq 1_{\mathfrak{C}}$ and $FG \simeq 1_{\mathfrak{D}}$. An object X of $\mathfrak{C}$ (respectively Y of $\mathfrak{D}$) is called F -reflexive (respectively, G -reflexive) in case δ_X (respectively, δ'_Y) is an isomorphism. An object X of $\mathfrak{C}$ (respectively Y of $\mathfrak{D}$) is called F -torsionless (respectively, G -torsionless) in

case δ_x (respectively, δ'_y) is a monomorphism. By $\text{Ref}(F)$ we will denote the full subcategory of all F -reflexive objects. As well by $\text{Ref}(G)$ we will denote the full subcategory of all G -reflexive objects. It is clear that the functors F and G induce a duality between the categories $\text{Ref}(F)$ and $\text{Ref}(G)$.

Let U be an object in $\mathfrak{C}$. For an object X in an abelian category $\mathfrak{C}$, we say

that X is U -generated if there is an exact sequence

$$U^{(I)} \longrightarrow X \longrightarrow 0,$$

where I is an index set and if there is an exact sequence

$$U^{(I_2)} \longrightarrow U^{(I_1)} \longrightarrow X \longrightarrow 0,$$

where each I_i is an index set, then X is said to be U -presented. We say that X is U -cogenerated if there is an exact sequence

$$0 \longrightarrow X \longrightarrow U^I,$$

where I is an index set and if there is an exact sequence

$$0 \longrightarrow X \longrightarrow U^{I_1} \longrightarrow U^{I_2},$$

where each I_i is an index set, then X is said to be U -copresented. We say that X is n - U -copresented if there is an exact sequence

$$0 \longrightarrow X \longrightarrow U^{I_0} \longrightarrow U^{I_1} \longrightarrow \dots \longrightarrow U^{I_{n-2}} \longrightarrow U^{I_{n-1}},$$

where each I_i is an index set and n is a positive integer. We denote by $\text{Cogen}(U)$, $\text{Copres}(U)$ and $n\text{-Copres}(U)$ the classes of all U -cogenerated, U -copresented and n - U -copresented objects respectively. It is clear that $(n+1)\text{-Copres}(U) \subseteq n\text{-Copres}(U)$, for every positive integer n .

An object U in $\mathfrak{C}$ is called co- F -small if for any set I , there is a canonical

isomorphism $F(U^I) \cong F(U)^{(I)}$. The object U is called n - F -quasi-injective if for any exact sequence

$$0 \longrightarrow X \longrightarrow U^I \longrightarrow Y \longrightarrow 0,$$

where $Y \in n\text{-Copres}(U)$, the sequence,

$$0 \longrightarrow F(Y) \longrightarrow F(U^I) \longrightarrow F(X) \longrightarrow 0,$$

is exact.

Let $F : \mathfrak{C} \longrightarrow \mathfrak{D}$ and $G : \mathfrak{D} \longrightarrow \mathfrak{C}$ be additive and contravariant functors between two abelian categories $\mathfrak{C}$ and $\mathfrak{D}$, such that they are adjoint on the right. Let $U \in \mathfrak{C}$ such that $V = F(U)$ is a projective object in $\mathfrak{D}$. If U is F -reflexive, the tuple (F, G, V, U) is called a Co- $\ast^n$ -tuple, where n is a positive integer, if:

(i) U is co- F -small,

- (ii) $(n+1)$ - F -quasi-injective,
- (iii) n - $Copres(U) = (n+1)$ - $Copres(U)$.

There are two examples of contravariant functors that could satisfy the conditions to be $\text{Co-}\ast^n$ -tuple. The first is the typical example and the second was exhibited by Castano-Iglesias in [4].

Let A be unital associative ring and M a right A -module. Suppose that $D = \text{End}_A(M)$, then it is clear that M is a D - A -bimodule. The contravariant functors $\Delta = \text{Hom}_A(, M) : \text{Mod-}A \longrightarrow D\text{-Mod}$ and $\Delta' = \text{Hom}_D(, M) : D\text{-Mod} \longrightarrow \text{Mod-}A$ are right adjoint.

