

Bayesian Inference Based on the Interval Grouped Data from the Weibull Model with Application

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ABSTRACT

In this paper, based on the interval grouped data, Bayesian approach is used to obtain estimator for the Weibull scale parameter, and some lifetime such as the reliability and hazard functions. The estimation procedures have been developed and modified under squared error loss function (SELF) and general entropy loss function (GELF). The estimators are derived using the inverted gamma conjugate prior. Credible intervals and high posterior density (HPD) credible intervals are also obtained. Prediction for the future number of failures in the corresponding intervals is presented. Finally, real life example is applied to illustrate the performance of the estimation procedures.

Keywords: Weibull distribution, Interval grouped data, Bayesian estimation, Loss functions, High posterior density estimator, Credible intervals, Prediction.

1. INTRODUCTION

The Weibull distribution has been extensively used in life testing and reliability studies. Initially, it was proposed by Weibull [1] for describing the fatigue failures from the wear out materials. Lieblein and Zelen [2] have used the model for the ball bearing failures. Recently, the Weibull statistics have become the most popular to describe the breakdown events in solid dielectrics [3]. Weibull distribution also provides a good model for lives of electrical and mechanical components or systems [4]. Reliability and survival analysis based on the weibull distribution are applied and discussed by many authors [5-9].

The density function for the Weibull distribution with parameters α, β is given by

$$f(t) = \frac{\beta}{\theta} t^{\beta-1} \exp\left(-\frac{t^\beta}{\theta}\right) \quad ; \quad t > 0 ; \theta > 0 ; \beta > 0 \quad (1)$$

Hence, the hazard rate and the reliability functions are given respectively by

$$h(t) = \frac{\beta}{\theta} t^{\beta-1} \quad , \quad t > 0 \quad (2)$$

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$$R(t) = \exp\left(-\frac{t^\beta}{\theta}\right), t > 0 \quad (3)$$

In the normal used conditions, the inspection units have long time for failure. Thus, having a complete failure data set is costly and time consuming. For this reason, censored life testing experiments are performed either by terminating the life test at a pre-assigned fixed single time, say T^* to have "Type I" censored data or after a pre-assigned number of failures at the random variable time $t(r)$ to have "Type II" censored data. Using both types of censoring, progressively censored data may be obtained when at different stages of the experiment; some of the surviving units remaining are censored from further observation. Examples are doubly, progressive type I and type II and hybrid progressive type I and type II censored data.

In other situations when continuous monitoring is not feasible, the intermittent inspection life testing is the only probable procedure by testing the failure units periodically. In the reliability and survival data analysis, the data obtained from the intermittent inspection are often called interval grouped data. Hence, the sample space of the life testing can be partitioned in different ways of selecting the end points to achieve the experimenter intention, the range and the source from which the data are collected. Interval grouped data are widely used in survival and reliability analysis [10-13].

Using different types of failure time's data, many researches are devoted for estimating the Weibull parameters. Nicholas et al. [14] has proved the uniqueness of the maximum likelihood estimators obtained from a complete data. He also presented a regression method approach using both complete and censored data. Using the censored data, maximum likelihood and least square regression estimators are applied to the Weibull parameters [15]. Nadarajah [16] has used progressively "Type II" censored data for the moments of the Weibull distribution. Kundu [17] presents maximum likelihood and Bayesian inference procedures for the parameters when the data are mixture of "Type I" and "Type II" censoring schemes. Other referees of interest may be found in Varian [18], Berger and Sun [19], Thompson and Basu [20], Zhang et al. [21], Chen et al. [22].

In order to select a single value δ as the Bayesian estimator of θ , a loss function must be specified. A frequently used symmetric loss function is the squared error loss function (SELF) defined by

$$L_s(\theta; \delta) = (\theta - \delta)^2 \quad (4)$$

Hence, the corresponding Bayesian estimator is the posterior mean [23].

