

The convolution sums $\sum_{m < \frac{n}{2}} m^2 \sigma_e(m) \sigma_f(n - 2m)$ and its applications

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Abstract

In this paper we obtain the convolution sum formulae of

$$\sum_{m < \frac{n}{2}} m^2 \sigma_e(m) \sigma_f(n - 2m)$$

for $n \in \mathbb{N}$ and an odd positive integer e and f . Moreover we obtain some identities induced from $\sum_{m < \frac{n}{2}} m^2 \sigma_e(m) \sigma_f(n - 2m)$.

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1 Introduction

For $n, m \in \mathbb{N}$, $s, r \in \mathbb{N} \cup \{0\}$, $q \in \mathbb{C}$ with $|q| < 1$, we define some necessary divisor functions and infinite product sums which also appear in many areas of number theory:

$$\begin{aligned}\sigma_s(n) &= \sum_{d|n} d^s, & \sigma_{s,r}(n; m) &= \sum_{\substack{d|n \\ d \equiv r \pmod{m}}} d^s, \\ \Delta(q) &:= \sum_{n=1}^{\infty} \tau(n) q^n = q \prod_{n=1}^{\infty} (1 - q^n)^{24}, \\ B(q) &:= \sum_{n=1}^{\infty} b(n) q^n = (\Delta(q) \Delta(q^2))^{\frac{1}{3}} = q \prod_{n=1}^{\infty} (1 - q^n)^8 (1 - q^{2n})^8, \\ C(q) &:= \sum_{n=1}^{\infty} c(n) q^n = (\Delta(q)^4 \Delta(q^2))^{\frac{1}{6}} = q \prod_{n=1}^{\infty} (1 - q^n)^{16} (1 - q^{2n})^4, \\ D(q) &:= \sum_{n=1}^{\infty} d(n) q^n = (\Delta(q)^2 \Delta(q^2) \Delta(q^4))^{\frac{1}{6}} = q^2 \prod_{n=1}^{\infty} (1 - q^n)^8 (1 - q^{2n})^4 (1 - q^{4n})^8.\end{aligned}\tag{1.1}$$

In general, it is satisfied that

$$b(n) = -8b\left(\frac{n}{2}\right)\tag{1.2}$$

for even n (see ([8], Remark 4.3)). We note that

$$\sigma_s(n) = \sigma_{s,1}(n; 2) + \sigma_{s,0}(n; 2)\tag{1.3}$$

and

$$\sigma_{s,1}(2n; 2) = \sigma_{s,1}(n; 2), \quad \sigma_{s,0}(2n; 2) = 2^s \sigma_s(n).\tag{1.4}$$

The Eisenstein series $L(q)$, $M(q)$, and $N(q)$ for $q \in \mathbb{C}$ with $|q| < 1$ are

$$L(q) = 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) q^n,\tag{1.5}$$

$$M(q) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n,\tag{1.6}$$

$$N(q) = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n\tag{1.7}$$

by ([2], p. 318). Lahiri ([9], p. 149) has derived the following identities from the work of Ramanujan [11]:

$$L^2(q) = 1 - 288 \sum_{n=1}^{\infty} n \sigma_1(n) q^n + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n,\tag{1.8}$$

$$L^3(q) = 1 - 1728 \sum_{n=1}^{\infty} n^2 \sigma_1(n) q^n + 2160 \sum_{n=1}^{\infty} n \sigma_3(n) q^n - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n,\tag{1.9}$$

$$L(q)M(q) = 1 + 720 \sum_{n=1}^{\infty} n \sigma_3(n) q^n - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n,\tag{1.10}$$

$$L^2(q)M(q) = 1 + 1728 \sum_{n=1}^{\infty} n^2 \sigma_3(n) q^n - 2016 \sum_{n=1}^{\infty} n \sigma_5(n) q^n + 480 \sum_{n=1}^{\infty} \sigma_7(n) q^n, \quad (1.11)$$

$$L(q)N(q) = 1 - 1008 \sum_{n=1}^{\infty} n \sigma_5(n) q^n + 480 \sum_{n=1}^{\infty} \sigma_7(n) q^n, \quad (1.12)$$

$$L^2(q)N(q) = 1 - 1728 \sum_{n=1}^{\infty} n^2 \sigma_5(n) q^n + 1440 \sum_{n=1}^{\infty} n \sigma_7(n) q^n - 264 \sum_{n=1}^{\infty} \sigma_9(n) q^n. \quad (1.13)$$

For $e, f, m, n \in \mathbb{N}$ and $k \in \mathbb{N} \cup \{0\}$ we define

$$\begin{aligned} I_{m,e,f}^k(n) &:= \sum_{m=1}^{n-1} m^k \sigma_e(m) \sigma_f(n-m), \\ I_{m,e,f}(n) &:= I_{m,e,f}^1(n) = \sum_{m=1}^{n-1} m \sigma_e(m) \sigma_f(n-m), \\ I_{e,f}(n) &:= I_{m,e,f}^0(n) = \sum_{m=1}^{n-1} \sigma_e(m) \sigma_f(n-m). \end{aligned} \quad (1.14)$$

Ramanujan [11] and Lahiri [9], [10] have shown that $I_{e,f}$ can be expressed as :

$$\begin{aligned} I_{1,1}(n) &= \frac{5}{12} \sigma_3(n) + \frac{(1-6n)}{12} \sigma_1(n), \\ I_{1,3}(n) &= \frac{7}{80} \sigma_5(n) + \frac{(1-3n)}{24} \sigma_3(n) - \frac{1}{240} \sigma_1(n), \\ I_{1,5}(n) &= \frac{5}{126} \sigma_7(n) + \frac{(1-2n)}{24} \sigma_5(n) + \frac{1}{504} \sigma_1(n), \\ I_{3,3}(n) &= \frac{1}{120} \sigma_7(n) - \frac{1}{120} \sigma_3(n), \\ I_{1,7}(n) &= \frac{11}{480} \sigma_9(n) + \frac{(2-3n)}{48} \sigma_7(n) - \frac{1}{480} \sigma_1(n), \\ I_{3,5}(n) &= \frac{11}{5040} \sigma_9(n) - \frac{1}{240} \sigma_5(n) + \frac{1}{504} \sigma_3(n), \\ I_{1,9}(n) &= \frac{455}{30404} \sigma_{11}(n) + \frac{(5-6n)}{120} \sigma_9(n) + \frac{1}{264} \sigma_1(n) - \frac{36}{3455} \tau(n), \\ I_{3,7}(n) &= \frac{91}{110560} \sigma_{11}(n) - \frac{1}{240} \sigma_7(n) - \frac{1}{480} \sigma_3(n) + \frac{15}{2764} \tau(n), \\ I_{5,5}(n) &= \frac{65}{174132} \sigma_{11}(n) + \frac{1}{252} \sigma_5(n) - \frac{3}{691} \tau(n), \\ I_{1,11}(n) &= \frac{691}{65520} \sigma_{13}(n) + \frac{(1-n)}{24} \sigma_{11}(n) - \frac{691}{65520} \tau(n), \\ I_{3,9}(n) &= \frac{1}{2640} \sigma_{13}(n) - \frac{1}{240} \sigma_9(n) + \frac{1}{264} \sigma_3(n), \\ I_{5,7}(n) &= \frac{1}{10080} \sigma_{13}(n) + \frac{1}{504} \sigma_7(n) - \frac{1}{480} \sigma_5(n). \end{aligned}$$

And we have already obtained :

Proposition 1.1. Let $n \in \mathbb{N}$. Then we have

(a) (See ([12], p. 155))

$$I_{m,1,1}(n) = \frac{1}{24}n \{5\sigma_3(n) - (6n-1)\sigma_1(n)\},$$

(b) (See ([12], p. 157))

$$I_{m,1,3}(n) = \frac{7}{240}n\sigma_5(n) - \frac{1}{40}n^2\sigma_3(n) - \frac{1}{240}n\sigma_1(n),$$

$$I_{m,3,1}(n) = \frac{7}{120}n\sigma_5(n) + \left(\frac{1}{24}n - \frac{1}{10}n^2\right)\sigma_3(n),$$

(c) (See ([5], Theorem 3.3, Theorem 3.1), ([12], p. 155))

$$I_{m,1,5}(n) = \frac{1}{504}n \{5\sigma_7(n) - 6n\sigma_5(n) + \sigma_1(n)\},$$

$$I_{m,3,3}(n) = \frac{1}{240}n \{\sigma_7(n) - \sigma_3(n)\},$$

$$I_{m,5,1}(n) = \frac{1}{168}n \{5\sigma_7(n) - (12n-7)\sigma_5(n)\},$$

(d) (See ([5], Theorem 3.8 (a), (b), Theorem 3.7, Theorem 3.5))

$$I_{m,1,7}(n) = \frac{1}{7200} \{33n\sigma_9(n) - 50n^2\sigma_7(n) - 15n\sigma_1(n) + 32\tau(n)\},$$

$$I_{m,3,5}(n) = \frac{1}{12600} \{11n\sigma_9(n) + 25n\sigma_3(n) - 36\tau(n)\},$$

$$I_{m,5,3}(n) = \frac{1}{8400} \{11n\sigma_9(n) - 35n\sigma_5(n) + 24\tau(n)\},$$

$$I_{m,7,1}(n) = \frac{1}{1800} \{33n\sigma_9(n) - 25n(4n-3)\sigma_7(n) - 8\tau(n)\}.$$

