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Three Dimensional Trans-Sasakian Manifold Admitting Quarter Symmetric Metric Connection

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Abstract

The object of this paper is to study quarter-symmetric metric connection on a trans-Sasakian manifold. Here we study locally ϕ-symmetric, ϕ-symmetric and concircular ϕ-symmetric trans-Sasakian manifold with respect to quarter symmetric metric connection and obtained some interesting results.

Keywords: Quarter symmetric metric connection, Trans-Sasakian manifold, ϕ-symmetry.
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1 Introduction

In 1985, Oubina J. A. (16) introduced a new class of almost Contact manifold namely trans-Sasakian manifold. Many geometers like (1), (2), (9), (13), (25), (26), (27), (30) have studied this manifold and obtained many interesting results. The notion of quarter-symmetric connection generalizes the semi-symmetric connection. In 1924, Friedman A. and Schouten J. A. (10), (24) introduced the notion of semi-symmetric linear connection on a differentiable manifold. In 1975, Golab S. (11) defined and studied quarter-symmetric connection in a differentiable manifold with affine connection. Rastogi S. C. (21), (22) continued the systematic study of quarter symmetric metric connection.

A linear connection $\widetilde{\nabla}$ in $(2n+1)$ dimensional differentiable manifold is said to be a quarter symmetric connection if its torsion tensor T is of the form

$$T(X,Y) = \widetilde{\nabla}_X Y - \widetilde{\nabla}_Y X - [X,Y], \tag{1.1}$$
$$= \eta(Y)\phi X - \eta(X)\phi Y, \tag{1.2}$$

where η is a 1-form and ϕ is a tensor of type $(1,1)$. In particular, if $\phi X = X$, then the quarter-symmetric connection reduces to the semi-symmetric connection. And if quarter-symmetric metric

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connection $\widetilde{\nabla}$ satisfies the condition

$$(\widetilde{\nabla}_X g)(Y, Z) = 0,$$

for all $X, Y, Z \in \chi(M)$, where $\chi(M)$ is the Lie algebra of vector fields on the manifold M, then $\widetilde{\nabla}$ is said to be a quarter symmetric metric connection.

In 1980, Mishra R. S. and Pandey S. N. (14) studied quarter symmetric metric connection in a Riemannian, Kaehlerian and Sasakian manifolds. In 1982, Yano K. and Imai T. (32) studied quarter symmetric metric connection in Hermition and Kaehlerian manifolds. In 1991, Mukhopadhyay S. et al. (15) studied quarter symmetric metric connection on a Riemannian manifold with an almost complex structure ϕ. Quarter symmetric metric connection was also studied by Singh R. N. (23). Biswas and De U. C. (3), De U. C. and Mondal A. K. (7) studied Quarter symmetric metric connection on SP-Sasakian manifold and Sasakian manifolds. Pradeep Kumar K. T., Bagewadi C. S. and Venkatesha (17) (18) and Prakasha D. G. (19) studied Quarter symmetric metric connection on K-contact manifolds and Kenmotsu manifolds. Shyamal Kumar Hui (28) studied ϕ-pseudo symmetric Kenmotsu manifolds with respect to quarter symmetric metric connection. Also Patra C. and Bhattacharya A. (20) studied trans-Sasakian manifold admitting quarter symmetric non-metric connection.

The notion of local symmetric of a Riemannian manifold has been weakened by many authors in several ways to a different extent. As a weaker version of local symmetric, Takahashi T. (29) introduced the notion of locally ϕ-symmetry on a Sasakian manifold. In the context of Contact geometry the notion of ϕ-symmetric is introduced and studied by Boeckx E. Buecken P. and Vanhecke L. (6) with several examples. Motivated by these ideas, in the present paper we like to study quarter symmetric metric connection on a trans-Sasakian manifold. The present paper is organized as follows.

In section 2, we recall some preliminary results. In section 3, we give the relation between the Levi-Civita connection and the quarter symmetric metric connection on three dimensional trans-Sasakian manifold. Section 4 deals with locally ϕ-symmetric trans-Sasakian manifold with respect to quarter symmetric metric connection. In the next section we study ϕ-symmetric trans-Sasakian manifold with respect to quarter symmetric metric connection. Finally in section 6, we study locally concircular ϕ-symmetric trans-Sasakian manifold with respect to quarter symmetric metric connection.