In many practical situations, it appears more realistic to express the loss function in terms of the ratio $\frac{\delta}{\theta}$. In this case, a useful symmetric loss function is the general entropy loss function (GELF) proposed by Calabria and Pulcini [24] defined by

$$L_g(\theta; \delta) = \left(\frac{\delta}{\theta}\right)^m - \theta \log\left(\frac{\delta}{\theta}\right) - 1 \quad (5)$$

Whose minimum occurs at $\delta = \theta$. This loss function is a generalization of the entropy loss which used by many authors [25-26]. Hence, the Bayesian estimator under GELF is obtained by

$$\theta_{Bg} = \left(E_{\tilde{\Pi}} (\theta^{-m}) \right)^{-\frac{1}{m}} \quad (6)$$

where $\tilde{\Pi}$ is the corresponding posterior.

2. LIKELIHOOD, PRIOR, AND POSTERIOR

Suppose that the lifetime T follow the Weibull distribution (θ, β) and assume the time scale line is divided into intervals by the constant inspection times a_j , $j = 1, 2, \dots, k$. Assuming $a_0 = 0$ and $a_{k+1} = \infty$, then the corresponding intervals are expressed as $I_1 = [0, a_1)$, $I_2 = [a_1, a_2)$, ..., $I_k = [a_{k-1}, a_k)$, $I_{k+1} = [a_k, \infty)$. If n units are put on the test for failure and the experimenter intermittently records the failure data which consists of the number of failures f_j in the intervals I_j , $j = 1, 2, \dots, k+1$ then the probability of failure in the corresponding intervals are

$$\begin{aligned} P_j(\theta, \beta) &= R(a_{j-1}) - R(a_j) = \exp\left(\frac{-a_{j-1}^\beta}{\theta}\right) - \exp\left(\frac{-a_j^\beta}{\theta}\right) \\ &= \exp\left(\frac{-a_{j-1}^\beta}{\theta}\right) \left(1 - \exp\left(\frac{a_j^\beta - a_{j-1}^\beta}{\theta}\right) \right), \quad j = 1, 2, \dots, k+1 \end{aligned} \quad (7)$$

Since $\sum_{j=1}^{k+1} f_j = n$, $\sum_{j=1}^{k+1} P_j(\theta) = 1$, the likelihood function given the interval grouped data is

$$L(\theta | \beta, G) \propto \exp\left(\frac{-\sum_{j=1}^{k+1} a_{j-1}^\beta f_j}{\theta}\right) \prod_{j=1}^k \left(1 - \exp\left(\frac{-(a_j^\beta - a_{j-1}^\beta)}{\theta}\right) \right)^{f_j} \quad (8)$$

A fundamental element in the Bayesian estimation is to specify a respective prior for the indexed parameter. In this research, we adopt the natural conjugate prior density for the Weibull scale parameter, namely the inverted gamma prior given by

$$\Pi(\theta) = \frac{b^a}{\Gamma(a)} \theta^{-(a+1)} \exp\left(\frac{-b}{\theta}\right), \quad a > 0, b > 0 \quad (9)$$

The mean and the variance of the above prior are

$$E(\theta) = \frac{b}{a-1}, \quad b > 0, a > 1 \quad (10)$$

$$V(\theta) = b^2 \left(\frac{1}{(a-1)^2(a-2)} \right), \quad b > 0, a > 2 \quad (11)$$

if $a = 0, b = 0$, then we get the non-informative prior.

Setting $S = \sum_{j=1}^{k+1} a_{j-1}^\beta f_j + b$ and combining (8) and (9), the posterior function of θ is

$$\Pi(\theta | G) = \frac{\theta^{-(a+1)} \exp\left(\frac{-S}{\theta}\right) \prod_{j=1}^k (1 - \exp\left(\frac{-(a_j^\beta - a_{j-1}^\beta)}{\theta}\right))^{f_j}}{\int_0^\infty \theta^{-(a+1)} \exp\left(\frac{-S}{\theta}\right) \prod_{j=1}^k (1 - \exp\left(\frac{-(a_j^\beta - a_{j-1}^\beta)}{\theta}\right))^{f_j} d\theta} \quad (12)$$