Next we can see the following proposition :

Proposition 1.2. Let $n \in \mathbb{N}$. Then we have

(a) (See ([12], p. 155))

$$I_{m,1,1}^2(n) = \frac{1}{8}n^2\sigma_3(n) + \left(\frac{1}{24}n^2 - \frac{1}{6}n^3\right)\sigma_1(n),$$

(b) (See ([12], p. 155~156))

$$I_{m,1,3}^2(n) = \frac{1}{80}n^2\sigma_5(n) - \frac{1}{120}n^3\sigma_3(n) - \frac{1}{240}n^2\sigma_1(n),$$

$$I_{m,3,1}^2(n) = \frac{1}{24}n^2\sigma_5(n) + \left(\frac{1}{24}n^2 - \frac{1}{12}n^3\right)\sigma_3(n),$$

(c) (See ([6], Theorem 1.1 (a), (b)))

$$I_{m,1,5}^2(n) = \frac{1}{3024} \{10n^2\sigma_7(n) - 9n^3\sigma_5(n) + 6n^2\sigma_1(n) - 7\tau(n)\},$$

$$I_{m,3,3}^2(n) = \frac{1}{2160} \{5n^2\sigma_7(n) - 9n^2\sigma_3(n) + 4\tau(n)\}.$$

Similarly, for $e, f, m, n \in \mathbb{N}$ and $k \in \mathbb{N} \cup \{0\}$ we set

$$\begin{aligned} T_{m,e,f}^k(n) &:= \sum_{m < \frac{n}{2}} m^k \sigma_e(m) \sigma_f(n-2m), \\ T_{m,e,f}(n) &:= T_{m,e,f}^1(n) = \sum_{m < \frac{n}{2}} m \sigma_e(m) \sigma_f(n-2m), \\ T_{e,f}(n) &:= T_{m,e,f}^0(n) = \sum_{m < \frac{n}{2}} \sigma_e(m) \sigma_f(n-2m). \end{aligned} \quad (1.15)$$

In 1997, Melfi considered especially the convolution sums $T_{1,1}(n)$, $T_{1,3}(n)$, $T_{3,1}(n)$, and for odd n , Melfi proved that

$$\begin{aligned} T_{1,1}(n) &= \frac{1}{12} \sigma_3(n) + \frac{(1-3n)}{24} \sigma_1(n), \quad \text{if } n \equiv 1 \pmod{2}, \\ T_{1,3}(n) &= \frac{1}{48} \sigma_5(n) + \frac{(2-3n)}{48} \sigma_3(n), \quad \text{if } n \equiv 1 \pmod{2}, \\ T_{3,1}(n) &= \frac{1}{240} \sigma_5(n) - \frac{1}{240} \sigma_1(n), \quad \text{if } n \equiv 1 \pmod{2}. \end{aligned}$$

In 2002 Huard, Ou, Spearman and Williams [4] evaluated $T_{1,1}(n)$, $T_{1,3}(n)$, and $T_{3,1}(n)$ for $n \in \mathbb{N}$, that is,

$$\begin{aligned} T_{1,1}(n) &= \frac{1}{12} \sigma_3(n) + \frac{1}{3} \sigma_3\left(\frac{n}{2}\right) + \frac{(1-3n)}{24} \sigma_1(n) + \frac{(1-6n)}{24} \sigma_1\left(\frac{n}{2}\right), \\ T_{1,3}(n) &= \frac{1}{48} \sigma_5(n) + \frac{1}{15} \sigma_5\left(\frac{n}{2}\right) + \frac{(2-3n)}{48} \sigma_3(n) - \frac{1}{240} \sigma_1\left(\frac{n}{2}\right), \\ T_{3,1}(n) &= \frac{1}{240} \sigma_5(n) + \frac{1}{12} \sigma_5\left(\frac{n}{2}\right) + \frac{(1-3n)}{24} \sigma_3\left(\frac{n}{2}\right) - \frac{1}{240} \sigma_1(n). \end{aligned}$$

For the other $T_{e,f}(n)$ we refer to ([3], p. 45–54) :

$$\begin{aligned}
 T_{5,1}(n) &= \frac{1}{2142}\sigma_7(n) + \frac{2}{51}\sigma_7\left(\frac{n}{2}\right) + \frac{(1-2n)}{24}\sigma_5\left(\frac{n}{2}\right) + \frac{1}{504}\sigma_1(n) - \frac{1}{408}b(n), \\
 T_{3,3}(n) &= \frac{1}{2040}\sigma_7(n) + \frac{2}{255}\sigma_7\left(\frac{n}{2}\right) - \frac{1}{240}\sigma_3(n) - \frac{1}{240}\sigma_3\left(\frac{n}{2}\right) + \frac{1}{272}b(n), \\
 T_{1,5}(n) &= \frac{1}{102}\sigma_7(n) + \frac{32}{1071}\sigma_7\left(\frac{n}{2}\right) + \frac{(1-n)}{24}\sigma_5(n) + \frac{1}{504}\sigma_1\left(\frac{n}{2}\right) - \frac{1}{102}b(n), \\
 T_{7,1}(n) &= \frac{1}{14880}\sigma_9(n) + \frac{17}{744}\sigma_9\left(\frac{n}{2}\right) + \frac{(2-3n)}{48}\sigma_7\left(\frac{n}{2}\right) - \frac{1}{480}\sigma_1(n) \\
 &\quad + \frac{1}{496}c(n) + \frac{2}{31}d(n), \\
 T_{5,3}(n) &= \frac{1}{31248}\sigma_9(n) + \frac{1}{465}\sigma_9\left(\frac{n}{2}\right) - \frac{1}{240}\sigma_5\left(\frac{n}{2}\right) + \frac{1}{504}\sigma_3(n) \\
 &\quad - \frac{1}{496}c(n) - \frac{2}{31}d(n), \\
 T_{3,5}(n) &= \frac{1}{7440}\sigma_9(n) + \frac{4}{1953}\sigma_9\left(\frac{n}{2}\right) - \frac{1}{240}\sigma_5(n) + \frac{1}{504}\sigma_3\left(\frac{n}{2}\right) \\
 &\quad + \frac{1}{248}c(n) + \frac{4}{31}d(n), \\
 T_{1,7}(n) &= \frac{17}{2976}\sigma_9(n) + \frac{8}{465}\sigma_9\left(\frac{n}{2}\right) + \frac{(4-3n)}{96}\sigma_7(n) - \frac{1}{480}\sigma_1\left(\frac{n}{2}\right) \\
 &\quad - \frac{1}{62}c(n) - \frac{16}{31}d(n), \\
 T_{9,1}(n) &= \frac{1}{91212}\sigma_{11}(n) + \frac{31}{2073}\sigma_{11}\left(\frac{n}{2}\right) + \frac{(5-6n)}{120}\sigma_9\left(\frac{n}{2}\right) + \frac{1}{264}\sigma_1(n) \\
 &\quad - \frac{21}{5528}\tau(n) - \frac{282}{3455}\tau\left(\frac{n}{2}\right), \\
 T_{7,3}(n) &= \frac{1}{331680}\sigma_{11}(n) + \frac{17}{20730}\sigma_{11}\left(\frac{n}{2}\right) - \frac{1}{240}\sigma_7\left(\frac{n}{2}\right) - \frac{1}{480}\sigma_3(n) \\
 &\quad + \frac{23}{11056}\tau(n) + \frac{91}{1382}\tau\left(\frac{n}{2}\right), \\
 T_{5,5}(n) &= \frac{1}{174132}\sigma_{11}(n) + \frac{16}{43533}\sigma_{11}\left(\frac{n}{2}\right) + \frac{1}{504}\sigma_5(n) + \frac{1}{504}\sigma_5\left(\frac{n}{2}\right) \\
 &\quad - \frac{11}{5528}\tau(n) - \frac{88}{691}\tau\left(\frac{n}{2}\right), \\
 T_{3,7}(n) &= \frac{17}{331680}\sigma_{11}(n) + \frac{8}{10365}\sigma_{11}\left(\frac{n}{2}\right) - \frac{1}{240}\sigma_7(n) - \frac{1}{480}\sigma_3\left(\frac{n}{2}\right) \\
 &\quad + \frac{91}{22112}\tau(n) + \frac{368}{691}\tau\left(\frac{n}{2}\right), \\
 T_{1,9}(n) &= \frac{31}{8292}\sigma_{11}(n) + \frac{256}{22803}\sigma_{11}\left(\frac{n}{2}\right) + \frac{(5-3n)}{120}\sigma_9(n) + \frac{1}{264}\sigma_1\left(\frac{n}{2}\right) \\
 &\quad - \frac{141}{6910}\tau(n) - \frac{2688}{691}\tau\left(\frac{n}{2}\right).
 \end{aligned}$$