2 Preliminaries

An $(2n+1)$ dimensional Riemannian manifold (M, g) is called an almost Contact manifold (4) if there exists on M, a $(1,1)$ tensor field ϕ, a vector field ξ and a 1-form η such that

$$\phi^2(X) = -X + \eta(X)\xi, \quad \eta(\phi X) = 0, \quad \phi\xi = 0, \quad \eta(\xi) = 1, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad g(X, \xi) = \eta(X), \tag{2.2}$$

$$g(X, \phi Y) = -g(\phi X, Y), \tag{2.3}$$

for any vector fields X, Y on M.

An almost Contact metric structure (ϕ, ξ, η, g) on M is called a trans-Sasakian structure (16), if $(M \times R, J, G)$ belongs to the class W_4 (12), where J is the almost complex structure on $M \times R$ defined by $J(X, f\frac{d}{dt}) = (\phi X - f\xi, \eta(X)\frac{d}{dt})$ for any vector field X on M and smooth function f on $M \times R$. This may be expressed by the condition (5),

$$(\nabla_X \phi)(Y) = \alpha\{g(X, Y)\xi - \eta(Y)X\} + \beta\{g(\phi X, Y)\xi - \eta(Y)\phi X\}, \tag{2.4}$$

for some smooth functions α and β on M, and we say that the trans-Sasakian structure is of type (α, β).

From (2.4), we have

$$(\nabla_X \xi) = -\alpha\phi X + \beta(X - \eta(X)\xi), \tag{2.5}$$
$$(\nabla_X \eta)(Y) = -\alpha g(\phi X, Y) + \beta g(\phi X, \phi Y). \tag{2.6}$$

In a 3*-dimensional trans-Sasakian manifold, from (2.4), (2.5) and (2.6), we can derive (8)*

$$\begin{aligned} R(X,Y)Z &= [\frac{r}{2} + 2\xi\beta - 2(\alpha^2 - \beta^2)](g(Y,Z)X - g(X,Z)Y) \\ &- g(Y,Z)[(\frac{r}{2} + \xi\beta - 3(\alpha^2 - \beta^2))\eta(X)\xi \\ &- \eta(X)(\phi grad\alpha - grad\beta) + (X\beta + (\phi X)\alpha)\xi] \\ &+ g(X,Z)[(\frac{r}{2} + 2\xi\beta - 3(\alpha^2 - \beta^2))\eta(Y)\xi \\ &- \eta(Y)(\phi grad\alpha - grad\beta) + (Y\beta + (\phi Y)\alpha)\xi], \end{aligned} \tag{2.7}$$

$$\begin{aligned} R(X,Y)\xi &= (\alpha^2 - \beta^2)\{\eta(Y)X - \eta(X)Y\} \\ &+ 2\alpha\beta\{\eta(Y)\phi X - \eta(X)\phi Y\} \\ &+ (Y\alpha)\phi X - (X\alpha)\phi Y + (Y\beta)\phi^2 X - (X\beta)\phi^2 Y, \end{aligned} \tag{2.8}$$

$$S(X,\xi) = \left(2(\alpha^2 - \beta^2) - \xi\beta\right)\eta(X) - (\phi X)\alpha - X\beta. \tag{2.9}$$

When $\phi(grad\alpha) = grad\beta$*, (2.9) reduces to*

$$S(X,\xi) = 2(\alpha^2 - \beta^2)\eta(X). \tag{2.10}$$

Definition 2.1. A Riemannian manifold M is said to be locally symmetric if $\nabla R = 0$, where R is the Riemannian curvature tensor of M and ∇ is Levi-Civita connection of M.