To obtain the integral in the denominator of (12), $\prod_{j=1}^k \left(1 - \exp\left(\frac{-(a_j^\beta - a_{j-1}^\beta)}{\theta}\right)\right)^{f_j}$ is

approximated by substituting the average $Z = \frac{1}{k} \sum_{j=1}^k (a_j^\beta - a_{j-1}^\beta)$ in both numerator and denominator of (12) and hence, the best posterior approximation can be given (see, the appendix (1)) by

$$\Pi(\theta | G) = \frac{\sum_{j=0}^r \binom{r}{j} (-1)^j \theta^{-(a+1)} \exp\left(\frac{-(jZ + S)}{\theta}\right)}{\sum_{j=0}^r \binom{r}{j} (-1)^j \int_0^\infty \theta^{-(a+1)} \exp\left(\frac{-(jZ + S)}{\theta}\right) d\theta} \quad (13)$$

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116

Where, $r = \sum_{j=1}^k f_j = n - f_{k+1}$

Using the transformations:

$$\theta = \frac{\beta t^{\beta-1}}{h(t)}, h(t) > 0, \quad \theta = \frac{t^{\beta-1}}{R(t)(\log(R(t)))^2}, 0 < R(t) < 1 \quad (14)$$

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The posterior functions for both the hazard and the reliability functions are readily derived.

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123

124 3. BAYESIAN ESTIMATION

125

In this section, based on the interval grouped data, the Bayesian and the HPD estimators for the parameter θ are obtained and modified. Under SELF, the Bayesian estimator is given by the mean of the posterior, define:

$$\Psi(r, h, w) = \sum_{j=0}^r \binom{r}{j} (-1)^j h^{-w} \quad (15)$$

and taking the mean of $\Pi(\theta | \beta, G)$, the Bayesian estimator of θ under SELF is given from the following formula.

131

$$\theta_{SB} = \frac{\Psi(r, jZ + S, a - 1)}{(a - 1)\Psi(r, jZ + S, a)} \quad (16)$$

134

135 and under GELF of shape parameter m , the Bayesian estimator

$$136 \quad \theta_{GB} = \left(\frac{\Gamma(a+m)}{\Gamma(a)} \right)^{\frac{-1}{m}} \left(\frac{\Psi(r, jZ+S, a+m)}{\Psi(r, jZ+S, a)} \right)^{\frac{-1}{m}} \quad (17)$$

137

138 To base the estimation on the sufficient statistics for θ given the interval grouped data

139 $S = \sum_{j=1}^{k+1} a_{j-1}^{\beta} f_j + b$, the posterior can be approximated by

$$140 \quad \Pi(\theta | \beta, G) = \frac{\theta^{-(a+r+1)} \exp\left(-\frac{S}{\theta}\right)}{\int_0^{\infty} \theta^{-(a+r+1)} \exp\left(-\frac{S}{\theta}\right) d\theta} \quad (18)$$

141

142 hence, Bayesian estimator under SELF is

$$143 \quad \theta_{SB} = \frac{S}{a+r-1} \quad (19)$$

144 and the corresponding Bayesian estimator under GELF is

$$145 \quad \theta_{GB} = S \left(\frac{\Gamma(a+r+m)}{\Gamma(a+r)} \right)^{\frac{-1}{m}} \quad (20)$$

146 **Mathematically, since the estimators in (19) and (20) are in closed form, they are sufficient**
 147 **for computing and modifications.**

148 If it is accepted for some specific loss function, the estimation can be based on the maximum
 149 likelihood principle. In the Bayesian inference, this leads to the mode of the posterior density
 150 or the HPD estimator. Since the posterior defined in (14) is unimodal, the HPD estimator of
 151 θ denoted by θ_{HPD} , can be obtained by solving the equation

$$152 \quad \exp\left(-\frac{S}{\theta}\right) \theta^{-(a+r+1)} \left(\frac{S}{\theta^2} - \frac{(a+r+1)}{\theta} \right) = 0 \quad (21)$$

153 This implies, the HPD estimator is given by

$$154 \quad \theta_{HPD} = \frac{S}{a+r+1} \quad (22)$$

155 using the transformations in (14), the Bayesian estimators for the hazard rate and the
 156 reliability functions under SELF at a given specified t are respectively

157

$$158 \quad h_s(t) = \frac{\beta(a+r)t^{\beta-1}}{S}, \quad R_s(t) = \left(\frac{S}{S+t^{\beta}} \right)^{a+r}, \quad t > 0 \quad (23)$$

