Complexiton solutions of nonlinear partial differential equations with a new auxiliary equation

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Abstract

In this paper, a novel auxiliary equation: $u'' = a + bu + cu^3$ which its' interaction solutions including trigonometric and hyperbolic function solutions are succeed to achieve are considered. Miscellaneous solutions corresponding to using this auxiliary equation for a series of partial differential equation models are easily, effective obtained. It is prime important to motivating authors to accurately analyze special phenomenon in their relevant aspects.

Keywords: Complexiton solution; Riccati function; $u'' = a + bu + cu^3$

1 Introduction

As it is well-known, many physics and nature science are usually characterized well by ubiquitous nonlinear dynamical equations. Soliton theory is one of significant fields in nonlinearity, travelling wave solutions of mathematical and physics nonlinear models especially the most representative equations like the KdV equation [1] and Schrödinger equation, they have been committed to assist people in describing the nature science better. The KdV equation is derived by Korteweg and de Vries to model the evolution of shallow water wave in 1895. The (2+1) dimensional KdV equation that model is more accurate to manifest the change of shallow water wave, it is obtained through potential function. Hirota-Satsuma equations is classified as a soliton equation by B. Fuchssteiner, it has a bi-hamiltonian formulation and obtains countably many conserved quantities and symmetry generators. Complete integrability of this equation is conjectured by Hirota and Satsuma [2].

In previously, the complexiton solutions(interaction solution) show that interact between different kinds of travelling waves of nonlinear evolution equation,they are

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attracted numerous attention, such as solutions involve trigonometric and hyperbolic structures, or contain hyperbolic and elliptic structures and so on. They are usually tended to uncovering potential meaningful application. As example, long time ago people have no practical countermeasures when suffer from tsunami and earthquake, through investigating soliton solutions of nonlinear correlative equation people will looking for their related properties to prevent large damage.

In currently, multifarious methods for searching analytical solutions of nonlinear equations have evolved from complicated process with heavy computation to simpleness and understandability, and advanced compute technology is in high-speed development along with our modernize society. Sub-equation approach which contain the the homogeneous balance method [3], sine-cosine method [4], the sech-function method [5], the hyperbolic tangent function method [6, 7], the multiple exp-function method [8, 9], the Riccati equations method [10] as widespread application to construct exact solutions. In 2008, the G'/G-expansion method was proposed by Wang [11] which arose a large of attention for its straightforward, simplification and applicability in obtaining analytical solutions of nonlinear equations. Subsequently authors developed the the G'/G-expansion method to improved G'/G method and extended G'/G methods. It has been successfully to get rational solutions, trigonometric and hyperbolic function solutions of many kinds of nonlinear evolution equations [12, 13, 14, 15, 16, 17, 18, 19] through the auxiliary equation $G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0$, where λ, μ are arbitrary constants. But the solutions of the solvable auxiliary equation are singular soliton solutions which are not contain complexiton solution in applying the basis G'/G-expansion and the other G'/G-expansion method. In this paper, the novel sub-equation can successfully to obtained interaction solutions.

Ma [20], Fan [21], Chen [22, 23], Chen [24, 25], Yan [26, 27] are devoted to constructing special soliton solutions by using combination of auxiliary equations and get great success.

In this paper, a new sub-equation: $u'' = a + bu + cu^3$ which we can seek out the particular nonlinear interactive wave solutions (interaction solution), they include trigonometric and hyperbolic function solutions. This method is proposed with a wide concrete applications that can propagate to many branches of nonlinear areas. Then solutions of examples are calculated through mathematical techniques and used to illustrate the effective of this method.