3 Curvature tensor of a three dimensional trans-Sasakian manifold with respect to quarter symmetric metric connection

Let $\widetilde{\nabla}$ *be the linear connection and* ∇ *be the Riemannian connection of an almost Contact metric manifold such that*

$$\widetilde{\nabla}_X Y = \nabla_X Y + H(X,Y), \tag{3.1}$$

where H *is the tensor field of type* $(1,1)$*. For* $\widetilde{\nabla}$ *to be quarter-symmetric metric connection on* M*, we have (11)*

$$H(X,Y) = \frac{1}{2}[T(X,Y) + T'(X,Y) + T'(Y,X)], \tag{3.2}$$
$$g(T'(X,Y),Z) = g(T'(Z,X),Y). \tag{3.3}$$

In view of equation (1.2) and (3.3), we obtain

$$T'(X,Y) = g(\phi Y, X)\xi - \eta(X)\phi Y. \tag{3.4}$$

Using (1.2) and (3.4) in (3.2), we get

$$H(X,Y) = -\eta(X)\phi Y.$$

Hence a quarter symmetric metric connection in a trans-Sasakian manifold is given by

$$\widetilde{\nabla}_X Y = \nabla_X Y - \eta(X)\phi Y. \tag{3.5}$$

This is the relation between quarter-symmetric metric connection and the Levi-Civita connection.

The curvature tensor $\widetilde{R}$ of M with respect to quarter symmetric metric connection $\widetilde{\nabla}$ is defined by

$$\widetilde{R}(X,Y)Z = \widetilde{\nabla}_X\widetilde{\nabla}_Y Z - \widetilde{\nabla}_Y\widetilde{\nabla}_X Z - \widetilde{\nabla}_{[X,Y]}Z.$$

In view of (3.5), above equation takes the form

$$\begin{aligned}\widetilde{R}(X,Y)Z &= R(X,Y)Z - 2\alpha g(\phi Y,X)\phi Z \\ &- [\alpha g(X,Z)+\beta g(\phi X,Z)]\eta(Y)\xi + [\alpha g(Y,Z)+\beta g(\phi Y,Z)]\eta(X)\xi \\ &+ (\alpha X+\beta\phi X)\eta(Y)\eta(Z) - (\alpha Y+\beta\phi Y)\eta(X)\eta(Z),\end{aligned} \tag{3.6}$$

where $\widetilde{R}$ and R are the Riemannian curvature tensor with respect to $\widetilde{\nabla}$ and ∇ respectively. From equation (3.6) it follows that

$$\widetilde{S}(Y,Z) = S(Y,Z) - \alpha g(Y,Z) + \beta g(\phi Y,Z) + 3\alpha\eta(Y)\eta(Z), \tag{3.7}$$

where $\widetilde{S}$ and S are the Ricci tensor of the connections $\widetilde{\nabla}$ and ∇ respectively.
Contracting (3.7), we get

$$\widetilde{r} = r, \tag{3.8}$$

where $\widetilde{r}$ and r are the scalar curvature of the connections $\widetilde{\nabla}$ and ∇ respectively.

4 Locally ϕ-symmetric trans-Sasakian manifold with respect to quarter symmetric metric connection

Definition 4.1. A trans-Sasakian manifold M is said to be locally ϕ-symmetric if

$$\phi^2((\nabla_W R)(X,Y)Z) = 0, \tag{4.1}$$

for any arbitrary vector fields X, Y, Z, W orthogonal to ξ.

Analogous to this definition, we define a locally ϕ-symmetric trans-Sasakian manifold with respect to the quarter-symmetric metric connection by

$$\phi^2((\widetilde{\nabla}_W\widetilde{R})(X,Y)Z) = 0, \tag{4.2}$$

for all vector fields X, Y, Z and W orthogonal to ξ.
Using (3.5), we can write

$$(\widetilde{\nabla}_W\widetilde{R})(X,Y)Z = (\nabla_W\widetilde{R})(X,Y)Z) - \eta(W)\phi\widetilde{R}(X,Y)Z. \tag{4.3}$$