159 Also, the Bayesian estimators for the hazard rate, and the reliability functions under GELF at
 160 a given specified time t are respectively

161

$$h_G(t) = \beta t^{\beta-1} S^{\frac{r+m}{m}} \left(\frac{\Gamma(a+r-m)}{\Gamma(a+r)} \right)^{\frac{-1}{m}}, \quad R_G(t) = \left(\frac{S}{S - mt^{\beta-1}} \right)^{\frac{-1}{m}} \quad (24)$$

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164 4. CREDIBLE INTERVALS

165

166 Another common Bayesian inference to obtain intervals $[C_1, C_2]$ for the unknown
 167 parameter θ is probably to lie. Based on the posterior distribution, the interval $[C_1, C_2]$ is said
 168 to be a $(1 - \alpha)\%$ credible interval for θ if

169

$$\int_{C_1}^{C_2} \Pi(\theta | G) d\theta = 1 - \alpha \quad (25)$$

171

172 In choosing a credible interval for θ , it is usually desirable to minimize its size subject to the
 173 condition (25) which requires

$$\Pi(C_1 | G) = \Pi(C_2 | G) \quad (26)$$

175 an interval $[C_1, C_2]$ which simultaneously satisfies (25), and (26) is called the $(1 - \alpha)\%$
 176 HPD credible interval. Substituting for $\Pi(\theta | G)$ from (18) and using the integral transform

$$\int_h^\infty \theta^{-(a+r+1)} \exp\left(-\frac{S}{\theta}\right) d\theta = S^{-(a+r)} \int_0^{\frac{S}{h}} w^{(a+r)-1} \exp(-w) dw$$

178

179 Implies HPD credible intervals can be obtained by choosing C_1, C_2 that satisfied the
 180 following two equations, simultaneously

181

$$I_g\left(\frac{S}{C_1}, a+r\right) - I_g\left(\frac{S}{C_2}, a+r\right) = 1 - \alpha \quad (27)$$

183

$$\left(\frac{C_1}{C_2}\right)^{-(a+r+1)} = \exp\left(S\left(\frac{1}{C_1} - \frac{1}{C_2}\right)\right) \quad (28)$$

184

185 where, $I_g(x, y) = \frac{1}{\Gamma(y)} \int_0^x t^{y-1} \exp(-t) dt$ is the incomplete gamma function

186

187 Using the transformations in (14), credible intervals and HPD credible intervals for the
 188 hazard and the reliability functions at a pre-given time t can be derived.

189

190 5. PREDICTION

191

192 In the Bayesian frame work related to life testing experiments, **prediction is mainly concern**
 193 **with** the future order failure **time** $t(r)$. In this research, we approach the prediction for the
 194 future number of failures in a pre-given interval.

195

Assume a new sample of size $\mathbf{N}$ is put on the test for failure, if the corresponding intervals $I_1 = [0, a_1)$, $I_2 = [a_1, a_2)$, ..., $I_k = [a_{k-1}, a_k)$, $I_{k+1} = [a_k, \infty)$ are still fixed and the number of failures $f_j, j=1,2,\dots,k+1$ in these intervals for the pre sample of size n are given, then the distribution for each of the new number of failures $f_j^*, j=1,2,\dots,k+1$ for a new sample of size N is binomial $(\mathbf{N}, P_j(\theta))$ where $P_j(\theta)$ is as defined in (7). This implies the predictive posterior of f_j^* which is given by

$$\Pi^*(f_j^* = x) = \frac{\binom{N}{x} \int_0^\infty (P_j(\theta))^x (1-P_j(\theta))^{N-x} \theta^{-(a+r+1)} \exp\left(-\frac{S}{\theta}\right) d\theta}{\int_0^\infty \theta^{-(a+r+1)} \exp\left(-\frac{S}{\theta}\right) d\theta}, x = 0,1,2,\dots,m \quad (29)$$

In particular, the predictive posterior of f_1^*, f_{k+1}^* are given by substituting $P_1(\theta) = 1 - \exp\left(-\frac{a_1^\beta}{\theta}\right)$, $P_{k+1}(\theta) = \exp\left(-\frac{a_k^\beta}{\theta}\right)$ for $P_j(\theta)$ in (29), respectively.