Let $\mathcal{G}$ be a group. For $\mathcal{G}$ -graded rings $A = \bigoplus_{x \in \mathcal{G}} A_x$ and $B = \bigoplus_{x \in \mathcal{G}} {}_x B$ we will denote by $\text{Mod}_{gr}\text{-}A$ (respectively, by $B\text{-Mod}_{gr}$) the category of all $\mathcal{G}$ -graded unital right A -modules (respectively, left B -modules). For $N, M \in \text{Mod}_{gr}\text{-}A$ we can consider the $\mathcal{G}$ -graded abelian group $\text{HOM}_A(N, M)$, whose x th homogeneous component is

$${}_x \text{HOM}_A(N, M) = \{f \in \text{Hom}_A(N, M) \mid f(N_y) \subseteq M_{xy}, \text{ for all } y \in \mathcal{G}\}.$$

If $N, M \in B\text{-Mod}_{gr}$ we can consider the $\mathcal{G}$ -graded abelian group $\text{HOM}_B(N, M)$, whose x th homogeneous component is

$$\text{HOM}_B(N, M)_x = \{f \in \text{Hom}_B(N, M) \mid f({}_y N) \subseteq {}_{yx} M, \text{ for all } y \in \mathcal{G}\}.$$

If we suppose that $B = \text{HOM}_A(M, M) = \text{END}(M_A)$, then B is a $\mathcal{G}$ -graded ring and M is a graded (B, A) -bimodule. Now we have a pair of contravariant functors

$$H_A^{gr} = \text{HOM}_A(, M) : \text{Mod}_{gr}\text{-}A \rightleftarrows B\text{-Mod}_{gr} : \text{HOM}_B(, M) =_B H^{gr}.$$

If for any object $X \in \mathfrak{C}$ there is a projective object $P \in \mathfrak{C}$ and an epimorphism $P \longrightarrow X \longrightarrow 0$, we say that $\mathfrak{C}$ has enough projectives. From now on we suppose that both $\mathfrak{C}$ and $\mathfrak{D}$ have enough projectives. It is clear that we can construct a projective resolution for any object X . Suppose we have a projective resolution of X in $\mathfrak{D}$.

$$P : \dots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow X \longrightarrow 0.$$

This gives rise to the chain complex

$$0 \longrightarrow G(X) \longrightarrow G(P_0) \longrightarrow G(P_1) \longrightarrow \dots,$$

for which we can compute its homology at the n -th spot (the kernel of the map from $G(P_n)$ modulo the image of the map to $G(P_{n+1})$) and denote it by $H^n(G(P))$. We define $R^n G(X) = H^n(G(P))$ as the n -th right derived functors of G . The

n -th right derived functors of F can be defined similarly. For the functor G and for a positive integer n , we define ${}^{\perp}T_G^{i \geq n} = \{X \in \mathfrak{D} : R^i G(X) = 0 \text{ for every } i \geq n\}$. In the same way one can define ${}^{\perp}T_F^{i \geq n}$.

Let

$$0 \longrightarrow Q \xrightarrow{f} U \longrightarrow V \longrightarrow 0$$

be an exact sequence in $\mathfrak{C}$. Applying the functor F we get the exact sequence

$$0 \longrightarrow F(V) \longrightarrow F(U) \xrightarrow{p} X \longrightarrow 0, \quad (1)$$

where $X = \text{Im}(F(f))$ and $F(f) = j \circ p$ is the canonical decomposition of $F(f)$, where $j : X \longrightarrow F(U)$ is the inclusion map. Applying the functor G to the sequence (1), we have the following exact sequence

$$0 \longrightarrow G(X) \xrightarrow{G(p)} GF(U) \longrightarrow GF(V).$$

Now if we put $\alpha = G(j) \circ \delta_Q$, then

$$G(p) \circ \alpha = G(p) \circ G(j) \circ \delta_Q = G(j \circ p) \circ \delta_Q = GF(f) \circ \delta_Q = \delta_U \circ f.$$

So we have the following commutative diagram with exact rows.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Q & \xrightarrow{f} & U & \longrightarrow & V & \longrightarrow & 0 \\ & & \downarrow \alpha & & \downarrow \delta_U & & \downarrow \delta_V & & \\ 0 & \longrightarrow & G(X) & \xrightarrow{G(p)} & GF(U) & \longrightarrow & GF(V) & & \end{array} \quad (2)$$

Lemma 1 [4, Lemma 1.1, Lemma 1.2] Let $F : \mathfrak{C} \longrightarrow \mathfrak{D}$ and $G : \mathfrak{D} \longrightarrow \mathfrak{C}$ be a pair of contravariant functors as above and assume that U and V are objects of $\mathfrak{C}$ and $\mathfrak{D}$, respectively. Then:

- (i) $F(X) \in \text{Copres}(F(U))$ whenever X is a U -presented object of $\mathfrak{C}$,
- (ii) $G(Y) \in \text{Copres}(G(V))$ whenever Y is a V -presented object of $\mathfrak{D}$.