Also we can find

$$T_{m,1,1}(n) = \frac{1}{48}n\sigma_3(n) - \frac{1}{48}n^2\sigma_1(n) + \frac{1}{12}n\sigma_3\left(\frac{n}{2}\right) + \left(\frac{1}{48}n - \frac{1}{12}n^2\right)\sigma_1\left(\frac{n}{2}\right) \quad (1.16)$$

in ([1], Theorem 4.1) and

Proposition 1.3. (See [7]) Let $n \in \mathbb{N}$. Then we have

(a)

$$T_{m,1,3}(n) = \frac{1}{1440} \left[n \left\{ 5\sigma_5(n) + 16\sigma_5\left(\frac{n}{2}\right) - 9n\sigma_3(n) - 3\sigma_1\left(\frac{n}{2}\right) \right\} + 4b(n) \right],$$

$$T_{m,3,1}(n) = \frac{1}{720} \left[n \left\{ \sigma_5(n) + 20\sigma_5\left(\frac{n}{2}\right) - (36n - 15)\sigma_3\left(\frac{n}{2}\right) \right\} - b(n) \right],$$

(b)

$$T_{m,1,5}(n) = \frac{1}{17136} \left\{ 21n\sigma_7(n) + 64n\sigma_7\left(\frac{n}{2}\right) - 51n^2\sigma_5(n) + 17n\sigma_1\left(\frac{n}{2}\right) + 1632d(n) \right. \\ \left. + 51c(n) - 21nb(n) \right\},$$

$$T_{m,3,3}(n) = \frac{1}{16320} \left\{ 2n\sigma_7(n) + 32n\sigma_7\left(\frac{n}{2}\right) - 34n\sigma_3\left(\frac{n}{2}\right) - 544d(n) - 17c(n) \right. \\ \left. + 15nb(n) \right\},$$

$$T_{m,5,1}(n) = \frac{1}{22848} \left\{ 4n\sigma_7(n) + 336n\sigma_7\left(\frac{n}{2}\right) - 816n^2\sigma_5\left(\frac{n}{2}\right) + 476n\sigma_5\left(\frac{n}{2}\right) + 544d(n) \right. \\ \left. + 17c(n) - 21nb(n) \right\},$$

(c)

$$T_{m,1,7}(n) = \frac{1}{446400} \left\{ 255n\sigma_9(n) + 768n\sigma_9\left(\frac{n}{2}\right) - 775n^2\sigma_7(n) - 465n\sigma_1\left(\frac{n}{2}\right) \right. \\ \left. + 1240\tau(n) + 190464\tau\left(\frac{n}{2}\right) - 23040nd(n) - 720nc(n) \right\},$$

$$T_{m,3,5}(n) = \frac{1}{781200} \left\{ 21n\sigma_9(n) + 320n\sigma_9\left(\frac{n}{2}\right) + 775n\sigma_3\left(\frac{n}{2}\right) - 651\tau(n) - 59520\tau\left(\frac{n}{2}\right) \right. \\ \left. + 20160nd(n) + 630nc(n) \right\},$$

$$T_{m,5,3}(n) = \frac{1}{520800} \left\{ 5n\sigma_9(n) + 336n\sigma_9\left(\frac{n}{2}\right) - 1085n\sigma_5\left(\frac{n}{2}\right) + 310\tau(n) + 13888\tau\left(\frac{n}{2}\right) \right. \\ \left. - 10080nd(n) - 315nc(n) \right\},$$

$$T_{m,7,1}(n) = \frac{1}{111600} \left\{ 3n\sigma_9(n) + 1020n\sigma_9\left(\frac{n}{2}\right) - 3100n^2\sigma_7\left(\frac{n}{2}\right) + 2325n\sigma_7\left(\frac{n}{2}\right) \right. \\ \left. - 93\tau(n) - 2480\tau\left(\frac{n}{2}\right) + 2880nd(n) + 90nc(n) \right\}.$$

In this paper we desire to expand Proposition 1.3 into $T_{m,e,f}^2(n)$ and so we obtain the following theorem :

Theorem 1.1. *Let $n \in \mathbb{N}$. Then we have*

(a)

$$T_{m,1,1}^2(n) = \sum_{m < \frac{n}{2}} m^2 \sigma_1(m) \sigma_1(n - 2m)$$

$$= \frac{1}{960} \left[n^2 \left\{ 6\sigma_3(n) + 24\sigma_3\left(\frac{n}{2}\right) - 5n\sigma_1(n) - 10(3n - 1)\sigma_1\left(\frac{n}{2}\right) \right\} - b(n) \right],$$

(b)

$$T_{m,1,3}^2(n) = \sum_{m < \frac{n}{2}} m^2 \sigma_1(m) \sigma_3(n - 2m)$$

$$= \frac{1}{6720} \left\{ 5n^2\sigma_5(n) + 16n^2\sigma_5\left(\frac{n}{2}\right) - 7n^3\sigma_3(n) - 7n^2\sigma_1\left(\frac{n}{2}\right) - 160d(n) \right. \\ \left. - 5c(n) + 7nb(n) \right\},$$

(c)

$$\begin{aligned} T_{m,3,1}^2(n) &= \sum_{m < \frac{n}{2}} m^2 \sigma_3(m) \sigma_1(n-2m) \\ &= \frac{1}{8064} \left\{ 4n^2 \sigma_5(n) + 80n^2 \sigma_5\left(\frac{n}{2}\right) - 168n^3 \sigma_3\left(\frac{n}{2}\right) + 84n^2 \sigma_3\left(\frac{n}{2}\right) + 96d(n) \right. \\ &\quad \left. + 3c(n) - 7nb(n) \right\}, \end{aligned}$$

(d)

$$\begin{aligned} T_{m,1,5}^2(n) &= \sum_{m < \frac{n}{2}} m^2 \sigma_1(m) \sigma_5(n-2m) \\ &= \frac{1}{2056320} \left\{ 420n^2 \sigma_7(n) + 1280n^2 \sigma_7\left(\frac{n}{2}\right) - 765n^3 \sigma_5(n) + 1020n^2 \sigma_1\left(\frac{n}{2}\right) \right. \\ &\quad \left. + 58752nd(n) + 1836nc(n) - 420n^2 b(n) - 1071\tau(n) - 121856\tau\left(\frac{n}{2}\right) \right\}, \end{aligned}$$

(e)

$$\begin{aligned} T_{m,3,3}^2(n) &= \sum_{m < \frac{n}{2}} m^2 \sigma_3(m) \sigma_3(n-2m) \\ &= \frac{1}{293760} \left\{ 10n^2 \sigma_7(n) + 160n^2 \sigma_7\left(\frac{n}{2}\right) - 306n^2 \sigma_3\left(\frac{n}{2}\right) - 4896nd(n) \right. \\ &\quad \left. - 153nc(n) + 75n^2 b(n) + 68\tau(n) + 4352\tau\left(\frac{n}{2}\right) \right\}, \end{aligned}$$

(f)

$$\begin{aligned} T_{m,5,1}^2(n) &= \sum_{m < \frac{n}{2}} m^2 \sigma_5(m) \sigma_1(n-2m) \\ &= \frac{1}{293760} \left\{ 20n^2 \sigma_7(n) + 1680n^2 \sigma_7\left(\frac{n}{2}\right) - 1530n^2 (3n-2) \sigma_5\left(\frac{n}{2}\right) + 4896nd(n) \right. \\ &\quad \left. + 153nc(n) - 105n^2 b(n) - 68\tau(n) - 2448\tau\left(\frac{n}{2}\right) \right\}. \end{aligned}$$

In addition, using Theorem 1.1 we can easily obtain some identities induced from $T_{m,e,f}^2(n)$, which are introduced in the following theorem :

Theorem 1.2. For $q \in \mathbb{C}$ with $|q| < 1$, we have

(a)

$$\begin{aligned} L^3(q)L(q^2) &= 1 + \frac{2016}{17} \sum_{n=1}^{\infty} \sigma_7(n) q^n + \frac{12288}{17} \sum_{n=1}^{\infty} \sigma_7(n) q^{2n} - 1244 \sum_{n=1}^{\infty} n \sigma_5(n) q^n \\ &\quad - 4608 \sum_{n=1}^{\infty} n \sigma_5(n) q^{2n} + \frac{18144}{5} \sum_{n=1}^{\infty} n^2 \sigma_3(n) q^n + \frac{82944}{5} \sum_{n=1}^{\infty} n^2 \sigma_3(n) q^{2n} \\ &\quad - 2592 \sum_{n=1}^{\infty} n^3 \sigma_1(n) q^n - 13824 \sum_{n=1}^{\infty} n^3 \sigma_1(n) q^{2n} - \frac{288}{85} \sum_{n=1}^{\infty} b(n) q^n, \end{aligned}$$