Now differentiating (3.6) with respect to W and using (2.4) and (2.6), we obtain

$$\begin{aligned}(\nabla_W \widetilde{R})(X,Y)Z &= (\nabla_W R)(X,Y)Z \\ &- 2\alpha g(\phi Y,X)[\alpha(g(W,Z)\xi - \eta(Z)W) + \beta(g(\phi W,Z)\xi \\ &- \eta(Z)\phi W)] - [\alpha g(X,Z) + \beta g(\phi X,Z)][\{-\alpha g(\phi W,Y) \\ &+ \beta g(\phi W,\phi Y)\}\xi + \eta(Y)\{-\alpha\phi W + \beta(W - \eta(W)\xi)\}] \\ &+ [\alpha g(Y,Z) + \beta g(\phi Y,Z)][\{-\alpha g(\phi W,X) + \beta g(\phi W,\phi X)\}\xi \\ &+ \eta(X)\{-\alpha\phi W + \beta(W - \eta(W)\xi)\}] \\ &+ (\alpha X + \beta\phi X)[\{-\alpha g(\phi W,Y) + \beta g(\phi W,\phi Y)\}\eta(Z) \\ &+ \eta(Y)\{-\alpha g(\phi W,Z) + \beta g(\phi W,\phi Z)\}] + \beta[\alpha(g(W,X)\xi \\ &- \eta(X)W) + \beta(g(\phi W,X)\xi - \eta(X)\phi W)]\eta(Y)\eta(Z) \\ &- (\alpha Y + \beta\phi Y)[\{-\alpha g(\phi W,X) + \beta g(\phi W,\phi X)\}\eta(Z) \\ &+ \eta(X)\{-\alpha g(\phi W,Z) + \beta g(\phi W,\phi Z)\}] - \beta[\alpha(g(W,Y)\xi \\ &- \eta(Y)W) + \beta(g(\phi W,Y)\xi - \eta(Y)\phi W)]\eta(X)\eta(Z).\end{aligned} \tag{4.4}$$

Using (2.1) and (4.4) in (4.3) and applying ϕ^2, we get

$$\begin{aligned}\phi^2((\widetilde{\nabla}_W \widetilde{R})(X,Y)Z) &= \phi^2((\nabla_W R)(X,Y)Z) \\ &+ 2\alpha g(\phi Y,X)[\alpha\phi^2 W + \beta\phi^2(\phi W)]\eta(Z) \\ &- \eta(Y)[\alpha g(X,Z) + \beta g(\phi X,Z)][-\alpha\phi^2(\phi W) + \beta\phi^2 W] \\ &+ \eta(X)[\alpha g(Y,Z) + \beta g(\phi Y,Z)][-\alpha\phi^2(\phi W) + \beta\phi^2 W] \\ &+ (\alpha\phi^2 X + \beta\phi^2(\phi X))[\{-\alpha g(\phi W,Y) + \beta g(\phi W,\phi Y)\}\eta(Z) \\ &+ \eta(Y)\{-\alpha g(\phi W,Z) + \beta g(\phi W,\phi Z)\}] \\ &- (\alpha\phi^2 Y + \beta\phi^2(\phi Y))[\{-\alpha g(\phi W,X) + \beta g(\phi W,\phi X)\}\eta(Z) \\ &+ \eta(X)\{-\alpha g(\phi W,Z) + \beta g(\phi W,\phi Z)\}] \\ &- \eta(W)\phi^2(\phi(\widetilde{R}(X,Y)Z)).\end{aligned} \tag{4.5}$$

If we consider X, Y, Z and W orthogonal to ξ then (4.5) reduces to

$$\phi^2((\widetilde{\nabla}_W \widetilde{R})(X,Y)Z) = \phi^2((\nabla_W R)(X,Y)Z). \tag{4.6}$$

Hence we can state:

Theorem 4.1. *In a three dimensional trans-Sasakian manifold, the locally ϕ-symmetric remains invariant under quarter-symmetric metric connection $\widetilde{\nabla}$ and Levi-Civita connection ∇.*

5 ϕ-Symmetric trans-Sasakian manifold with respect to the quarter-symmetric metric connection

Definition 5.1. A trans-Sasakian manifold M is said to be ϕ-symmetric if

$$\phi^2((\nabla_W R)(X,Y)Z) = 0, \tag{5.1}$$

for any arbitrary vector fields X, Y, Z and W on M.