For the next statements, we assume that U and V are generators of $\mathfrak{C}$ and $\mathfrak{D}$, respectively.

- (iii) X is F -torsionless if and only if $X \in \text{Cogen}(G(V))$, for every $X \in \mathfrak{C}$,
- (iv) Y is G -torsionless if and only if $Y \in \text{Cogen}(F(U))$, for every $Y \in \mathfrak{D}$.

Lemma 2 Let F and G be a pair of contravariant functors as above. Let V be a projective generator in $\mathfrak{D}$ with $G(V) = U$ and let n be a positive integer. For any $Y \in \mathfrak{D}$, if $R^i G(Y) = 0$ for $1 \leq i \leq n$, then $G(Y) \in (n+2)\text{-Copres}(U)$.

3 Co- $\ast^n$ -tuple Pair of Contravariant Functors

Throughout this section, let $F : \mathfrak{C} \rightarrow \mathfrak{D}$ and $G : \mathfrak{D} \rightarrow \mathfrak{C}$ be a pair of additive and contravariant functors which are adjoint on the right, between abelian categories. As well, let U be a F -reflexive object in $\mathfrak{C}$ with $F(U) = V$ be a projective generator in $\mathfrak{D}$. Moreover, we consider a positive integer n .

Proposition 3 *Suppose that (F, G, V, U) is a Co- $\ast^n$ -tuple. Then for any $X \in n\text{-Copres}(U)$, δ_X is an isomorphism and $R^i G(F(X)) = 0$, for every $i \geq 1$.*

Proof. Let $X \in n\text{-Copres}(U)$. It follows that $X \in (n+1)\text{-Copres}(U)$, by assumptions. Hence there is an exact sequence

$$0 \longrightarrow X \longrightarrow U^I \longrightarrow Y \longrightarrow 0,$$

where $Y \in n\text{-Copres}(U)$. Since (F, G, V, U) is a Co- $\ast^n$ -tuple we have the exact sequence

$$0 \longrightarrow F(Y) \longrightarrow F(U^I) \longrightarrow F(X) \longrightarrow 0,$$

after applying the functor F . Applying the functor G to the last sequence and taking into account that V is projective, hence $V^{(I)} = F(U)^{(I)} = F(U^I)$ is also projective, we get an exact sequence

$$0 \longrightarrow GF(X) \longrightarrow GF(U^I) \longrightarrow GF(Y) \longrightarrow R^1 G(F(X)) \longrightarrow 0,$$

and the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \longrightarrow & U^I & \longrightarrow & Y & \longrightarrow & 0 \\ & & \downarrow \delta_X & & \downarrow \delta_{U^I} & & \downarrow \delta_Y & & \\ 0 & \longrightarrow & GF(X) & \longrightarrow & GF(U^I) & \longrightarrow & GF(Y) & \longrightarrow & R^1 G(F(X)) \longrightarrow 0. \end{array}$$

By Lemma 1, δ_Y is a monomorphism. By Snake Lemma, it follows that δ_X is an isomorphism since δ_{U^I} is an isomorphism. Then δ_Y is also an isomorphism by a similar argument. Hence, $R^1 G(F(X)) = 0$, by commutativity of the right square. Since $Y \in n\text{-Copres}(U)$, $R^1 G(F(Y)) = 0$. Then we can get the assertion inductively. ■

Theorem 4 *The following conditions are equivalent*

- (1) (F, G, V, U) is a Co- $\ast^n$ -tuple,
- (2) i) U is co- F -small
- (ii) For any exact sequence $0 \longrightarrow X \longrightarrow U^I \longrightarrow Y \longrightarrow 0$, where $X \in n\text{-Copres}(U)$ and I is a set, it remains exact after applying the functor F if and only if $Y \in n\text{-Copres}(U)$,

