STABILITY ANALYSIS OF A SEIV EPIDEMIC MODEL WITH SATURATED INCIDENCE RATE

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ABSTRACT

In this paper, a SEIV epidemic model with saturated incidence rate that incorporates polynomial information on current and past states of the disease is investigated. The model exhibits two equilibria, disease-free equilibrium (DFE) and the endemic equilibrium (EE). It is shown that if the basic reproduction number, $R_0 < a_0$, where $a_0 < 1$, the DFE is locally asymptotically stable and by the use of Lyapunov function, DFE is globally asymptotically stable and in such a case, the EE is unstable. Moreover, if $R_0 > a_0$, the endemic equilibrium is locally asymptotically stable.

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1. INTRODUCTION

Vaccinating susceptible against disease infections is an effective measure to control and prevent the spread of the infection. Kribs-Zaleta and Velasco-Hemadez (2000) investigated SIS model with vaccination, standard incidence and no disease-induced deaths. Arino et al., (2003) formulated an SIRS model with vaccinations standard incidence and no disease-induced deaths. Li et al., (2006) studied an SIS model with vaccination, standard incidence and the disease induced death. Li and Ma (2002, 2004b) all analyzed global behaviour of simple SIS vaccination epidemic models under the condition that the vaccine is perfectly efficient. Li and Ma (2004a) introduced a vaccination compartment into an SIS model. They assumed that in vaccine takes effect in duration of length τ .

Long and Xiang (2011) studied an SEIV epidemic model with vaccination and vertical transmission. Their results were written in form of basic reproduction number, R_0 and they carried out a bifurcation analysis and obtain the conditions ensuring that the system exhibits backward bifurcation. Chika and Ezeofor (2011) extended Cai and Li (2009) model by incorporating important practical factors as the proportion of recruited individuals that are exposed or infectious, the recovery rate of exposed individuals and the mortality rate due to the infection. Abdulrazak et al. (2012) discussed a vaccination model with non-linear incidence rate and vaccination waning period like (Cai and Li, 2009) but they numerically simulated the model to see how variations in rate of the variables. Capasso and Serio (1978) introduced a saturated incidence rate g (I) S into epidemic models, where g(I) tends to a saturation level when I gets large, i.e. $g(I) = \frac{kI}{1+\alpha I}$ where kI measures the infection force of the disease and $\frac{1}{1+\alpha I}$ measures the inhibition effect from the behavioural change of the susceptible individuals when their number increases or from the crowding effect of the infective individual. In this paper, we extend the work of Cai and Li (2009) to incorporate a saturated incidence rate as used by Capasso and Serio (1978) with a condition that $a_0 > 1$.

$$\frac{dS}{dt} = (1-\rho)\pi - \frac{\beta SI}{a_0 + \alpha I} - \mu S + \omega V$$
$$\frac{dE}{dt} = \frac{\beta SI}{a_0 + \alpha I} - (\mu + \sigma)E \qquad (1.1)$$
$$\frac{dI}{dt} = \sigma E - (\mu + \gamma)I$$
$$\frac{dV}{dt} = \rho\pi - \mu V + \gamma I - \omega V$$

where

π = recruitment of individuals that includes new born and immigrants into the susceptible population S, ρ = fraction of recruited individuals who are vaccinated, β **= rate at which susceptible individual become infected by those who are infectious**, μ = natural death rate, σ = rate at which exposed individuals become infectious, γ= rate at which infected individuals are treated or recovered, $\frac{1}{a_0 + \alpha(I)}$ = measurement of psychological or inhibition effect of the behavioural change of the susceptible individuals when there is an increase in the number of infective individuals, ω = rate at which vaccine wanes

2. THE BASIC REPRODUCTION NUMBER

It is easy to see that the region $\{(S,E,I,V), S > 0, I \geq 0, E \geq 0, V > 0\}$ is positively invariant for the model (1.1). Summing up the four equations in model (1.1), we have

$$\frac{d}{dt}(S + E + I + V) = \mu\,[\frac{\pi}{\mu} - (S + E + I + V)].$$

Then, $\lim_{t\to\infty} \sup\,(S + E + I + V) \leq \frac{\pi}{\mu}$. So, we study the dynamic behaviour of model (1.1) on the region

$$\sum = \{(S,E,I,V) \;\; S>0, I\geq 0, E\geq 0, V>0, \; S + E + I + V \leq \frac{\pi}{\mu}\},$$

which is a positive invariant set for (1.1)

Corresponding to E = I = 0, model (1.1) always has a disease-free equilibrium,

$$P_0 = \left(\frac{(\mu+\omega-\rho\mu)\pi}{\mu(\mu+\omega)}, \quad 0, \quad 0 \;, \frac{\rho\pi}{\mu+\omega} \right).$$