2 The new interaction solutions of $u'' = a + bu + cu^3$

For desired equation read

$$u'' = a + bu + cu^3, \tag{1}$$

where $u'' = u''(\xi)$, in order to work out $u(\xi)$, hypothesis are taken as following:

$$u(\xi) = a_0 + \frac{a_1 G(\xi)H(\xi) + a_2 G'(\xi)H'(\xi)}{a_3 F(\xi) + 1}, \tag{2}$$

where $F(\xi), G(\xi), H(\xi)$ are functions satisfying the Riccati equation respectively. In addition a_0, a_1, a_2, a_3 are constant to be determined later.

$$F'(\xi) = A_1 + B_1 F(\xi) + C_1 F^2(\xi)$$

$$G'(\xi) = A_2 + B_2 G(\xi) + C_2 G^2(\xi)$$

$$H'(\xi) = A_3 + B_3 H(\xi) + C_3 H^2(\xi), \tag{3}$$

where $A_1, A_2, A_3, B_1, B_2, B_3, C_1, C_2, C_3$, are free constant.

Inserting eqno(2) into eqno(1) with the related Riccati auxiliary equations eqno(3), then setting the coefficients of $F^i(\xi)G^j(\xi)H^s(\xi)$ $(0 \leq i, j, s \leq 6)$ equate to zero, we derive a system of over determined linear equations with a_0, a_1, a_2, a_3 $A_1, A_2, A_3, B_1, B_2, B_3, C_1, C_2, C_3$.

The precondition relationship:

$A_1 = A_1, B_1 = B_1, C_1 = -a_3^2 A_1 + a_3 B_1, A_2 = A_2, B_2 = B_2, C_2, A_3 = 0, B_3 = -\frac{B_1}{2} + a_3 A_1, C_3 = 0,\ a = -\frac{(B_1^2 - 4a_3 B_1 A_1 + 4a_3^2 A_1^2)}{4} a_0, b = \frac{B_1^2 - a_3 B_1 A_1 + a_3^2 A_1^2}{4}, c = 0\ a_0 = a_0, a_1 = a_1, a_2 = 0, a_3 = a_3.$

Type 1: When $A_1 = \frac{1}{2}, B_1 = 0, C_1 = \frac{1}{2}$

$$u_1 = a_0 + \frac{a_1(\tan(\xi) \pm \sec(\xi))(\sinh(B_3\xi) + \cosh(B_3\xi))}{a_3(\tan(\xi) \pm \sec(\xi)) + 1} \tag{4}$$

$$u_2 = a_0 + \frac{a_1(\csc(\xi) - \cot(\xi))(\sinh(B_3\xi) + \cosh(B_3\xi))}{a_3(\csc(\xi) - \cot(\xi)) + 1} \tag{5}$$

$$u_3 = a_0 + \frac{a_1(\frac{\tan(\xi)}{1 \pm \sec(\xi)})(\sinh(B_3\xi) + \cosh(B_3\xi))}{a_3(\frac{\tan(\xi)}{1 \pm \sec(\xi)}) + 1} \tag{6}$$

Type 2: When $A_1 = -\frac{1}{2}, B_1 = 0, C_1 = -\frac{1}{2}$

$$u_4 = a_0 + \frac{a_1(\cot(\xi) \pm \csc(\xi))(\sinh(B_3\xi) + \cosh(B_3\xi))}{a_3(\cot(\xi) \pm \csc(\xi)) + 1} \tag{7}$$

$$u_5 = a_0 + \frac{a_1(\sec(\xi) - \tan(\xi))(\sinh(B_3\xi) + \cosh(B_3\xi))}{a_3(\sec(\xi) - \tan(\xi)) + 1} \tag{8}$$

$$u_6 = a_0 + \frac{a_1(\frac{\cot(\xi)}{1 \pm \csc(\xi)})(\sinh(B_3\xi) + \cosh(B_3\xi))}{a_3(\frac{\cot(\xi)}{1 \pm \csc(\xi)}) + 1} \tag{9}$$

Type 3: When $A_1 = 1, B_1 = 0, C_1 = 4$

$$u_7 = a_0 + \frac{a_1(\frac{\tan(\xi)}{1 - \tan^2(\xi)})(\sinh(B_3\xi) + \cosh(B_3\xi))}{a_3(\frac{\tan(\xi)}{1 - \tan^2(\xi)}) + 1} \tag{10}$$