Proof. (1) $\Rightarrow$ (2) Suppose that we have an exact sequence $0 \longrightarrow X \longrightarrow U^I \longrightarrow Y \longrightarrow 0$, where $X \in n\text{-Copres}(U)$ and I a set. Assume that $Y \in n\text{-Copres}(U)$. Since (F, G, V, U) is a $\text{Co-}\ast^n$ -tuple, we get the exact sequence

$$0 \longrightarrow F(Y) \longrightarrow F(U^I) \longrightarrow F(X) \longrightarrow 0.$$

Conversely, assume that the sequence

$$0 \longrightarrow F(Y) \longrightarrow F(U^I) \longrightarrow F(X) \longrightarrow 0$$

is exact. Applying the functor G we get the following long exact sequence

$$0 \longrightarrow GF(X) \longrightarrow GF(U^I) \longrightarrow GF(Y) \longrightarrow R^1G(F(X)) \longrightarrow R^1G(F(U^I)) \longrightarrow R^1G(F(Y)) \longrightarrow \dots \quad (3)$$

By Proposition 3, δ_X is an isomorphism and $R^iG(F(X)) = 0$ for any $i \geq 1$. Thus, we have the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \longrightarrow & U^I & \longrightarrow & Y & \longrightarrow & 0 \\ & & \downarrow \delta_X & & \downarrow \delta_{U^I} & & \downarrow \delta_Y & & \\ 0 & \longrightarrow & GF(X) & \longrightarrow & GF(U^I) & \longrightarrow & GF(Y) & \longrightarrow & 0 \end{array}$$

It is clear, by Snake Lemma, that δ_Y is an isomorphism, which means that

$Y \cong GF(Y)$. From the exactness of sequence (3) we conclude that $R^iG(F(Y)) \cong R^iG(F(U^I)) = 0$ for any $i \geq 1$, so by Lemma 2, $Y \cong GF(Y) \in n\text{-Copres}(U)$.

(2) $\Rightarrow$ (1) It is enough to prove $n\text{-Copres}(U) = (n+1)\text{-Copres}(U)$. If $X \in n\text{-Copres}(U)$, then $F(X)$ is V -generated over $\mathfrak{D}$, thus by [4, Lemma 2.2], there exists an exact sequence $0 \longrightarrow X \longrightarrow U^I \longrightarrow Y \longrightarrow 0$, which remains exact after applying the functor F . Then $Y \in n\text{-Copres}(U)$, hence $X \in (n+1)\text{-Copres}(U)$. ■

Proposition 5 Suppose that (F, G, V, U) be a $\text{Co-}\ast^n$ -tuple. Then G is an exact functor in $F(n\text{-Copres}(U))$. Moreover $F(n\text{-Copres}(U)) = {}^\perp T_G^{i \geq 1}$.

Proof. By Proposition 3 we have $F(n\text{-Copres}(U)) \subseteq {}^\perp T_G^{i \geq 1}$ and G is an exact functor in $F(n\text{-Copres}(U))$. Conversely, for any $X \in {}^\perp T_G^{i \geq 1}$, by Lemma 2, $G(X) \in n\text{-Copres}(U)$. Since V is a generator in $\mathfrak{D}$, there is an exact sequence $0 \longrightarrow Y \xrightarrow{f} V^{(I)} \xrightarrow{g} X \longrightarrow 0$, where I is a set. If we apply the functor G we get the long exact sequence

$$0 \longrightarrow G(X) \longrightarrow G(V^{(I)}) \longrightarrow G(Y) \longrightarrow R^1G((X)) \longrightarrow R^1G((V^{(I)})) \longrightarrow R^1G((Y)) \longrightarrow \dots$$

By assumption $R^iG((X)) = 0$ for any $i \geq 1$. Since $R^iG((V^{(I)})) = 0$, for any $i \geq 1$, $R^iG((Y)) = 0$ for any $i \geq 1$, by the exactness. Thus $Y \in {}^\perp T_G^{i \geq 1}$ and

hence by Lemma 2, $G(Y) \in n\text{-Copres}(U)$. Since (F, G, V, U) is a $\text{Co-}\ast^n$ -tuple, applying the functor F to the following sequence

$$0 \longrightarrow G(X) \longrightarrow G(V^{(I)}) \longrightarrow G(Y) \longrightarrow 0$$

we get the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \longrightarrow & V^{(I)} & \longrightarrow & X & \longrightarrow & 0 \\ & & \downarrow \delta'_Y & & \downarrow \delta'_{V^{(I)}} & & \downarrow \delta'_X & & \\ 0 & \longrightarrow & FG(Y) & \longrightarrow & FG(V^{(I)}) & \longrightarrow & FG(X) & \longrightarrow & 0 \end{array}$$