Let $x = (S\,, E, I\,, V)^T$ **.Then the model (1.1) can be written as** $\frac{dx}{dt} = F\,(x) - V(x)$

where $F(x) = \begin{pmatrix} \frac{\beta SI}{a_0+\alpha I} \\ 0 \\ 0 \\ 0 \end{pmatrix}$ **,** $V(x) = \begin{pmatrix} \mu E + \sigma E \\ \gamma I + \mu I - \sigma E \\ -(1-\rho)\pi + \frac{\beta SI}{a_0+\alpha I} - \omega V + \mu S \\ \mu V + \omega V - \rho\pi - \gamma I \end{pmatrix}$

We have $F = \begin{pmatrix} 0 & \frac{\beta S_0}{a_0} \\ 0 & 0 \end{pmatrix}$ $V = \begin{pmatrix} \mu+\sigma & 0 \\ -\sigma & \mu+\gamma \end{pmatrix}$

So, $V^{-1} = \begin{pmatrix} \frac{1}{\mu+\sigma} & 0 \\ \frac{\sigma}{(\mu+\sigma)(\mu+\gamma)} & \frac{1}{\mu+\gamma} \end{pmatrix}$

$$FV^{-1} = \begin{pmatrix} \frac{\sigma\beta S_0}{a_0(\mu+\sigma)(\mu+\gamma)} & \frac{\beta S_0}{a_0(\mu+\gamma)} \\ 0 & 0 \end{pmatrix}$$

In van den Driessche and Watmough (2002), the basic reproduction number is defined as the spectral radius of the next generation matrix

$$FV^{-1} \quad \left(\rho\left(FV^{-1}\right)\right).$$

So, according to Theorem (2) in van den Driessche and Watmough (2002), the basic reproduction number of model (1.1),is

$$R_0 = \ell\left(FV^{-1}\right) = \frac{\sigma\beta S_0}{a_0(\mu+\sigma)(\mu+\gamma)}$$

where $S_0 = \dfrac{(\mu+\sigma)(1-\rho)\pi + \omega\rho\pi}{\mu(\mu+\omega)}$

Therefore,

$$R_0 = \frac{\sigma\beta[(\mu+\omega)(1-\rho)\pi + \omega\rho\pi]}{a_0\mu(\mu+\omega)(\mu+\sigma)(\mu+\gamma)}$$

Note: The spectral radius $\left(\rho[FV^{-1}]\right)$ **is the dominant eigenvalue of the matrix FV^{-1}.**

3. LOCAL AND GLOBAL STABILITY OF EQUILIBRIA

Let $x = S - S_0, \qquad I = I, \quad E = E, \qquad y = V - V_0$ (3.1)

Then, equation (1.1) becomes

$$\frac{dS}{dt} = (1-\rho)\pi - \beta I\left(x + \frac{(\mu+\omega)(1-\rho)\pi + \omega\rho\pi}{\mu(\mu+\omega)} \right)(a_0 + \alpha I)^{-1} -$$

$$\mu\left(x + \frac{(\mu+\omega)(1-\rho)\pi + \omega\rho\pi}{\mu(\mu+\omega)} \right) + \omega\left(y + \frac{\rho\pi}{\mu+\omega} \right)$$

$$\frac{dE}{dt} = \beta I\left(x + \frac{(\mu+\omega)(1-\rho)\pi + \omega\rho\pi}{\mu(\mu+\omega)} \right)(a_0 + \alpha I)^{-1} - (\mu+\sigma)E \qquad (3.2)$$

$$\frac{dI}{dt} = \sigma E - (\mu+\gamma)I$$

$$\frac{dV}{dt} = \rho\pi - (\mu+\omega)\left(y + \frac{\rho\pi}{\mu+\omega} \right) + \gamma I$$

The Jacobian matrix of equation (3.2) after linearization is

$$A = \begin{pmatrix} -\mu & 0 & \dfrac{-\beta[(\mu+\omega)(1-\rho)\pi + \omega\rho\pi]}{a_0\mu(\mu+\omega)} & \omega \\ 0 & -(\mu+\sigma) & \dfrac{\beta[(\mu+\omega)(1-\rho)\pi + \omega\rho\pi]}{a_0\mu(\mu+\omega)} & 0 \\ 0 & \sigma & -(\mu+\gamma) & 0 \\ 0 & 0 & \gamma & -(\mu+\omega) \end{pmatrix}$$

Let $\dfrac{\beta[(\mu+\omega)(1-\rho)\pi + \omega\rho\pi]}{\mu(\mu+\omega)} = \Gamma$

$$|A - \lambda I| = \begin{vmatrix} -(\mu+\lambda) & 0 & \dfrac{-\Gamma}{a_0} & \omega \\ 0 & -(\mu+\sigma+\lambda) & \dfrac{\Gamma}{a_0} & 0 \\ 0 & \sigma & -(\mu+\gamma+\lambda) & 0 \\ 0 & 0 & \gamma & -(\mu+\omega+\lambda) \end{vmatrix} = 0$$