Similarly, a trans-Sasakian manifold M is said to be ϕ-symmetric with respect to the quarter-symmetric metric connection if

$$\phi^2((\widetilde{\nabla}_W \widetilde{R})(X,Y)Z) = 0, \tag{5.2}$$

for arbitrary vector fields X, Y, Z and W.
Now by virtue of (2.1), (5.2) gives

$$-(\widetilde{\nabla}_W \widetilde{R})(X,Y)Z + \eta((\widetilde{\nabla}_W \widetilde{R})(X,Y)Z)\xi = 0, \tag{5.3}$$

from which it follows that

$$-g((\widetilde{\nabla}_W \widetilde{R})(X,Y)Z,U) + \eta((\widetilde{\nabla}_W \widetilde{R})(X,Y)Z)\eta(U) = 0. \tag{5.4}$$

Let $\{e_i\}, i = 1,2,3$ be an orthonormal basis of the tangent space at any point of the space form. Then putting $X = U = e_i$, in (5.4) and taking summation over i, $1 \leq i \leq 3$, we get

$$-(\widetilde{\nabla}_W \widetilde{S})(Y,Z) + \sum \eta((\widetilde{\nabla}_W \widetilde{R})(e_i,Y)Z)\eta(e_i) = 0. \tag{5.5}$$

The second term of (5.5) by putting $Z = \xi$ takes the form

$$\eta((\widetilde{\nabla}_W \widetilde{R})(e_i,Y)\xi)\eta(e_i) = g((\widetilde{\nabla}_W \widetilde{R})(e_i,Y)\xi,\xi)g(e_i,\xi). \tag{5.6}$$

By using (3.5) and (4.3), we get

$$g((\widetilde{\nabla}_W \widetilde{R})(e_i,Y)\xi,\xi) = g((\nabla_W \widetilde{R})(e_i,Y)\xi,\xi) - \eta(W)\eta(\phi\widetilde{R}(e_i,Y)\xi). \tag{5.7}$$

On simplification we obtain from (5.7) that

$$g((\widetilde{\nabla}_W \widetilde{R})(e_i,Y)\xi,\xi) = g((\nabla_W R)(e_i,Y)\xi,\xi). \tag{5.8}$$

In a trans-Sasakian manifold M, we have $g((\nabla_W R)(e_i,Y)\xi,\xi) = 0$ and so from (5.8), we have

$$g((\widetilde{\nabla}_W \widetilde{R})(e_i,Y)\xi,\xi) = 0. \tag{5.9}$$

By replacing Z by ξ in (5.5) and using (5.9), we get

$$(\widetilde{\nabla}_W \widetilde{S})(Y,\xi) = 0. \tag{5.10}$$

We know that

$$(\widetilde{\nabla}_W \widetilde{S})(Y,\xi) = \widetilde{\nabla}_W \widetilde{S}(Y,\xi) - \widetilde{S}(\widetilde{\nabla}_W Y,\xi) - \widetilde{S}(Y,\widetilde{\nabla}_W \xi). \tag{5.11}$$

By making use of (2.4), (2.9), (3.5) and (3.7) in (5.11), we get

$$\begin{aligned} \beta S(Y,W) - \alpha S(Y,\phi W) &= 2(\alpha^2 - \beta^2 + \alpha)[-\alpha g(Y,\phi W) + \beta g(\phi Y,\phi W)] \\ &- (\alpha^2 - \beta^2)g(Y,\phi W) + 2\alpha\beta g(\phi Y,\phi W) \\ &+ 2\beta(\alpha^2 - \beta^2)\eta(W)\eta(Y). \end{aligned} \tag{5.12}$$

Replacing Y by ϕY and W by ϕW, we get

$$\begin{aligned} &\beta[S(Y,W) - 2(\alpha^2 - \beta^2 + \alpha)g(Y,W) + 2n\alpha\eta(Y)\eta(W) + \beta g(W,\phi Y) - \alpha g(\phi Y,\phi W)] \\ + \quad &\alpha[S(\phi Y,W) - 2(\alpha^2 - \beta^2 + \alpha)g(\phi Y,W) - \beta g(\phi W,\phi Y) - \alpha g(\phi Y,W)] = 0. \end{aligned}$$