(b)

$$\begin{aligned}
 & L^3(q^2)M(q) \\
 &= 1 - \frac{120}{31} \sum_{n=1}^{\infty} \sigma_9(n)q^n - \frac{8064}{31} \sum_{n=1}^{\infty} \sigma_9(n)q^{2n} + \frac{1080}{17} \sum_{n=1}^{\infty} n\sigma_7(n)q^n \\
 &+ \frac{34560}{17} \sum_{n=1}^{\infty} n\sigma_7(n)q^{2n} - \frac{2160}{7} \sum_{n=1}^{\infty} n^2\sigma_5(n)q^n - \frac{27648}{7} \sum_{n=1}^{\infty} n^2\sigma_5(n)q^{2n} \\
 &+ 432 \sum_{n=1}^{\infty} n^3\sigma_3(n)q^n + \frac{86400}{217} \sum_{n=1}^{\infty} d(n)q^n + \frac{2700}{217} \sum_{n=1}^{\infty} c(n)q^n \\
 &+ \frac{756}{17} \sum_{n=1}^{\infty} nb(n)q^n,
 \end{aligned}$$

(c)

$$\begin{aligned}
 & L^3(q)M(q^2) \\
 &= 1 - \frac{504}{31} \sum_{n=1}^{\infty} \sigma_9(n)q^n - \frac{7680}{31} \sum_{n=1}^{\infty} \sigma_9(n)q^{2n} + \frac{2160}{17} \sum_{n=1}^{\infty} n\sigma_7(n)q^n \\
 &+ \frac{69120}{17} \sum_{n=1}^{\infty} n\sigma_7(n)q^{2n} - \frac{1728}{7} \sum_{n=1}^{\infty} n^2\sigma_5(n)q^n - \frac{138240}{7} \sum_{n=1}^{\infty} n^2\sigma_5(n)q^{2n} \\
 &+ 27648 \sum_{n=1}^{\infty} n^3\sigma_3(n)q^{2n} - \frac{172800}{217} \sum_{n=1}^{\infty} d(n)q^n - \frac{5400}{217} \sum_{n=1}^{\infty} c(n)q^n \\
 &+ \frac{1512}{17} \sum_{n=1}^{\infty} nb(n)q^n,
 \end{aligned}$$

(d)

$$\begin{aligned}
 & L(q)L^2(q^2)N(q) \\
 &= 1 + \frac{4080}{691} \sum_{n=1}^{\infty} \sigma_{11}(n)q^n + \frac{61440}{691} \sum_{n=1}^{\infty} \sigma_{11}(n)q^{2n} - \frac{15264}{155} \sum_{n=1}^{\infty} n\sigma_9(n)q^n \\
 &- \frac{165888}{155} \sum_{n=1}^{\infty} n\sigma_9(n)q^{2n} + \frac{8112}{17} \sum_{n=1}^{\infty} n^2\sigma_7(n)q^n + \frac{49152}{17} \sum_{n=1}^{\infty} n^2\sigma_7(n)q^{2n} \\
 &- 648 \sum_{n=1}^{\infty} n^3\sigma_5(n)q^n - \frac{82944}{155} \sum_{n=1}^{\infty} nd(n)q^n - \frac{2592}{155} \sum_{n=1}^{\infty} nc(n)q^n \\
 &- \frac{4032}{17} \sum_{n=1}^{\infty} n^2b(n)q^n - \frac{36984}{3455} \sum_{n=1}^{\infty} \tau(n)q^n - \frac{61440}{691} \sum_{n=1}^{\infty} \tau(n)q^{2n},
 \end{aligned}$$

(e)

$$\begin{aligned}
 & L^2(q^2)M(q)M(q^2) \\
 &= 1 + \frac{240}{691} \sum_{n=1}^{\infty} \sigma_{11}(n)q^n + \frac{65280}{691} \sum_{n=1}^{\infty} \sigma_{11}(n)q^{2n} - \frac{144}{31} \sum_{n=1}^{\infty} n\sigma_9(n)q^n \\
 &\quad - \frac{96768}{155} \sum_{n=1}^{\infty} n\sigma_9(n)q^{2n} + \frac{240}{17} \sum_{n=1}^{\infty} n^2\sigma_7(n)q^n + \frac{15360}{17} \sum_{n=1}^{\infty} n^2\sigma_7(n)q^{2n} \\
 &\quad + \frac{76032}{31} \sum_{n=1}^{\infty} nd(n)q^n + \frac{2376}{31} \sum_{n=1}^{\infty} nc(n)q^n + \frac{1800}{17} \sum_{n=1}^{\infty} n^2b(n)q^n \\
 &\quad + \frac{32928}{691} \sum_{n=1}^{\infty} \tau(n)q^n + \frac{2857728}{3455} \sum_{n=1}^{\infty} \tau(n)q^{2n},
 \end{aligned}$$

(f)

$$\begin{aligned}
 & L(q)L^2(q^2)N(q^2) \\
 &= 1 + \frac{48}{691} \sum_{n=1}^{\infty} \sigma_{11}(n)q^n + \frac{65472}{691} \sum_{n=1}^{\infty} \sigma_{11}(n)q^{2n} - \frac{144}{155} \sum_{n=1}^{\infty} n\sigma_9(n)q^n \\
 &\quad - \frac{196128}{155} \sum_{n=1}^{\infty} n\sigma_9(n)q^{2n} + \frac{48}{17} \sum_{n=1}^{\infty} n^2\sigma_7(n)q^n + \frac{81408}{17} \sum_{n=1}^{\infty} n^2\sigma_7(n)q^{2n} \\
 &\quad - 5184 \sum_{n=1}^{\infty} n^3\sigma_5(n)q^{2n} - \frac{31104}{155} \sum_{n=1}^{\infty} nd(n)q^n - \frac{972}{155} \sum_{n=1}^{\infty} nc(n)q^n \\
 &\quad - \frac{252}{17} \sum_{n=1}^{\infty} n^2b(n)q^n - \frac{16824}{3455} \sum_{n=1}^{\infty} \tau(n)q^n - \frac{65472}{691} \sum_{n=1}^{\infty} \tau(n)q^{2n}.
 \end{aligned}$$

Finally, we can deduce Theorem 1.3 :

Theorem 1.3. *Let $n \in \mathbb{N}$. Then we have*

(a)

$$\begin{aligned}
 & \sum_{\substack{(a,b,c) \in \mathbb{N}^3 \\ a+b+c=n}} a\sigma_{1,1}(a;2)\sigma_{3,1}(b;2)\sigma_1(c) = \sum_{m=1}^{n-1} \sum_{k=1}^{m-1} k\sigma_{1,1}(k;2)\sigma_{3,1}(m-k;2)\sigma_1(n-m) \\
 &= \frac{1}{685440} \left\{ 133n\sigma_7(n) + 4032n\sigma_7\left(\frac{n}{2}\right) - 17n(12n-7)\sigma_5(n) - 816n(12n-7)\sigma_5\left(\frac{n}{2}\right) \right. \\
 &\quad + 2499n\sigma_3(n) - 6664n\sigma_3\left(\frac{n}{2}\right) - 833n(4n-1)\sigma_1(n) + 1666n(4n-1)\sigma_1\left(\frac{n}{2}\right) \\
 &\quad \left. - 8704d(n) - 272c(n) + 56(21n-17)b(n) \right\},
 \end{aligned}$$

(b)

$$\begin{aligned}
 & \sum_{\substack{(a,b,c) \in \mathbb{N}^3 \\ a+b+c=n}} a\sigma_{1,1}(a;2)\sigma_{3,0}(b;2)\sigma_1(c) \\
 &= \frac{1}{514080} \left\{ 126n\sigma_7(n) - 3696n\sigma_7\left(\frac{n}{2}\right) - 34n(12n-7)\sigma_5(n) + 748n(12n-7)\sigma_5\left(\frac{n}{2}\right) \right. \\
 &\quad - 2142n\sigma_3(n) + 5712n\sigma_3\left(\frac{n}{2}\right) + 714n(4n-1)\sigma_1(n) - 1428n(4n-1)\sigma_1\left(\frac{n}{2}\right) \\
 &\quad \left. + 1632d(n) + 51c(n) - 7(69n-68)b(n) \right\},
 \end{aligned}$$

(c)

$$\begin{aligned}
& \sum_{\substack{(a,b,c) \in \mathbb{N}^3 \\ a+b+c=n}} a\sigma_{1,0}(a;2)\sigma_{3,0}(b;2)\sigma_1(c) \\
&= \frac{1}{42840} \left\{ 7n\sigma_7(n) + 588n\sigma_7\left(\frac{n}{2}\right) - 17n^2\sigma_5(n) - 17n(104n-49)\sigma_5\left(\frac{n}{2}\right) \right. \\
&\quad - 119n\sigma_3(n) + 119n(6n^2-3n-4)\sigma_3\left(\frac{n}{2}\right) + 119n^2\sigma_1(n) \\
&\quad \left. + 119n(4n-1)\sigma_1\left(\frac{n}{2}\right) + 544d(n) + 17c(n) - 7nb(n) \right\},
\end{aligned}$$

(d)

$$\begin{aligned}
& \sum_{\substack{(a,b,c) \in \mathbb{N}^3 \\ a+b+c=n}} a\sigma_{1,0}(a;2)\sigma_{3,1}(b;2)\sigma_1(c) \\
&= \frac{1}{1028160} \left\{ 273n\sigma_7(n) - 12768n\sigma_7\left(\frac{n}{2}\right) - 17n(99n-35)\sigma_5(n) \right. \\
&\quad + 136n(288n-133)\sigma_5\left(\frac{n}{2}\right) + 357n(6n^2-3n+7)\sigma_3(n) \\
&\quad - 1428n(12n^2-6n-7)\sigma_3\left(\frac{n}{2}\right) - 2499n^2\sigma_1(n) - 2499n(4n-1)\sigma_1\left(\frac{n}{2}\right) \\
&\quad \left. - 3264d(n) - 102c(n) - 14(45n-34)b(n) \right\}.
\end{aligned}$$

Remark 1.1. **Theorem 1.1 and Theorem 1.2 help us to approach easily** to another convolution sums, for example, $T_{m,e,f}^3(n)$, $T_{m,e,f}^4(n)$, etc., and we expect that this work allows us to find some relations between divisor functions and the infinite product sums.