Type 4: When $A_1 = -1, B_1 = 0, C_1 = -4$

$$u_8 = a_0 + \frac{a_1(\frac{\tan(\xi)}{1-\cot^2(\xi)})(\sinh(B_3\xi) + \cosh(B_3\xi))}{a_3(\frac{\tan(\xi)}{1-\cot^2(\xi)}) + 1} \quad (11),$$

where a_0, a_2, a_3, B_2, B_3 are arbitrary constant. Typical explicit solutions are taken into consideration above, and there exists abundant interaction solutions of eqno(1) we omit in the paper. In the wake of such work, we aim at wielding the auxiliary equation to the governing equations as followings.

3 Concrete applications of this sub-equation

Example 1 Consider the (2+1) dimensional KdV equation[28]

$$U_t + U_{xxx} - 3U_xV - 3UV_x = 0$$
$$U_x - V_y = 0, \quad (12)$$

here, to start off, we have the hypothesis in the following terms is obtained:

$$U(\xi) = \sum_{p=0}^{m} m_p u^p(\xi),$$
$$V(\xi) = \sum_{q=0}^{n} n_q u^q(\xi), u(\xi) = x + ky - vt, \quad (13)$$

here k, v are constant, v represent as the wave speed. Where m, n are positive integer and equate to 2 respectively which are determined by the principle of homogeneous balance. $u(\xi)$ satisfy the sub-equation: $u'' = a + bu + cu^3$.

$$U(\xi) = m_0 + m_1u(\xi) + m_2u^2(\xi),$$
$$V(\xi) = n_0 + n_1u(\xi) + n_2u^2(\xi) \quad (14)$$

$m_0, m_1, m_2, n_0, n_1, n_2$ are all acquired in the later. Hence, when we substitute eqno(14) into eqno(13) along with aid of auxiliary equation. Equating the coefficients of $u^\alpha(\xi)u'(\xi)$ $(0 \leq \alpha \leq 3)$ to zero, a set of algebraic equations are yielded that unknown parameters $m_0, m_1, m_2, n_0, n_1, n_2, k, c, v$ are able to solve through using the computation of Maple.

Then analytical interaction solutions of system eqno(12): when $k = k, v = v, m_0 = \frac{1}{3}k(-v + 4b - 3n_0), m_1 = 0, m_2 = n_2, n_0 = n_0, n_1 = 0, n_2 = n_2$ with type1, type2, type3, type4 as explicit solutions of eqno(1):

$$U_1 = \frac{1}{3}k(-v + 4b - 3n_0) + n_2(a_0 + \frac{a_1(\tan(\xi) \pm \sec(\xi))(\sinh(B_3\xi) + \cosh(B_3\xi))}{a_3(\tan(\xi) \pm \sec(\xi)) + 1})^2$$

$$V_1 = n_0 + n_2(a_0 + \frac{a_1(\tan(\xi) \pm \sec(\xi))(\sinh(B_3\xi) + \cosh(B_3\xi))}{a_3(\tan(\xi) \pm \sec(\xi)) + 1})^2 \tag{15}$$

$$U_2 = \frac{1}{3}k(-v + 4b - 3n_0) + n_2(a_0 + \frac{a_1(\csc(\xi) - \cot(\xi))(\sinh(B_3\xi) + \cosh(B_3\xi))}{a_3(\csc(\xi) - \cot(\xi)) + 1})^2$$

$$V_2 = n_0 + n_2(a_0 + \frac{a_1(\csc(\xi) - \cot(\xi))(\sinh(B_3\xi) + \cosh(B_3\xi))}{a_3(\csc(\xi) - \cot(\xi)) + 1})^2 \tag{16}$$