This implies, the expected prediction value of f_1, f_{k+1} are

$$E_{\Pi^*}(f_1^*) = \frac{S^{a+r}}{\Gamma(a+r)} \sum_{x=0}^N \binom{N}{x} \frac{x q_1^x S^{a+r} \Gamma(a+r+x)}{(S + a_1^\beta (N-x))^{a+r+1}} \quad (30)$$

$$E_{\Pi^*}(f_{k+1}^*) = \frac{S^{a+r}}{\Gamma(a+r)} \sum_{x=0}^N \binom{N}{x} \frac{x q_{k+1}^x S^{a+r} \Gamma(a+r+x)}{(S + a_{k+1}^\beta (N-x))^{a+r+1}} \quad (31)$$

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211 6. APPLICATION AND CONCLUSION

212

To illustrate the performance of the Bayesian inference, real life example is presented in this section [26]. The data represent number of million revolutions before failure for 23 ball bearings dividing each by 17. The data are as follows:

216 1.05, 1.70, 1.94, 2.44, 2.48, 2.68, 2.85, 3.05, 3.06, 3.18, 3.27, 3.99, 4.04, 4.04, 4.05, 4.95,
217 5.48, 5.80, 6.18, 6.23, 7.52, 7.53, 10.20.

218

219 Since the interval grouped data have specific loss of information about the exact failure
220 times, the obtained Bayesian estimator derived in this paper are compared with the
221 maximum likelihood estimator using the complete ungrouped data. The above ungrouped
222 data are fit to the Weibull distribution by the maximum likelihood estimation method using
223 Minitab with 95% confidence interval. The corresponding maximum likelihood estimators
224 using the complete ungrouped data for the scale and the shape parameters are
225 $\hat{\alpha}=481615, \hat{\beta}=21018$ implies the value of θ , $\hat{\theta}_{mle} = 27.2228$. Then the data are grouped
226 into intervals of fixed length = 1.5, This length is probable to the above data range to
227 terminate the experiment at the times $a_k = 4.5, 6$, and 7.5 to have 3,4,5 intervals. Two
228 cases for the informative priors are considered. In the first case assuming $\text{var}(\theta)=10$ and
229 in the second case $\text{var}(\theta)=5$. In both cases, the prior mean is assumed to
230 satisfy $E_{\Pi}(\theta) = \hat{\theta}_{mle} = 27.2228$. Thus, the available prior information indicates **a= 76.108**,
231 **b= 2044.65** for the first case and **a= 150.2161, b= 4062.08** for the second. For the general

loss the value of m is fixed to be 2 in both cases. The Bayesian estimators θ_{SB} , θ_{GB} and θ_{HPD} are computed assuming the end interval points $a_k = 4.5, 6$, and 7.5 using equations (19), (20), and (22) and their efficiencies is computed as the ratio of their values to $\hat{\theta}_{mle}$. 90% symmetric Credible Intervals are derived using (27) and (28). The predicted values for the number of failures are also computed using (30) and (31) as illustrated in Tables (1) and (2).

Table (1): Case 1 performance with $a= 76.108$, and $b= 2044.65$

a_j	θ_{SB} efficiency		θ_{GB} efficiency		θ_{HPD} efficiency		90% Credible Interval	Real and Predicted values of f_{k+1} and $\hat{f}_{k+1}$	
4.5	24.5267	90.09%	24.1254	88.62%	23.9941	88.14%	(20.60,29.40)	8	7.35
6	24.6645	90.60%	24.2738	89.16%	24.1459	88.69%	(20.75,29.33)	5	4.42
7.5	26.3243	96.69%	25.9160	95.19%	25.7822	94.71%	(22.55,29.21)	3	2.85

Table (2): Case 2 performance with $a= 150.2161$, and $b= 4062.08$

a_j	θ_{SB} efficiency		θ_{GB} efficiency		θ_{HPD} efficiency		90% Credible Interval	Real and Predicted values of f_{k+1} and $\hat{f}_{k+1}$	
4.5	25.7434	94.55%	25.5105	93.71%	25.4336	93.42%	(22.62,29.24)	8	7.76
6	26.2696	96.49%	25.5691	93.92%	25.9535	95.33%	(22.70,29.23)	5	4.71
7.5	27.2000	99.91%	26.4832	97.28%	26.2312	96.35%	(23.53,29.20)	3	2.96