2 The various convolution sums multiplying m^2

We introduce Proposition 2.1 to obtain some various convolution sum formulae.

Proposition 2.1. (See ([1], Theorem 1.1)) For $q \in \mathbb{C}$ with $|q| < 1$, we have

(a)

$$L(q)M(q^2) = 1 - 24 \sum_{n=1}^{\infty} \sigma_5(n)q^n + 1440 \sum_{n=1}^{\infty} n\sigma_3(n)q^{2n} - 480 \sum_{n=1}^{\infty} \sigma_5(n)q^{2n},$$

(b)

$$L(q^2)M(q) = 1 + 360 \sum_{n=1}^{\infty} n\sigma_3(n)q^n - 120 \sum_{n=1}^{\infty} \sigma_5(n)q^n - 384 \sum_{n=1}^{\infty} \sigma_5(n)q^{2n},$$

(c)

$$\begin{aligned}
L(q)L^2(q^2) &= 1 - 144 \sum_{n=1}^{\infty} n^2\sigma_1(n)q^n + 144 \sum_{n=1}^{\infty} n\sigma_3(n)q^n - 24 \sum_{n=1}^{\infty} \sigma_5(n)q^n \\
&\quad - 2304 \sum_{n=1}^{\infty} n^2\sigma_1(n)q^{2n} + 2592 \sum_{n=1}^{\infty} n\sigma_3(n)q^{2n} - 480 \sum_{n=1}^{\infty} \sigma_5(n)q^{2n},
\end{aligned}$$

(d)

$$L^2(q)L(q^2) = 1 - 576 \sum_{n=1}^{\infty} n^2 \sigma_1(n) q^n + 648 \sum_{n=1}^{\infty} n \sigma_3(n) q^n - 120 \sum_{n=1}^{\infty} \sigma_5(n) q^n \\ - 2304 \sum_{n=1}^{\infty} n^2 \sigma_1(n) q^{2n} + 2304 \sum_{n=1}^{\infty} n \sigma_3(n) q^{2n} - 384 \sum_{n=1}^{\infty} \sigma_5(n) q^{2n}.$$

Proof of Theorem 1.1. (a) By (1.5) and (1.9), we have

$$L^3(q)L(q^2) = L^3(q) \cdot L(q^2) \\ = \left(1 - 1728 \sum_{n=1}^{\infty} n^2 \sigma_1(n) q^n + 2160 \sum_{n=1}^{\infty} n \sigma_3(n) q^n - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n \right) \\ \times \left(1 - 24 \sum_{m=1}^{\infty} \sigma_1(m) q^{2m} \right) \\ = 1 + \sum_{N=1}^{\infty} \left\{ -1728 N^2 \sigma_1(N) + 2160 N \sigma_3(N) - 504 \sigma_5(N) - 24 \sigma_1\left(\frac{N}{2}\right) \right. \\ + 1728 \cdot 24 \sum_{m < \frac{N}{2}} (N-2m)^2 \sigma_1(N-2m) \sigma_1(m) \\ - 2160 \cdot 24 \sum_{m < \frac{N}{2}} (N-2m) \sigma_3(N-2m) \sigma_1(m) \\ \left. + 504 \cdot 24 \sum_{m < \frac{N}{2}} \sigma_5(N-2m) \sigma_1(m) \right\} q^N \\ = 1 + \sum_{N=1}^{\infty} \left\{ -1728 N^2 \sigma_1(N) + 2160 N \sigma_3(N) - 504 \sigma_5(N) - 24 \sigma_1\left(\frac{N}{2}\right) \right. \\ + 1728 \cdot 24 N^2 \cdot T_{1,2}(N) - 1728 \cdot 24 \cdot 4N \cdot T_{m,1,1}(N) \\ + 1728 \cdot 24 \cdot 4 \sum_{m < \frac{N}{2}} m^2 \sigma_1(m) \sigma_1(N-2m) - 2160 \cdot 24N \cdot T_{1,3}(N) \\ \left. + 2160 \cdot 24 \cdot 2 \cdot T_{m,1,3}(N) + 504 \cdot 24 \cdot T_{1,5}(N) \right\} q^N. \quad (2.1)$$

Also, by (1.5) and Proposition 2.1 (d), we note that

$$\begin{aligned}
 L^3(q)L(q^2) &= L(q) \cdot L^2(q)L(q^2) \\
 &= \left(1 - 24 \sum_{n=1}^{\infty} \sigma_1(n)q^n\right) \left(1 - 576 \sum_{m=1}^{\infty} m^2 \sigma_1(m)q^m + 648 \sum_{m=1}^{\infty} m \sigma_3(m)q^m \right. \\
 &\quad \left. - 120 \sum_{m=1}^{\infty} \sigma_5(m)q^m - 2304 \sum_{m=1}^{\infty} m^2 \sigma_1(m)q^{2m} + 2304 \sum_{m=1}^{\infty} m \sigma_3(m)q^{2m} \right. \\
 &\quad \left. - 384 \sum_{m=1}^{\infty} \sigma_5(m)q^{2m}\right) \\
 &= 1 + \sum_{N=1}^{\infty} \left\{ -576N^2 \sigma_1(N) + 648N \sigma_3(N) - 120 \sigma_5(N) - 576N^2 \sigma_1\left(\frac{N}{2}\right) \right. \\
 &\quad + 1152N \sigma_3\left(\frac{N}{2}\right) - 384 \sigma_5\left(\frac{N}{2}\right) - 24 \sigma_1(N) + 24 \cdot 576 \sum_{m=1}^{N-1} \sigma_1(N-m) \cdot m^2 \sigma_1(m) \\
 &\quad - 24 \cdot 648 \sum_{m=1}^{N-1} \sigma_1(N-m) \cdot m \sigma_3(m) + 24 \cdot 120 \sum_{m=1}^{N-1} \sigma_1(N-m) \sigma_5(m) \\
 &\quad + 24 \cdot 2304 \sum_{m < \frac{N}{2}} \sigma_1(N-2m) \cdot m^2 \sigma_1(m) \\
 &\quad \left. - 24 \cdot 2304 \sum_{m < \frac{N}{2}} \sigma_1(N-2m) \cdot m \sigma_3(m) + 24 \cdot 384 \sum_{m < \frac{N}{2}} \sigma_1(N-2m) \sigma_5(m) \right\} q^N \\
 &= 1 + \sum_{N=1}^{\infty} \left\{ -576N^2 \sigma_1(N) + 648N \sigma_3(N) - 120 \sigma_5(N) - 576N^2 \sigma_1\left(\frac{N}{2}\right) \right. \\
 &\quad + 1152N \sigma_3\left(\frac{N}{2}\right) - 384 \sigma_5\left(\frac{N}{2}\right) - 24 \sigma_1(N) + 24 \cdot 576 \cdot I_{m,1,1}^2(N) \\
 &\quad - 24 \cdot 648 \cdot I_{m,3,1}(N) + 24 \cdot 120 \cdot I_{1,5}(N) \\
 &\quad + 24 \cdot 2304 \sum_{m < \frac{N}{2}} m^2 \sigma_1(m) \sigma_1(N-2m) - 24 \cdot 2304 \cdot T_{m,3,1}(N) \\
 &\quad \left. + 24 \cdot 384 \cdot T_{5,1}(N) \right\} q^N.
 \end{aligned} \tag{2.2}$$

Therefore, equating (2.1) with (2.2) and using Proposition 1.1 (b), Proposition 1.2 (a), (1.16), and Proposition 1.3 (a), we obtain the proof.