$$U_3 = \frac{1}{3}k(-v + 4b - 3n_0) + n_2(a_0 + \frac{a_1(\frac{\tan(\xi)}{1\pm\sec(\xi)})(\sinh(B_3\xi) + \cosh(B_3\xi))}{a_3(\frac{\tan(\xi)}{1\pm\sec(\xi)}) + 1})^2$$

$$V_3 = n_0 + n_2(a_0 + \frac{a_1(\frac{\tan(\xi)}{1\pm\sec(\xi)})(\sinh(B_3\xi) + \cosh(B_3\xi))}{a_3(\frac{\tan(\xi)}{1\pm\sec(\xi)}) + 1})^2 \tag{17}$$

$$U_4 = \frac{1}{3}k(-v + 4b - 3n_0) + n_2(a_0 + \frac{a_1(\cot(\xi) \pm \csc(\xi))(\sinh(B_3\xi) + \cosh(B_3\xi))}{a_3(\cot(\xi) \pm \csc(\xi)) + 1})^2$$

$$V_4 = n_0 + n_2(a_0 + \frac{a_1(\cot(\xi) \pm \csc(\xi))(\sinh(B_3\xi) + \cosh(B_3\xi))}{a_3(\cot(\xi) \pm \csc(\xi)) + 1})^2 \tag{18}$$

$$U_5 = \frac{1}{3}k(-v + 4b - 3n_0) + n_2(a_0 + \frac{a_1(\sec(\xi) - \tan(\xi))(\sinh(B_3\xi) + \cosh(B_3\xi))}{a_3(\sec(\xi) - \tan(\xi)) + 1})^2$$

$$V_5 = n_0 + n_2(a_0 + \frac{a_1(\sec(\xi) - \tan(\xi))(\sinh(B_3\xi) + \cosh(B_3\xi))}{a_3(\sec(\xi) - \tan(\xi)) + 1})^2 \tag{19}$$

$$U_6 = \frac{1}{3}k(-v + 4b - 3n_0) + n_2(a_0 + \frac{a_1(\frac{\cot(\xi)}{1\pm\csc(\xi)})(\sinh(B_3\xi) + \cosh(B_3\xi))}{a_3(\frac{\cot(\xi)}{1\pm\csc(\xi)}) + 1})^2$$

$$V_6 = n_0 + n_2(a_0 + \frac{a_1(\frac{\cot(\xi)}{1\pm\csc(\xi)})(\sinh(B_3\xi) + \cosh(B_3\xi))}{a_3(\frac{\cot(\xi)}{1\pm\csc(\xi)}) + 1})^2 \tag{20}$$

$$U_7 = \frac{1}{3}k(-v + 4b - 3n_0) + n_2(a_0 + \frac{a_1(\frac{\tan(\xi)}{1-\tan^2(\xi)})(\sinh(B_3\xi) + \cosh(B_3\xi))}{a_3(\frac{\tan(\xi)}{1-\tan^2(\xi)}) + 1})^2$$

$$V_7 = n_0 + n_2(a_0 + \frac{a_1(\frac{\tan(\xi)}{1-\tan^2(\xi)})(\sinh(B_3\xi) + \cosh(B_3\xi))}{a_3(\frac{\tan(\xi)}{1-\tan^2(\xi)}) + 1})^2 \tag{21}$$