Hence by Snake Lemma, δ'_X is an epimorphism and $\text{Ker}(\delta'_X) \cong \text{Coker}(\delta'_Y)$, since $\delta'_{V^{(I)}}$ is an isomorphism. Similarly δ'_Y is also an epimorphism. Thus, δ'_X is an isomorphism and therefore $X \cong FG(X) \in F(n\text{-Copres}(U))$. So $F(n\text{-Copres}(U)) = {}^\perp T_G^{i \geq 1}$. ■

Proposition 6 Suppose that (F, G, V, U) be a $\text{Co-}\ast^n$ -tuple. Then F preserves any exact sequence in $n\text{-Copres}(U)$.

Proof. Let $0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$ be an exact sequence in $n\text{-Copres}(U)$. Applying the functor F we get the following long exact sequence

$$0 \longrightarrow F(Z) \xrightarrow{F(g)} F(Y) \xrightarrow{F(f)} F(X) \xrightarrow{\alpha} R^1 F(Z) \longrightarrow \dots$$

Thus we can get the following two exact sequences

$$0 \longrightarrow W \longrightarrow F(X) \longrightarrow Q \longrightarrow 0,$$

$$0 \longrightarrow F(Z) \longrightarrow F(Y) \longrightarrow W \longrightarrow 0,$$

where $Q = \text{Im } \alpha$ and $W = \text{Im } F(f)$. Applying the functor G to the last sequence we get the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0 \\ & & \downarrow & & \downarrow \delta_Y & & \downarrow \delta_Z & & \\ 0 & \longrightarrow & G(W) & \longrightarrow & GF(Y) & \longrightarrow & GF(Z) & \longrightarrow & R^1 G(W) \longrightarrow 0 \end{array}.$$

It is clear by Proposition 3 that δ_Y and δ_Z are isomorphisms and $R^i G(F(Y)) =$

$0 = R^i G(F(Z))$, for any $i \geq 1$. By Snake Lemma, $X \cong G(W)$ and by the exactness $R^i G(W) = 0$ for any $i \geq 1$. Hence by Proposition 5, $W = F(D)$ for some $D \in n\text{-Copres}(U)$. Therefore

$$W = F(D) \cong F(GF(D)) = FG(F(D)) = FG(W) \cong F(X).$$

Hence $F(f)$ is an epimorphism, and therefore the induced sequence

$$0 \longrightarrow F(Z) \xrightarrow{F(g)} F(Y) \xrightarrow{F(f)} F(X) \longrightarrow 0$$

is exact. ■

Theorem 7 *The following conditions are equivalent:*

(1) (F, G, V, U) is a $Co\text{-}\ast^n$ -tuple.

(2) (i) U is $co\text{-}F$ -small;

(ii) For any exact sequence $0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$ with $X, Y \in n\text{-Copres}(U)$, we have $Z \in n\text{-Copres}(U)$ if and only if $0 \longrightarrow F(Z) \xrightarrow{F(g)} F(Y) \xrightarrow{F(f)} F(X) \longrightarrow 0$ is exact.

Proof. (1) $\Rightarrow$ (2) The necessity follows from Proposition 6 and the sufficiency from a similar proof to that of (1) $\Rightarrow$ (2) in Theorem 4.

(2) $\Rightarrow$ (1) It follows from (2) $\Rightarrow$ (1) in Theorem 4. ■

Proposition 8 *Suppose that (F, G, V, U) is a $Co\text{-}\ast^n$ -tuple. Then $n\text{-Copres}(U)$ is closed under extensions if and only if $n\text{-Copres}(U) \subseteq^\perp T_F^1 = \{X \in \mathfrak{C} : R^1F(X) = 0\}$.*

Proof. Suppose that $n\text{-Copres}(U)$ is closed under extensions. For any $X \in n\text{-Copres}(U)$ one can construct an exact sequence using the canonical maps to get an extension $0 \longrightarrow U \longrightarrow Y \longrightarrow X \longrightarrow 0$ of U by X . We have $Y \in n\text{-Copres}(U)$, by assumption. By Proposition 6, F preserves any exact sequence in $n\text{-Copres}(U)$, so applying F to the last exact sequence we get the exact sequence

$$0 \longrightarrow F(X) \longrightarrow F(Y) \longrightarrow F(U) \longrightarrow 0,$$

thus by the exactness, $R^1F(X) = 0$, so $X \in^\perp T_F^1$ and hence $n\text{-Copres}(U) \subseteq^\perp T_F^1$.