(b) By (1.6) and (1.9), we have

$$\begin{aligned}
 L^3(q^2)M(q) &= M(q) \cdot L^3(q^2) \\
 &= \left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n)q^n \right) \\
 &\quad \times \left(1 - 1728 \sum_{m=1}^{\infty} m^2 \sigma_1(m)q^{2m} + 2160 \sum_{n=1}^{\infty} m \sigma_3(m)q^{2m} - 504 \sum_{m=1}^{\infty} \sigma_5(m)q^{2m} \right) \\
 &= 1 + \sum_{N=1}^{\infty} \left\{ -432N^2 \sigma_1\left(\frac{N}{2}\right) + 1080N \sigma_3\left(\frac{N}{2}\right) - 504 \sigma_5\left(\frac{N}{2}\right) + 240 \sigma_3(N) \right. \\
 &\quad - 240 \cdot 1728 \sum_{m < \frac{N}{2}} \sigma_3(N-2m) \cdot m^2 \sigma_1(m) \\
 &\quad + 240 \cdot 2160 \sum_{m < \frac{N}{2}} \sigma_3(N-2m) \cdot m \sigma_3(m) \\
 &\quad \left. - 240 \cdot 504 \sum_{m < \frac{N}{2}} \sigma_3(N-2m) \sigma_5(m) \right\} q^N \\
 &= 1 + \sum_{N=1}^{\infty} \left\{ -432N^2 \sigma_1\left(\frac{N}{2}\right) + 1080N \sigma_3\left(\frac{N}{2}\right) - 504 \sigma_5\left(\frac{N}{2}\right) + 240 \sigma_3(N) \right. \\
 &\quad - 240 \cdot 1728 \sum_{m < \frac{N}{2}} m^2 \sigma_1(m) \sigma_3(N-2m) + 240 \cdot 2160 \cdot T_{m,3,3}(N) \\
 &\quad \left. - 240 \cdot 504 \cdot T_{5,3}(N) \right\} q^N.
 \end{aligned} \tag{2.3}$$

And by (1.8) and Proposition 2.1 (b), we obtain

$$\begin{aligned}
 L^3(q^2)M(q) &= L(q^2)M(q) \cdot L^2(q^2) \\
 &= \left(1 + 360 \sum_{n=1}^{\infty} n\sigma_3(n)q^n - 120 \sum_{n=1}^{\infty} \sigma_5(n)q^n - 384 \sum_{n=1}^{\infty} \sigma_5(n)q^{2n} \right) \\
 &\quad \times \left(1 - 288 \sum_{m=1}^{\infty} m\sigma_1(m)q^{2m} + 240 \sum_{m=1}^{\infty} \sigma_3(m)q^{2m} \right) \\
 &= 1 + \sum_{N=1}^{\infty} \left\{ -144N\sigma_1\left(\frac{N}{2}\right) + 240\sigma_3\left(\frac{N}{2}\right) + 360N\sigma_3(N) \right. \\
 &\quad - 360 \cdot 288 \sum_{m < \frac{N}{2}} (N-2m)\sigma_3(N-2m) \cdot m\sigma_1(m) \\
 &\quad + 360 \cdot 240 \sum_{m < \frac{N}{2}} (N-2m)\sigma_3(N-2m)\sigma_3(m) - 120\sigma_5(N) \\
 &\quad + 120 \cdot 288 \sum_{m < \frac{N}{2}} \sigma_5(N-2m) \cdot m\sigma_1(m) - 120 \cdot 240 \sum_{m < \frac{N}{2}} \sigma_5(N-2m)\sigma_3(m) \\
 &\quad - 384\sigma_5\left(\frac{N}{2}\right) + 384 \cdot 288 \sum_{m < \frac{N}{2}} \sigma_5\left(\frac{N}{2} - m\right) \cdot m\sigma_1(m) \\
 &\quad \left. - 384 \cdot 240 \sum_{m < \frac{N}{2}} \sigma_5\left(\frac{N}{2} - m\right)\sigma_3(m) \right\} q^N \\
 &= 1 + \sum_{N=1}^{\infty} \left\{ -144N\sigma_1\left(\frac{N}{2}\right) + 240\sigma_3\left(\frac{N}{2}\right) + 360N\sigma_3(N) - 360 \cdot 288N \cdot T_{m,1,3}(N) \right. \\
 &\quad + 360 \cdot 288 \cdot 2 \sum_{m < \frac{N}{2}} m^2\sigma_1(m)\sigma_3(N-2m) + 360 \cdot 240N \cdot T_{3,3}(N) \\
 &\quad - 360 \cdot 240 \cdot 2 \cdot T_{m,3,3}(N) - 120\sigma_5(N) + 120 \cdot 288 \cdot T_{m,1,5}(N) \\
 &\quad - 120 \cdot 240 \cdot T_{3,5}(N) - 384\sigma_5\left(\frac{N}{2}\right) + 384 \cdot 288 \cdot I_{m,1,5}\left(\frac{N}{2}\right) \\
 &\quad \left. - 384 \cdot 240 \cdot I_{3,5}\left(\frac{N}{2}\right) \right\} q^N.
 \end{aligned} \tag{2.4}$$

So we equate (2.3) with (2.4) and refer to Proposition 1.1 (c), Proposition 1.3 (a) and (b).

(c) From (1.6) and (1.9) we note that

$$\begin{aligned}
 L^3(q)M(q^2) &= L^3(q) \cdot M(q^2) \\
 &= \left(1 - 1728 \sum_{n=1}^{\infty} n^2 \sigma_1(n) q^n + 2160 \sum_{n=1}^{\infty} n \sigma_3(n) q^n - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n \right) \\
 &\quad \times \left(1 + 240 \sum_{m=1}^{\infty} \sigma_3(m) q^{2m} \right) \\
 &= 1 + \sum_{N=1}^{\infty} \left\{ -1728 N^2 \sigma_1(N) + 2160 N \sigma_3(N) - 504 \sigma_5(N) + 240 \sigma_3\left(\frac{N}{2}\right) \right. \\
 &\quad - 1728 \cdot 240 \sum_{m < \frac{N}{2}} (N-2m)^2 \sigma_1(N-2m) \sigma_3(m) \\
 &\quad + 2160 \cdot 240 \sum_{m < \frac{N}{2}} (N-2m) \sigma_3(N-2m) \sigma_3(m) \\
 &\quad \left. - 504 \cdot 240 \sum_{m < \frac{N}{2}} \sigma_5(N-2m) \sigma_3(m) \right\} q^N \\
 &= 1 + \sum_{N=1}^{\infty} \left\{ -1728 N^2 \sigma_1(N) + 2160 N \sigma_3(N) - 504 \sigma_5(N) + 240 \sigma_3\left(\frac{N}{2}\right) \right. \\
 &\quad - 1728 \cdot 240 N^2 \cdot T_{3,1}(N) + 1728 \cdot 240 \cdot 4N \cdot T_{m,3,1}(N) \\
 &\quad - 1728 \cdot 240 \cdot 4 \sum_{m < \frac{N}{2}} m^2 \sigma_3(m) \sigma_1(N-2m) + 2160 \cdot 240 N \cdot T_{3,3}(N) \\
 &\quad \left. - 2160 \cdot 240 \cdot 2 \cdot T_{m,3,3}(N) - 504 \cdot 240 \cdot T_{3,5}(N) \right\} q^N.
 \end{aligned} \tag{2.5}$$

Similarly, by (1.8) and Proposition 2.1 (a), we have

$$\begin{aligned}
 L^3(q)M(q^2) &= L^2(q) \cdot L(q)M(q^2) \\
 &= \left(1 - 288 \sum_{n=1}^{\infty} n\sigma_1(n)q^n + 240 \sum_{n=1}^{\infty} \sigma_3(n)q^n\right) \\
 &\quad \times \left(1 - 24 \sum_{m=1}^{\infty} \sigma_5(m)q^m + 1440 \sum_{m=1}^{\infty} m\sigma_3(m)q^{2m} - 480 \sum_{m=1}^{\infty} \sigma_5(m)q^{2m}\right) \\
 &= 1 + \sum_{N=1}^{\infty} \left\{ -24\sigma_5(N) + 720N\sigma_3\left(\frac{N}{2}\right) - 480\sigma_5\left(\frac{N}{2}\right) - 288N\sigma_1(N) \right. \\
 &\quad + 288 \cdot 24 \sum_{m=1}^{N-1} (N-m)\sigma_1(N-m)\sigma_5(m) \\
 &\quad - 288 \cdot 1440 \sum_{m < \frac{N}{2}} (N-2m)\sigma_1(N-2m) \cdot m\sigma_3(m) \\
 &\quad + 288 \cdot 480 \sum_{m < \frac{N}{2}} (N-2m)\sigma_1(N-2m)\sigma_5(m) + 240\sigma_3(N) \\
 &\quad - 240 \cdot 24 \sum_{m=1}^{n-1} \sigma_3(N-m)\sigma_5(m) + 240 \cdot 1440 \sum_{m < \frac{N}{2}} \sigma_3(N-2m) \cdot m\sigma_3(m) \\
 &\quad \left. - 240 \cdot 480 \sum_{m < \frac{N}{2}} \sigma_3(N-2m)\sigma_5(m) \right\} q^N \\
 &= 1 + \sum_{N=1}^{\infty} \left\{ -24\sigma_5(N) + 720N\sigma_3\left(\frac{N}{2}\right) - 480\sigma_5\left(\frac{N}{2}\right) - 288N\sigma_1(N) + 240\sigma_3(N) \right. \\
 &\quad + 288 \cdot 24N \cdot I_{1,5}(N) - 288 \cdot 24 \cdot I_{m,5,1}(N) - 288 \cdot 1440N \cdot T_{m,3,1}(N) \\
 &\quad + 288 \cdot 1440 \cdot 2 \sum_{m < \frac{N}{2}} m^2\sigma_3(m)\sigma_1(N-2m) + 288 \cdot 480N \cdot T_{5,1}(N) \\
 &\quad - 288 \cdot 480 \cdot 2 \cdot T_{m,5,1}(N) - 240 \cdot 24 \cdot I_{3,5}(N) + 240 \cdot 1440 \cdot T_{m,3,3}(N) \\
 &\quad \left. - 240 \cdot 480 \cdot T_{5,3}(N) \right\} q^N.
 \end{aligned} \tag{2.6}$$

Therefore, equating (2.5) with (2.6) and appealing to Proposition 1.1 (c), Proposition 1.3 (a) and (b), we obtain the proof.