Despite there is a considerable loss of information about the exact failure times. From the results in Tables (1) and (2), Bayesian inference based on the interval grouped data show significance efficiency as compared to the maximum likelihood estimation using the complete ungrouped data. This efficiency increases as the number of intervals increases. This result is satisfied, as the number of interval increases, the loss of information about the exact failure times decreases. The available prior information clearly affects the accuracy of the Bayesian estimates. This depends on the variance of the indicated priors. The developed computations for both the posterior and the corresponding estimators and prediction gave a high precision, and thus, can be used for any further inference.

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- 324

325 Appendix (1)

326

327 Proof for the posterior approximation in equations (12), and (13)

328 Let;

329

330 $w1 = \min\{(a_j^\theta - a_{j-1}^\theta), j = 1, 2, \dots, k\}$

331 $w2 = \max\{(a_j^\theta - a_{j-1}^\theta), j = 1, 2, \dots, k\}$

332 $Z = \text{average}\{(a_j^\theta - a_{j-1}^\theta), j = 1, 2, \dots, k\} = \frac{\sum_{j=1}^k (a_j^\theta - a_{j-1}^\theta)}{k}$

333

334 This implies,

335 $1 - e^{-\frac{w2}{\theta}} \leq 1 - e^{-\frac{(a_j^\theta - a_{j-1}^\theta)}{\theta}} \leq 1 - e^{-\frac{Z}{\theta}} \leq 1 - e^{-\frac{w1}{\theta}}, j = 1, 2, \dots, k$

336

337 This implies,

338
$$\prod_{j=1}^k (1 - e^{-\frac{w2}{\theta}})^{f_j} \leq \prod_{j=1}^k \left(1 - e^{-\frac{(a_j^\theta - a_{j-1}^\theta)}{\theta}}\right)^{f_j} \leq \prod_{j=1}^k \left(1 - e^{-\frac{Z}{\theta}}\right)^{f_j}$$

$$\leq \prod_{j=1}^k (1 - e^{-\frac{w1}{\theta}})^{f_j}$$

339

340 Thus, substitute for $\prod_{j=1}^k \left(1 - e^{-\frac{(a_j^\theta - a_{j-1}^\theta)}{\theta}}\right)^{f_j}$

341 $\prod_{j=1}^k (1 - e^{-\frac{w2}{\theta}})^{f_j}$ in the denominator of (12)

342 $\prod_{j=1}^k (1 - e^{-\frac{w1}{\theta}})^{f_j}$ in the numerator of (12)

343 We will have a lower bound for the posterior in (12) which gives a lower bound estimate for
344 the Bayesian estimator.

345 Similarly, substitute for $\prod_{j=1}^k \left(1 - e^{-\frac{(a_j^\theta - a_{j-1}^\theta)}{\theta}}\right)^{f_j}$

346 $\prod_{j=1}^k (1 - e^{-\frac{w1}{\theta}})^{f_j}$ in the denominator of (12)

347 $\prod_{j=1}^k (1 - e^{-\frac{w2}{\theta}})^{f_j}$ in the numerator of (12)

348 Then, we will have an upper bound for the posterior which gives an upper bound estimate for
349 the Bayesian estimator.

350

Therefore to have a best approximation estimate for the posterior and the Bayesian estimator, substitute for

353 $\prod_{j=1}^k \left(1 - e^{-\frac{(a_j^\theta - a_{j-1}^\theta)}{\theta}}\right)^{f_j}, \prod_{j=1}^k \left(1 - e^{-\frac{Z}{\theta}}\right)^{f_j}$ in both numerator and denominator

354 of (12), and set $r = n - f_{k+1}$ and use the binomial expansion

355
$$\prod_{j=1}^k \left(1 - e^{-\frac{Z}{\theta}}\right)^{f_j} = (1 - e^{-Z})^{n - f_{k+1}} = \sum_{j=0}^{n - f_{k+1}} (-1)^j \binom{n - f_{k+1}}{j} e^{-jZ}$$

356 Then, simplify, you will have exactly equation (13).

357

358 Note: $e^x = \exp(x)$

359