Conversely. For any extension $0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$, of X by Z , where $X, Z \in n\text{-Copres}(U)$, the induced sequence

$$0 \longrightarrow F(Z) \longrightarrow F(Y) \longrightarrow F(X) \longrightarrow 0,$$

is exact by assumption. According to Proposition 3, both δ_X and δ_Z are isomorphisms and $F(X), F(Z) \in^\perp T_G^{i \geq 1}$. Then it is clear that δ_Y is an isomorphism and $F(Y) \in^\perp T_G^{i \geq 1}$. Hence by Lemma 2, we have $Y \cong GF(Y) \in n\text{-Copres}(U)$. ■

Theorem 9 *The following conditions are equivalent:*

(1) (F, G, V, U) is a $Co\text{-}\ast^n$ -tuple.

(2) There is a duality

$$G :^\perp T_G^{i \geq 1} \rightleftarrows n\text{-Copres}(U) : F$$

Proof. (1) $\Rightarrow$ (2) By Propositions 3 and Propositions 5.

(2) $\Rightarrow$ (1) Since $V^{(I)} \in^\perp T_G^{i \geq 1}$, we get $F(U^I) \cong F(G(V)^I) \cong F(G(V^{(I)})) \cong FG(V^{(I)}) \cong V^{(I)} \cong F(U)^{(I)}$. So U is $co\text{-}F$ -small. For any $X \in n\text{-Copres}(U)$, by assumption $X \cong G(F(X))$ and $F(X) \in^\perp T_G^{i \geq 1}$, thus $X \in (n+1)\text{-Copres}(U)$, by Lemma 2. So $n\text{-Copres}(U) = (n+1)\text{-Copres}(U)$. Now let $0 \longrightarrow X \xrightarrow{f} U^I \longrightarrow$

$Y \longrightarrow 0$ be an exact sequence, with $Y \in (n+1)\text{-Copres}(U)$ and I a set. We can get the following exact sequence

$$0 \longrightarrow F(Y) \longrightarrow F(U^I) \xrightarrow{F(f)} F(X) \longrightarrow Q \longrightarrow 0,$$

where $Q = \text{Im } \alpha$, where $\alpha : F(X) \longrightarrow R^1 F(Y)$. By using argument similar to that in Proposition 6, we get the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{f} & U^I & \longrightarrow & Y & \longrightarrow & 0 \\ & & \downarrow & & \downarrow \delta_{U^I} & & \downarrow \delta_Y & & \\ 0 & \longrightarrow & G(W) & \longrightarrow & GF(U^I) & \longrightarrow & GF(Y) & \longrightarrow & R^1 G(W) \longrightarrow 0 \end{array},$$

where $W = \text{Im } F(f)$, and we conclude that $Q = 0$, which means that we have an exact sequence $0 \longrightarrow F(Y) \longrightarrow F(U^I) \longrightarrow F(X) \longrightarrow 0$. Thus (F, G, V, U) is a $\text{Co-}^*{}^n$ -tuple. ■

Proposition 10 Suppose that U is a co- F -small. Assume that $n\text{-Copres}(U) = {}^\perp T_F^{i \geq 1}$. Then (F, G, V, U) is a $\text{Co-}^*{}^n$ -tuple.

Proof. Let $0 \longrightarrow X \longrightarrow U^I \longrightarrow Y \longrightarrow 0$, be an exact sequence with $X \in n\text{-Copres}(U)$ and I a set. We can get the following long exact sequence

$$0 \longrightarrow F(Y) \longrightarrow F(U^I) \longrightarrow F(X) \longrightarrow R^1 F(Y) \longrightarrow R^1 F(U^I) \longrightarrow R^1 F(X) \longrightarrow \dots$$

Note that $X, U^I \in n\text{-Copres}(U) = {}^\perp T_F^{i \geq 1}$, so by exactness, $R^i F(Y) = 0$, for every $i \geq 2$. Now $R^1 F(Y) = 0$ if and only if $Y \in {}^\perp T_F^{i \geq 1} = n\text{-Copres}(U)$. So by Theorem 4 we get the desired result. ■

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