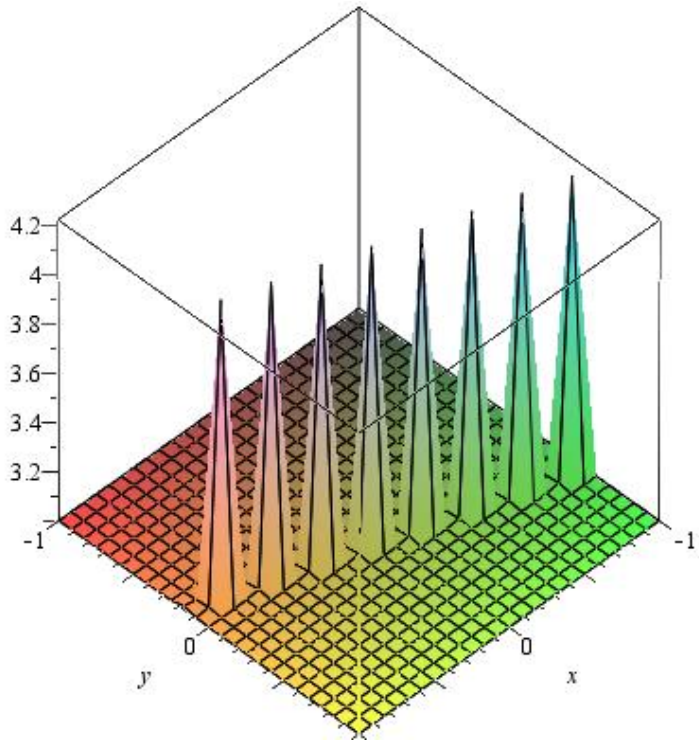

Figure 1: solution U_1 in eqno.(15) of eqno.(14) corresponding to $n_0 = 0, n_2 = 1, a_0 = 0, a_1 = 1, a_3 = 1, k = 3, v = 1$

$$U_8 = \frac{1}{3}k(-v + 4b - 3n_0) + n_2(a_0 + \frac{a_1(\frac{\tan(\xi)}{1-\cot^2(\xi)})(\sinh(B_3\xi) + \cosh(B_3\xi))}{a_3(\frac{\tan(\xi)}{1-\cot^2(\xi)}) + 1})^2$$

$$V_8 = n_0 + n_2(a_0 + \frac{a_1(\frac{\tan(\xi)}{1-\cot^2(\xi)})(\sinh(B_3\xi) + \cosh(B_3\xi))}{a_3(\frac{\tan(\xi)}{1-\cot^2(\xi)}) + 1})^2, \tag{22}$$

where B_3 have the same meaning as before in eqno(3). Figure.1, Figure.2 represent the complexiton solution of (2+1) dimensional KdV equation. The shape is created by interacting hyperbolic between trigonometric function of solutions when t is constant.

Example 2 Consider the Hirota-Satsuma equation[29].

$$U_t + U_{xxx} + 6UU_x - 6VV_x = 0$$

$$V_t - 2V_{xxx} - 6UV_x = 0, \tag{23}$$

we handing eqno(23) as the same of eqno(12) with hypothesis eqno(13), $m = 2, n = 2$ according the homogeneous balance law.