This implies

$$\begin{aligned} &S(Y,W) - 2(\alpha^2 - \beta^2 + \alpha)g(Y,W) + 2\alpha\eta(Y)\eta(W) + \beta g(W,\phi Y) \\ - \quad &\alpha g(\phi Y,\phi W) = 0 \end{aligned} \tag{5.13}$$

$$\begin{aligned} &and \\ &S(\phi Y,W) - 2(\alpha^2 - \beta^2 + \alpha)g(\phi Y,W) - \beta g(\phi W,\phi Y) - \alpha g(\phi Y,W) = 0. \end{aligned} \tag{5.14}$$

Contracting (5.13) and (5.14), we get

$$r = 6(\alpha^2 - \beta^2 + \alpha). \tag{5.15}$$

Therefore we can state,

Theorem 5.1. *Let M be a three dimensional ϕ-symmetric trans-Sasakian manifold with respect to quarter symmetric metric connection $\widetilde{\nabla}$. Then the manifold has a scalar curvature r with respect to Levi-Civita connection ∇ of M given by (5.15).*

Case I: *Considering $\alpha = 1$ and $\beta = 0$ in (5.15) we get $r = 12$ which is proved by Pradeep Kumar K. T., Bagewadi C. S. and Venkatesha in (17) for a Sasakian manifold.*

Case II: *And considering $\alpha = 0$ and $\beta = 1$ in (5.15) we get $r = -6$ which is proved by Prakasha D. G. in (19) for a Kenmotsu manifold.*

6 Locally concircular ϕ-symmetric trans-Sasakian manifold with respect to quarter symmetric metric connection

Definition 6.1. A trans-Sasakian manifold M is said to be locally concircular ϕ-symmetric if

$$\phi^2((\nabla_W \widetilde{C})(X,Y)Z) = 0, \tag{6.1}$$

for any arbitrary vector fields X, Y, Z, W orthogonal to ξ, where $\widetilde{C}$ is the concircular curvature tensor given by (31).

$$\widetilde{C}(X,Y)Z = R(X,Y)Z - \frac{r}{6}[g(Y,Z)X - g(X,Z)Y]. \tag{6.2}$$

A trans-Sasakian manifold M is said to be locally concircular ϕ-symmetric with respect to the quarter-symmetric metric connection if

$$\phi^2((\widetilde{\nabla}_W \widetilde{\widetilde{C}})(X,Y)Z) \quad = \quad 0. \tag{6.3}$$

for all vector field X, Y, Z and W orthogonal to ξ, where $\widetilde{\widetilde{C}}$ is the concircular curvature tensor with respect to quarter symmetric metric connection given by

$$\widetilde{\widetilde{C}}(X,Y)Z = \widetilde{R}(X,Y)Z - \frac{\widetilde{r}}{6}[g(Y,Z)X - g(X,Z)Y], \tag{6.4}$$

where $\widetilde{R}$ and $\widetilde{r}$ are the Riemannian curvature tensor and the scalar curvature with respect to quarter symmetric metric connection.
Using (3.5), we can write

$$(\widetilde{\nabla}_W \widetilde{\widetilde{C}})(X,Y)Z = (\nabla_W \widetilde{\widetilde{C}})(X,Y)Z) - \eta(W)\phi\widetilde{\widetilde{C}}(X,Y)Z. \tag{6.5}$$

Now differentiating (6.4) covariantly with respect to W, we obtain

$$(\widetilde{\nabla}_W \widetilde{\widetilde{C}})(X,Y)Z = (\nabla_W \widetilde{R})(X,Y)Z - \frac{\nabla_W \widetilde{r}}{6}[g(Y,Z)X - g(X,Z)Y]. \tag{6.6}$$