Therefore, $\lambda_1 = -\mu$, $\lambda_2 = -(\mu+\omega)$, and the roots of the quadratic

$$\lambda^2 + (2\mu+\sigma+\gamma)\lambda + \mu^2 + \mu\gamma + \sigma\mu + \sigma\gamma - \frac{\sigma\Gamma}{a_0} = 0$$

For negative roots, we must have by Descartes' rule of signs

$$\mu^2 + \mu\gamma + \sigma\mu + \sigma\gamma - \frac{\sigma\Gamma}{a_0} > 0$$

That is,

$$(\mu+\sigma)(\mu+\gamma) > \sigma\beta\frac{[(\mu+\omega)(1-\rho)\pi + \omega\rho\pi]}{a_0\mu(\mu+\omega)}$$

$$\sigma\beta\frac{[(\mu+\omega)(1-\rho)\pi + \omega\rho\pi]}{a_0\mu(\mu+\omega)(\mu+\sigma)(\mu+\gamma)} < 1$$

Therefore,

$$R_0 = \frac{\sigma\beta\,[(\mu+\omega)(1-\rho)\pi + \omega\rho\pi]}{a_0\mu(\mu+\omega)(\mu+\sigma)(\mu+\gamma)} < 1$$

Theorem 3.1: If $R_0<a_0$, the disease free equilibrium P_0 is locally asymptotically stable; if $R_0=a_0$, P_0 is stable; if $R_0>a_0$, P_0 is unstable

Proof: We shall check the stability of the disease-free equilibrium P_0, from the model, thus the linearization of model of the disease-free equilibrium P_0 gives the following characteristics equation.

$$-(\mu+\lambda)(\mu+\omega+\lambda)\left[(\mu+\gamma+\lambda)(\mu+\sigma+\lambda) - \frac{\sigma\beta[(\mu+\omega)(1-\rho)\pi + \omega\rho\pi]}{a_0\mu(\mu+\omega)}\right] = 0 \qquad (3.3)$$

Thus, we have

$$-(\mu+\lambda)(\mu+\omega+\lambda)\left[\lambda^2+(2\mu+\gamma+\sigma)\lambda+\mu^2+\mu\gamma+\sigma\mu+\sigma\gamma-\frac{\sigma\beta[(\mu+\omega)(1-\rho)\pi+\omega\rho\pi]}{a_0\mu(\mu+\omega)}\right]=0 \quad (3.4)$$

From equation (3.4), it can be seen that $\lambda_1=-\mu, \quad \lambda_2=-(\mu+\omega)$ are two of the eigenvalues and they are always negative. To obtain the other eigenvalues of equation (3.4) we consider the equation

$$\lambda^2+(2\mu+\gamma+\sigma)\lambda+\mu^2+\mu\gamma+\sigma\mu+\sigma\gamma-\frac{\sigma\beta[(\mu+\omega)(1-\rho)\pi+\omega\rho\pi]}{a_0\mu(\mu+\omega)}=0 \quad (3.5)$$

From equation (3.5) we see that all roots have negative real parts if

$$\mu^2+\mu\gamma+\sigma\mu+\sigma\gamma-\frac{\sigma\beta[(\mu+\omega)(1-\rho)\pi+\omega\rho\pi]}{a_0\mu(\mu+\omega)}>0$$

That is, if $R_0<a_0$.

:. The disease-free equilibrium P_0, is locally asymptotically stable.

If $R_0=a_0$, one eigenvalue of equation (3.5) is zero and it is simple. Then P_0 is stable.

If $R_0>a_0$, one of the roots of equation (3.5) has a positive real part, then P_0 is unstable.

Theorem 3.2: If $R_0<a_0$, the disease-free equilibrium P_0 is globally asymptotically stable.

Proof: Consider the Lyapunov function $L(t) = \sigma E + (\mu+\sigma)\, I$. Its derivative along the system (1.1) is

$$L^1(t)=\frac{\sigma\beta S_0 I}{a_0+\alpha I}-(\mu+\sigma)(\mu+\gamma)I$$

$$L^1(t) = [\frac{\sigma\beta S_0}{(a_0+\alpha I)\,(\mu+\sigma)(\mu+\gamma)}-1]\,I$$

$$L^1(t) = [\frac{R_0}{(a_0+\alpha I)}-1]$$

If $R_0 < a_0$, $I \neq 0$, $L^1(t) < 0$

Therefore, disease-free equilibrium is globally asymptotically stable

4. CONCLUSION AND DISCUSSION

This paper has considered a SEIV model that incorporates an incidence rate of the form $\frac{\beta SI}{a_0 + \alpha I}$ where $a_0 > 1$. We discussed the local and global stabilities of the disease-free equilibrium. If $a_0 + \alpha I + \ldots.\ where\ a_0 \geq 1$, which gives information about the states of the disease, then the basic reproduction number is proportional to $\frac{1}{a_0}$. Since the basic reproduction number R_0 of an infection is the mean number of secondary cases a typical single infected case will cause in a population with no immunity to the disease in the absence of interventions to control the infection, the information on past history of the disease is a good way to control the disease.

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