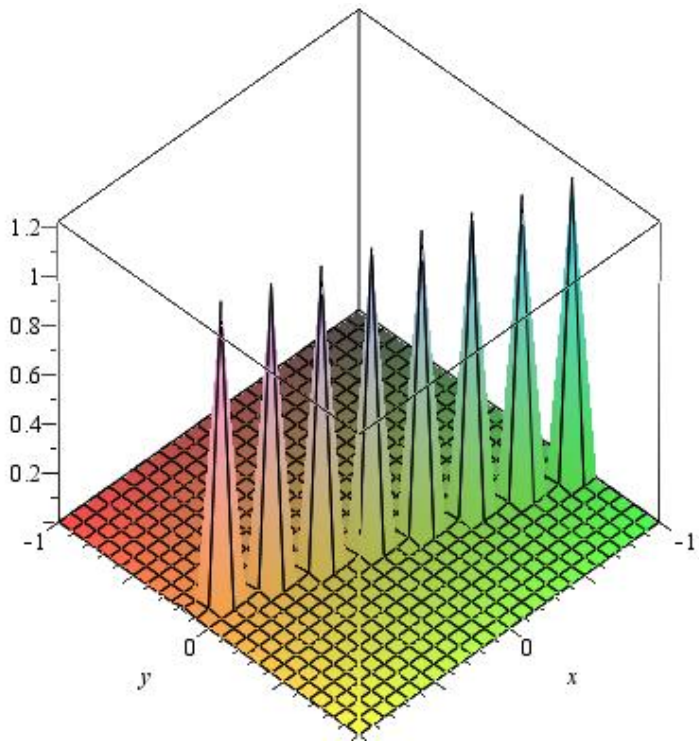

Figure 2: solution V_1 in eqno.(15) of eqno.(14) corresponding to $n_0 = 0, n_2 = 1, a_0 = 0, a_1 = 1, a_3 = 1, k = 3, v = 1$

$$U(\xi) = m_0 + m_1 u(\xi) + m_2 u^2(\xi)$$
$$V(\xi) = n_0 + n_1 u(\xi) + n_2 u^2(\xi), \tag{24}$$

where $m_0, m_1, m_2, n_0, n_1, n_2$ are constant, they will be determined by the following work. Here $u(\xi)$ fulfills the ordinary sub-equation $u'' = a + bu + cu^3$ ($u(\xi) = u(x - lt)$). With the help of solutions of $u'' = a + bu + cu^3$, collecting the coefficients of all power of $u(\xi)u'(\xi)$ equate to zero after taking the eqno(24) into eqno(23).

Two series of free constants are obtained:

Case 1: $v = v, m_0 = m_0, m_1 = 0, m_2 = RootOf(Z^2 - 2, label =_L 1)n_2$, $n_0 = -\frac{1}{4}RootOf(Z^2 - 2, label =_L 1)(l - 2m_0), n_1 = 0, n_2 = n_2$

Case 2:$v = \frac{-(n_1^2+2m_0m_2)}{m_2}, m_0 = m_0, m_1 = 0, m_2 = m_2, n_0 = n_0, n_1 = n_1, n_2 = 0$

Travelling exact solutions of eqno(23) in case 2:

$$U_1 = m_0 + m_2(a_0 + \frac{a_1(\tan(\xi) \pm \sec(\xi))(\sinh(B_3\xi) + \cosh(B_3\xi))}{a_3(\tan(\xi) \pm \sec(\xi)) + 1})^2$$
$$V_1 = n_0 + n_1(a_0 + \frac{a_1(\tan(\xi) \pm \sec(\xi))(\sinh(B_3\xi) + \cosh(B_3\xi))}{a_3(\tan(\xi) \pm \sec(\xi)) + 1}) \tag{25}$$

$$U_2 = m_0 + m_2(a_0 + \frac{a_1(\csc(\xi) - \cot(\xi))(\sinh(B_3\xi) + \cosh(B_3\xi))}{a_3(\csc(\xi) - \cot(\xi)) + 1})^2$$
$$V_2 = n_0 + n_1(a_0 + \frac{a_1(\csc(\xi) - \cot(\xi))(\sinh(B_3\xi) + \cosh(B_3\xi))}{a_3(\csc(\xi) - \cot(\xi)) + 1}) \tag{26}$$

$$U_3 = m_0 + m_2(a_0 + \frac{a_1(\frac{\tan(\xi)}{1\pm\sec(\xi)})(\sinh(B_3\xi) + \cosh(B_3\xi))}{a_3(\frac{\tan(\xi)}{1\pm\sec(\xi)}) + 1})^2$$
$$V_3 = n_0 + n_1(a_0 + \frac{a_1(\frac{\tan(\xi)}{1\pm\sec(\xi)})(\sinh(B_3\xi) + \cosh(B_3\xi))}{a_3(\frac{\tan(\xi)}{1\pm\sec(\xi)}) + 1}) \tag{27}$$