(d) By (1.8) and (1.12), we obtain

$$\begin{aligned}
 L(q)L^2(q^2)N(q) &= L(q)N(q) \cdot L^2(q^2) \\
 &= \left(1 - 1008 \sum_{n=1}^{\infty} n\sigma_5(n)q^n + 480 \sum_{n=1}^{\infty} \sigma_7(n)q^n\right) \\
 &\quad \times \left(1 - 288 \sum_{m=1}^{\infty} m\sigma_1(m)q^{2m} + 240 \sum_{m=1}^{\infty} \sigma_3(m)q^{2m}\right) \\
 &= 1 + \sum_{N=1}^{\infty} \left\{ -144N\sigma_1\left(\frac{N}{2}\right) + 240\sigma_3\left(\frac{N}{2}\right) - 1008N\sigma_5(N) \right. \\
 &\quad + 1008 \cdot 288 \sum_{m < \frac{N}{2}} (N-2m)\sigma_5(N-2m) \cdot m\sigma_1(m) \\
 &\quad - 1008 \cdot 240 \sum_{m < \frac{N}{2}} (N-2m)\sigma_5(N-2m)\sigma_3(m) + 480\sigma_7(N) \\
 &\quad - 480 \cdot 288 \sum_{m < \frac{N}{2}} \sigma_7(N-2m) \cdot m\sigma_1(m) \\
 &\quad \left. + 480 \cdot 240 \sum_{m < \frac{N}{2}} \sigma_7(N-2m)\sigma_3(m) \right\} q^N \\
 &= 1 + \sum_{N=1}^{\infty} \left\{ -144N\sigma_1\left(\frac{N}{2}\right) + 240\sigma_3\left(\frac{N}{2}\right) - 1008N\sigma_5(N) + 1008 \cdot 288N \cdot T_{m,1,5}(N) \right. \\
 &\quad - 1008 \cdot 288 \cdot 2 \sum_{m < \frac{N}{2}} m^2\sigma_1(m)\sigma_5(N-2m) - 1008 \cdot 240N \cdot T_{3,5}(N) \\
 &\quad + 1008 \cdot 240 \cdot 2 \cdot T_{m,3,5}(N) + 480\sigma_7(N) - 480 \cdot 288 \cdot T_{m,1,7}(N) \\
 &\quad \left. + 480 \cdot 240 \cdot T_{3,7}(N) \right\} q^N.
 \end{aligned} \tag{2.7}$$

Also by (1.7) and Proposition 2.1 (c), we observe that

$$\begin{aligned}
 L(q)L^2(q^2)N(q) &= N(q) \cdot L(q)L^2(q^2) \\
 &= \left(1 - 504 \sum_{n=1}^{\infty} \sigma_5(n)q^n\right) \left(1 - 144 \sum_{m=1}^{\infty} m^2 \sigma_1(m)q^m + 144 \sum_{m=1}^{\infty} m \sigma_3(m)q^m \right. \\
 &\quad \left. - 24 \sum_{m=1}^{\infty} \sigma_5(m)q^m - 2304 \sum_{m=1}^{\infty} m^2 \sigma_1(m)q^{2m} + 2592 \sum_{m=1}^{\infty} m \sigma_3(m)q^{2m} \right. \\
 &\quad \left. - 480 \sum_{m=1}^{\infty} \sigma_5(m)q^{2m}\right) \\
 &= 1 + \sum_{N=1}^{\infty} \left\{ -144N^2 \sigma_1(N) + 144N \sigma_3(N) - 24 \sigma_5(N) - 576N^2 \sigma_1\left(\frac{N}{2}\right) \right. \\
 &\quad \left. + 1296N \sigma_3\left(\frac{N}{2}\right) - 480 \sigma_5\left(\frac{N}{2}\right) - 504 \sigma_5(N) \right. \\
 &\quad \left. + 504 \cdot 144 \sum_{m=1}^{N-1} \sigma_5(N-m) \cdot m^2 \sigma_1(m) \right. \\
 &\quad \left. - 504 \cdot 144 \sum_{m=1}^{N-1} \sigma_5(N-m) \cdot m \sigma_3(m) + 504 \cdot 24 \sum_{m=1}^{N-1} \sigma_5(N-m) \sigma_5(m) \right. \\
 &\quad \left. + 504 \cdot 2304 \sum_{m < \frac{N}{2}} \sigma_5(N-2m) \cdot m^2 \sigma_1(m) \right. \\
 &\quad \left. - 504 \cdot 2592 \sum_{m < \frac{N}{2}} \sigma_5(N-2m) \cdot m \sigma_3(m) \right. \\
 &\quad \left. + 504 \cdot 480 \sum_{m < \frac{N}{2}} \sigma_5(N-2m) \sigma_5(m) \right\} q^N \\
 &= 1 + \sum_{N=1}^{\infty} \left\{ -144N^2 \sigma_1(N) + 144N \sigma_3(N) - 528 \sigma_5(N) - 576N^2 \sigma_1\left(\frac{N}{2}\right) \right. \\
 &\quad \left. + 1296N \sigma_3\left(\frac{N}{2}\right) - 480 \sigma_5\left(\frac{N}{2}\right) + 504 \cdot 144 \cdot I_{m,1,5}^2(N) - 504 \cdot 144 \cdot I_{m,3,5}(N) \right. \\
 &\quad \left. + 504 \cdot 24 \cdot I_{5,5}(N) + 504 \cdot 2304 \sum_{m < \frac{N}{2}} m^2 \sigma_1(m) \sigma_5(N-2m) \right. \\
 &\quad \left. - 504 \cdot 2592 \cdot T_{m,3,5}(N) + 504 \cdot 480 \cdot T_{5,5}(N) \right\} q^N.
 \end{aligned} \tag{2.8}$$

Thus we equate (2.7) with (2.8) and use Proposition 1.1 (d), Proposition 1.2 (c), Proposition 1.3 (b) and (c).

(e) From (1.6) and (1.11) we have

$$\begin{aligned}
 L^2(q^2)M(q)M(q^2) &= M(q) \cdot L^2(q^2)M(q^2) \\
 &= \left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n)q^n\right) \\
 &\quad \times \left(1 + 1728 \sum_{m=1}^{\infty} m^2 \sigma_3(m)q^{2m} - 2016 \sum_{m=1}^{\infty} m \sigma_5(m)q^{2m} + 480 \sum_{m=1}^{\infty} \sigma_7(m)q^{2m}\right) \\
 &= 1 + \sum_{N=1}^{\infty} \left\{ 432N^2 \sigma_3\left(\frac{N}{2}\right) - 1008N \sigma_5\left(\frac{N}{2}\right) + 480 \sigma_7\left(\frac{N}{2}\right) + 240 \sigma_3(N) \right. \\
 &\quad + 240 \cdot 1728 \sum_{m < \frac{N}{2}} \sigma_3(N-2m) \cdot m^2 \sigma_3(m) \\
 &\quad - 240 \cdot 2016 \sum_{m < \frac{N}{2}} \sigma_3(N-2m) \cdot m \sigma_5(m) \\
 &\quad \left. + 240 \cdot 480 \sum_{m < \frac{N}{2}} \sigma_3(N-2m) \sigma_7(m) \right\} q^N \\
 &= 1 + \sum_{N=1}^{\infty} \left\{ 432N^2 \sigma_3\left(\frac{N}{2}\right) - 1008N \sigma_5\left(\frac{N}{2}\right) + 480 \sigma_7\left(\frac{N}{2}\right) + 240 \sigma_3(N) \right. \\
 &\quad + 240 \cdot 1728 \sum_{m < \frac{N}{2}} m^2 \sigma_3(m) \sigma_3(N-2m) - 240 \cdot 2016 \cdot T_{m,5,3}(N) \\
 &\quad \left. + 240 \cdot 480 \cdot T_{7,3}(N) \right\} q^N.
 \end{aligned} \tag{2.9}$$