Using (3.8) and (4.4) in (6.6), we have

$$\begin{aligned}(\nabla_W \widetilde{\widetilde{C}})(X,Y)Z &= (\nabla_W R)(X,Y)Z - 2\alpha g(\phi Y,X)[\alpha(g(W,Z)\xi \\ &- \eta(Z)W) + \beta(g(\phi W,Z)\xi - \eta(Z)\phi W)] \\ &- [\alpha g(X,Z) + \beta g(\phi X,Z)][\{-\alpha g(\phi W,Y) \\ &+ \beta g(\phi W,\phi Y)\}\xi + \eta(Y)\{-\alpha\phi W + \beta(W - \eta(W)\xi)\}] \\ &+ [\alpha g(Y,Z) + \beta g(\phi Y,Z)][\{-\alpha g(\phi W,X) \\ &+ \beta g(\phi W,\phi X)\}\xi + \eta(X)\{-\alpha\phi W + \beta(W - \eta(W)\xi)\}] \\ &+ (\alpha X + \beta\phi X)[\{-\alpha g(\phi W,Y) + \beta g(\phi W,\phi Y)\}\eta(Z) \\ &+ \eta(Y)\{-\alpha g(\phi W,Z) + \beta g(\phi W,\phi Z)\}] + \beta[\alpha(g(W,X)\xi \\ &- \eta(X)W) + \beta(g(\phi W,X)\xi - \eta(X)\phi W)]\eta(Y)\eta(Z) \\ &- (\alpha Y + \beta\phi Y)[\{-\alpha g(\phi W,X) + \beta g(\phi W,\phi X)\}\eta(Z) \\ &+ \eta(X)\{-\alpha g(\phi W,Z) + \beta g(\phi W,\phi Z)\}] - \beta[\alpha(g(W,Y)\xi \\ &- \eta(Y)W) + \beta(g(\phi W,Y)\xi - \eta(Y)\phi W)]\eta(X)\eta(Z) \\ &- \frac{\nabla_W \widetilde{r}}{6}[g(Y,Z)X - g(X,Z)Y].\end{aligned} \tag{6.7}$$

Again using (6.2), the equation (6.7) reduces to

$$\begin{aligned}(\nabla_W \widetilde{\widetilde{C}})(X,Y)Z &= (\nabla_W C)(X,Y)Z - 2\alpha g(\phi Y,X)[\alpha(g(W,Z)\xi \\ &- \eta(Z)W) + \beta(g(\phi W,Z)\xi - \eta(Z)\phi W)] \\ &- [\alpha g(X,Z) + \beta g(\phi X,Z)][\{-\alpha g(\phi W,Y) \\ &+ \beta g(\phi W,\phi Y)\}\xi + \eta(Y)\{-\alpha\phi W + \beta(W - \eta(W)\xi)\}] \\ &+ [\alpha g(Y,Z) + \beta g(\phi Y,Z)][\{-\alpha g(\phi W,X) \\ &+ \beta g(\phi W,\phi X)\}\xi + \eta(X)\{-\alpha\phi W + \beta(W - \eta(W)\xi)\}] \\ &+ (\alpha X + \beta\phi X)[\{-\alpha g(\phi W,Y) + \beta g(\phi W,\phi Y)\}\eta(Z) \\ &+ \eta(Y)\{-\alpha g(\phi W,Z) + \beta g(\phi W,\phi Z)\}] + \beta[\alpha(g(W,X)\xi \\ &- \eta(X)W) + \beta(g(\phi W,X)\xi - \eta(X)\phi W)]\eta(Y)\eta(Z) \\ &- (\alpha Y + \beta\phi Y)[\{-\alpha g(\phi W,X) + \beta g(\phi W,\phi X)\}\eta(Z) \\ &+ \eta(X)\{-\alpha g(\phi W,Z) + \beta g(\phi W,\phi Z)\}] - \beta[\alpha(g(W,Y)\xi \\ &- \eta(Y)W) + \beta(g(\phi W,Y)\xi - \eta(Y)\phi W)]\eta(X)\eta(Z).\end{aligned} \tag{6.8}$$