$$U_4 = m_0 + m_2(a_0 + \frac{a_1(\cot(\xi) \pm \csc(\xi))(\sinh(B_3\xi) + \cosh(B_3\xi))}{a_3(\cot(\xi) \pm \csc(\xi)) + 1})^2$$
$$V_4 = n_0 + n_1(a_0 + \frac{a_1(\cot(\xi) \pm \csc(\xi))(\sinh(B_3\xi) + \cosh(B_3\xi))}{a_3(\cot(\xi) \pm \csc(\xi)) + 1}) \tag{28}$$

$$U_5 = m_0 + m_2(a_0 + \frac{a_1(\sec(\xi) - \tan(\xi))(\sinh(B_3\xi) + \cosh(B_3\xi))}{a_3(\sec(\xi) - \tan(\xi)) + 1})^2$$
$$V_5 = n_0 + n_1(a_0 + \frac{a_1(\sec(\xi) - \tan(\xi))(\sinh(B_3\xi) + \cosh(B_3\xi))}{a_3(\sec(\xi) - \tan(\xi)) + 1}) \tag{29}$$

$$U_6 = m_0 + m_2(a_0 + \frac{a_1(\frac{\cot(\xi)}{1\pm\csc(\xi)})(\sinh(B_3\xi) + \cosh(B_3\xi))}{a_3(\frac{\cot(\xi)}{1\pm\csc(\xi)}) + 1})^2$$

$$V_6 = n_0 + n_1(a_0 + \frac{a_1(\frac{\cot(\xi)}{1\pm\csc(\xi)})(\sinh(B_3\xi) + \cosh(B_3\xi))}{a_3(\frac{\cot(\xi)}{1\pm\csc(\xi)}) + 1}) \tag{30}$$

$$U_7 = m_0 + m_2(a_0 + \frac{a_1(\frac{\tan(\xi)}{1-\tan^2(\xi)})(\sinh(B_3\xi) + \cosh(B_3\xi))}{a_3(\frac{\tan(\xi)}{1-\tan^2(\xi)}) + 1})^2$$

$$V_7 = n_0 + n_1(a_0 + \frac{a_1(\frac{\tan(\xi)}{1-\tan^2(\xi)})(\sinh(B_3\xi) + \cosh(B_3\xi))}{a_3(\frac{\tan(\xi)}{1-\tan^2(\xi)}) + 1}) \tag{31}$$

$$U_8 = m_0 + m_2(a_0 + \frac{a_1(\frac{\tan(\xi)}{1-\cot^2(\xi)})(\sinh(B_3\xi) + \cosh(B_3\xi))}{a_3(\frac{\tan(\xi)}{1-\cot^2(\xi)}) + 1})^2$$

$$V_8 = n_0 + n_1(a_0 + \frac{a_1(\frac{\tan(\xi)}{1-\cot^2(\xi)})(\sinh(B_3\xi) + \cosh(B_3\xi))}{a_3(\frac{\tan(\xi)}{1-\cot^2(\xi)}) + 1}), \tag{32}$$

where $m_0, m_2, n_0, n_1, a_0, a_2, B_2, B_3$, are arbitrary constant. Figure.3, Figure.4 represent the complexiton solution of Hirota-Satsuma equation. The shape is created by interacting hyperbolic with trigonometric function in solutions.

$U(\xi) = U(x - vt)$, v is propagation speed of soliton waves. Particular obtained solutions $U_i, V_i (1 \leq i \leq 8)$ work in concert with solutions of solvable ordinary eqno(1) which coefficients need to satisfying $a = -ca_0^3 - ba_0, b = b, c = c$. There some more interaction solutions of eqno(23), we ignore in this paper as in case 1 for simplification.

4 Conclusion and discussion

In this paper, two physics models response the novel auxiliary equation that carry a major contribution for searching interaction wave solutions. Trigonometric and hyperbolic function solutions are not generated in ref [28], [30]. It will draw great attention as solutions of the novel auxiliary equation themselves are interactive, the situation is not appeared in previous method as like the (G'/G)-expansion method. Complicate physical phenomena in nonlinear model systems will be study well in the future when apply the typical interaction solutions we get.

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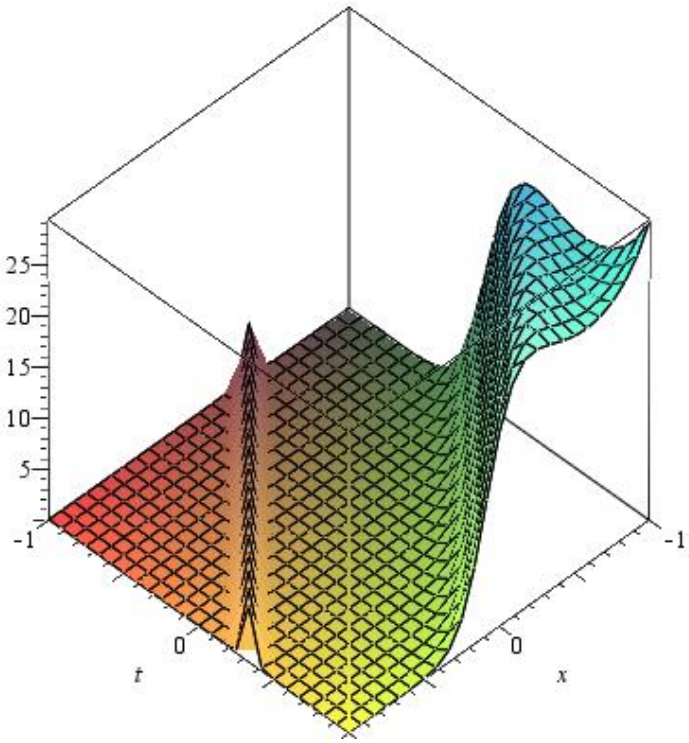

Figure 3: solution U_8 in eqno.(32) of eqno.(23) corresponding to $m_0 = 0, m_2 = 1, a_0 = 0, a_1 = 1, a_3 = 1, v = 1$

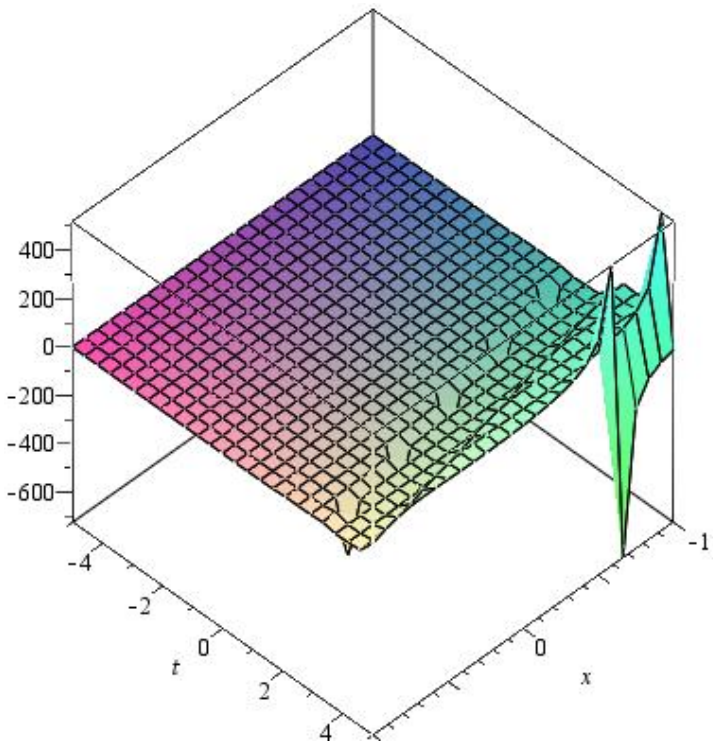

Figure 4: solution V_8 in eqno.(32) of eqno.(23) corresponding to $n_0 = 0, n_2 = 1, a_0 = 0, a_1 = 1, a_3 = 1, k = 3, v = 1$

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