Using (2.1) and (6.8) in (6.5) and applying ϕ^2, we get

$$\begin{aligned}\phi^2(\widetilde{\nabla}_W \widetilde{\widetilde{C}})(X,Y)Z &= \phi^2((\nabla_W \widetilde{C})(X,Y)Z) + 2\alpha g(\phi Y,X)[\alpha\phi^2 W + \beta\phi^2(\phi W)]\eta(Z) \\ &- \eta(Y)[\alpha g(X,Z) + \beta g(\phi X,Z)][-\alpha\phi^2(\phi W) + \beta\phi^2 W] \\ &+ \eta(X)[\alpha g(Y,Z) + \beta g(\phi Y,Z)][-\alpha\phi^2(\phi W) + \beta\phi^2 W] \\ &+ (\alpha\phi^2 X + \beta\phi^2(\phi X))[\{-\alpha g(\phi W,Y) + \beta g(\phi W,\phi Y)\}\eta(Z) \\ &+ \eta(Y)\{-\alpha g(\phi W,Z) + \beta g(\phi W,\phi Z)\}] \\ &- (\alpha\phi^2 Y + \beta\phi^2(\phi Y))[\{-\alpha g(\phi W,X) + \beta g(\phi W,\phi X)\}\eta(Z) \\ &+ \eta(X)\{-\alpha g(\phi W,Z) + \beta g(\phi W,\phi Z)\}] \\ &- \eta(W)\phi^2(\phi(\widetilde{\widetilde{C}}(X,Y)Z)).\end{aligned} \tag{6.9}$$

If we consider X, Y, Z and W orthogonal to ξ, then (6.9) reduces to

$$\phi^2(\widetilde{\nabla}_W \widetilde{\widetilde{C}})(X,Y)Z = \phi^2((\nabla_W \widetilde{C})(X,Y)Z). \tag{6.10}$$

Hence we state:

Theorem 6.1. *In a three dimensional trans-Sasakian manifold, locally concircular ϕ-symmetric remains invariant under quarter symmetric metric $\widetilde{\nabla}$ and Levi-Civita connection ∇.*

In view of (2.1) and (6.7), we have from (6.5) that

$$\begin{aligned}\phi^2(\widetilde{\nabla}_W\widetilde{\widetilde{C}})(X,Y)Z &= \phi^2((\nabla_W R)(X,Y)Z) + 2\alpha g(\phi Y,X)[\alpha\phi^2 W + \beta\phi^2(\phi W)]\eta(Z) \\ &- \eta(Y)[\alpha g(X,Z) + \beta g(\phi X,Z)][-\alpha\phi^2(\phi W) + \beta\phi^2 W] \\ &+ \eta(X)[\alpha g(Y,Z) + \beta g(\phi Y,Z)][-\alpha\phi^2(\phi W) + \beta\phi^2 W] \\ &+ (\alpha\phi^2 X + \beta\phi^2(\phi X))[\{-\alpha g(\phi W,Y) + \beta g(\phi W,\phi Y)\}\eta(Z) \\ &+ \eta(Y)\{-\alpha g(\phi W,Z) + \beta g(\phi W,\phi Z)\}] \\ &- (\alpha\phi^2 Y + \beta\phi^2(\phi Y))[\{-\alpha g(\phi W,X) + \beta g(\phi W,\phi X)\}\eta(Z) \\ &+ \eta(X)\{-\alpha g(\phi W,Z) + \beta g(\phi W,\phi Z)\}] \\ &- \eta(W)\phi^2(\phi(\widetilde{\widetilde{C}}(X,Y)Z)) \\ &- \frac{\nabla_W\widetilde{r}}{6}[g(Y,Z)\phi^2 X - g(X,Z)\phi^2 Y].\end{aligned} \tag{6.11}$$

If we consider X, Y, Z and W orthogonal to ξ, then (6.11) reduces to

$$\begin{aligned}\phi^2(\widetilde{\nabla}_W\widetilde{\widetilde{C}})(X,Y)Z &= \phi^2((\nabla_W R)(X,Y)Z) \\ &- \frac{\nabla_W\widetilde{r}}{6}[g(Y,Z)\phi^2 X - g(X,Z)\phi^2 Y].\end{aligned} \tag{6.12}$$

If r is constant, then $\nabla_W r = 0$. Therefore (6.12) yields

$$\phi^2(\widetilde{\nabla}_W\widetilde{\widetilde{C}})(X,Y)Z = \phi^2((\nabla_W R)(X,Y)Z). \tag{6.13}$$

Hence we state:

Theorem 6.2. *A three dimensional locally concircular ϕ-symmetric trans-Sasakian manifold admitting quarter symmetric metric connection is locally ϕ-symmetric if and only if the scalar curvature r is constant with respect to Levi-Civita connection.*

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