Similarly, by (1.10) and Proposition 2.1 (b), we obtain

$$\begin{aligned}
L^2(q^2)M(q)M(q^2) &= L(q^2)M(q) \cdot L(q^2)M(q^2) \\
&= \left(1 + 360 \sum_{n=1}^{\infty} n\sigma_3(n)q^n - 120 \sum_{n=1}^{\infty} \sigma_5(n)q^n - 384 \sum_{n=1}^{\infty} \sigma_5(n)q^{2n}\right) \\
&\quad \times \left(1 + 720 \sum_{m=1}^{\infty} m\sigma_3(m)q^{2m} - 504 \sum_{m=1}^{\infty} \sigma_5(m)q^{2m}\right) \\
&= 1 + \sum_{N=1}^{\infty} \left\{ 360N\sigma_3\left(\frac{N}{2}\right) - 504\sigma_5\left(\frac{N}{2}\right) + 360N\sigma_3(N) \right. \\
&\quad + 360 \cdot 720 \sum_{m < \frac{N}{2}} (N-2m)\sigma_3(N-2m) \cdot m\sigma_3(m) \\
&\quad - 360 \cdot 504 \sum_{m < \frac{N}{2}} (N-2m)\sigma_3(N-2m)\sigma_5(m) - 120\sigma_5(N) \\
&\quad - 120 \cdot 720 \sum_{m < \frac{N}{2}} \sigma_5(N-2m) \cdot m\sigma_3(m) \\
&\quad + 120 \cdot 504 \sum_{m < \frac{N}{2}} \sigma_5(N-2m)\sigma_5(m) - 384\sigma_5\left(\frac{N}{2}\right) \\
&\quad \left. - 384 \cdot 720 \sum_{m < \frac{N}{2}} \sigma_5\left(\frac{N}{2} - m\right) \cdot m\sigma_3(m) + 384 \cdot 504 \sum_{m < \frac{N}{2}} \sigma_5\left(\frac{N}{2} - m\right)\sigma_5(m) \right\} q^N \\
&= 1 + \sum_{N=1}^{\infty} \left\{ 360N\sigma_3\left(\frac{N}{2}\right) - 888\sigma_5\left(\frac{N}{2}\right) + 360N\sigma_3(N) - 120\sigma_5(N) \right. \\
&\quad + 360 \cdot 720N \cdot T_{m,3,3}(N) - 360 \cdot 720 \cdot 2 \sum_{m < \frac{N}{2}} m^2\sigma_3(m)\sigma_3(N-2m) \\
&\quad - 360 \cdot 504N \cdot T_{5,3}(N) + 360 \cdot 504 \cdot 2 \cdot T_{m,5,3}(N) - 120 \cdot 720 \cdot T_{m,3,5}(N) \\
&\quad \left. + 120 \cdot 504 \cdot T_{5,5}(N) - 384 \cdot 720 \cdot I_{m,3,5}\left(\frac{N}{2}\right) + 384 \cdot 504 \cdot I_{5,5}\left(\frac{N}{2}\right) \right\} q^N.
\end{aligned} \tag{2.10}$$

So equating (2.9) with (2.10) and using Proposition 1.1 (d), Proposition 1.3 (b) and (c), we can complete the proof.

(f) By (1.5) and (1.13), we note that

$$\begin{aligned}
 L(q)L^2(q^2)N(q^2) &= L(q) \cdot L^2(q^2)N(q^2) \\
 &= \left(1 - 24 \sum_{n=1}^{\infty} \sigma_1(n)q^n\right) \\
 &\quad \times \left(1 - 1728 \sum_{m=1}^{\infty} m^2 \sigma_5(m)q^{2m} + 1440 \sum_{m=1}^{\infty} m \sigma_7(m)q^{2m} - 264 \sum_{m=1}^{\infty} \sigma_9(m)q^{2m}\right) \\
 &= 1 + \sum_{N=1}^{\infty} \left\{ -432N^2 \sigma_5\left(\frac{N}{2}\right) + 720N \sigma_7\left(\frac{N}{2}\right) - 264 \sigma_9\left(\frac{N}{2}\right) - 24 \sigma_1(N) \right. \\
 &\quad + 24 \cdot 1728 \sum_{m < \frac{N}{2}} \sigma_1(N-2m) \cdot m^2 \sigma_5(m) - 24 \cdot 1440 \sum_{m < \frac{N}{2}} \sigma_1(N-2m) \cdot m \sigma_7(m) \\
 &\quad \left. + 24 \cdot 264 \sum_{m < \frac{N}{2}} \sigma_1(N-2m) \sigma_9(m) \right\} q^N \\
 &= 1 + \sum_{N=1}^{\infty} \left\{ -432N^2 \sigma_5\left(\frac{N}{2}\right) + 720N \sigma_7\left(\frac{N}{2}\right) - 264 \sigma_9\left(\frac{N}{2}\right) - 24 \sigma_1(N) \right. \\
 &\quad + 24 \cdot 1728 \sum_{m < \frac{N}{2}} m^2 \sigma_5(m) \sigma_1(N-2m) - 24 \cdot 1440 \cdot T_{m,7,1}(N) \\
 &\quad \left. + 24 \cdot 264 \cdot T_{9,1}(N) \right\} q^N.
 \end{aligned} \tag{2.11}$$

And by (1.7) and Proposition 2.1 (c), we obtain

$$\begin{aligned}
 L(q)L^2(q^2)N(q^2) &= L(q)L^2(q^2) \cdot N(q^2) \\
 &= \left(1 - 144 \sum_{n=1}^{\infty} n^2 \sigma_1(n) q^n + 144 \sum_{n=1}^{\infty} n \sigma_3(n) q^n - 24 \sum_{n=1}^{\infty} \sigma_5(n) q^n \right. \\
 &\quad \left. - 2304 \sum_{n=1}^{\infty} n^2 \sigma_1(n) q^{2n} + 2592 \sum_{n=1}^{\infty} n \sigma_3(n) q^{2n} - 480 \sum_{n=1}^{\infty} \sigma_5(n) q^{2n} \right) \\
 &\quad \times \left(1 - 504 \sum_{m=1}^{\infty} \sigma_5(m) q^{2m}\right) \\
 &= 1 + \sum_{N=1}^{\infty} \left\{ -144N^2 \sigma_1(N) + 144N \sigma_3(N) - 24 \sigma_5(N) - 576N^2 \sigma_1\left(\frac{N}{2}\right) \right. \\
 &\quad + 1296N \sigma_3\left(\frac{N}{2}\right) - 480 \sigma_5\left(\frac{N}{2}\right) - 504 \sigma_5\left(\frac{N}{2}\right) \\
 &\quad + 144 \cdot 504 \sum_{m < \frac{N}{2}} (N-2m)^2 \sigma_1(N-2m) \sigma_5(m) \\
 &\quad - 144 \cdot 504 \sum_{m < \frac{N}{2}} (N-2m) \sigma_3(N-2m) \sigma_5(m) \\
 &\quad + 24 \cdot 504 \sum_{m < \frac{N}{2}} \sigma_5(N-2m) \sigma_5(m) + 2304 \cdot 504 \sum_{n < \frac{N}{2}} n^2 \sigma_1(n) \sigma_5\left(\frac{N}{2} - n\right) \\
 &\quad \left. - 2592 \cdot 504 \sum_{n < \frac{N}{2}} n \sigma_3(n) \sigma_5\left(\frac{N}{2} - n\right) + 480 \cdot 504 \sum_{m < \frac{N}{2}} \sigma_5\left(\frac{N}{2} - m\right) \sigma_5(m) \right\} q^N \\
 &= 1 + \sum_{N=1}^{\infty} \left\{ -144N^2 \sigma_1(N) + 144N \sigma_3(N) - 24 \sigma_5(N) - 576N^2 \sigma_1\left(\frac{N}{2}\right) \right. \\
 &\quad + 1296N \sigma_3\left(\frac{N}{2}\right) - 984 \sigma_5\left(\frac{N}{2}\right) + 144 \cdot 504N^2 \cdot T_{5,1}(N) \\
 &\quad - 144 \cdot 504 \cdot 4N \cdot T_{m,5,1}(N) + 144 \cdot 504 \cdot 4 \sum_{m < \frac{N}{2}} m^2 \sigma_5(m) \sigma_1(N-2m) \\
 &\quad - 144 \cdot 504N \cdot T_{5,3}(N) + 144 \cdot 504 \cdot 2 \cdot T_{m,5,3}(N) + 24 \cdot 504 \cdot T_{5,5}(N) \\
 &\quad \left. + 2304 \cdot 504 \cdot I_{m,1,5}^2\left(\frac{N}{2}\right) - 2592 \cdot 504 \cdot I_{m,3,5}\left(\frac{N}{2}\right) + 480 \cdot 504 \cdot I_{5,5}\left(\frac{N}{2}\right) \right\} q^N.
 \end{aligned} \tag{2.12}$$

Therefore we equate (2.11) with (2.12) and refer to Proposition 1.1 (d), Proposition 1.2 (c), Proposition 1.3 (b) and (c). □

3 The induced identities from $\sum_{m < \frac{n}{2}} m^2 \sigma_e(m) \sigma_f(n-2m)$

In Theorem 1.2 we obtain some identities induced from Theorem 1.1 :

Proof of Theorem 1.2. (a) Insert Theorem 1.1 (a) into (2.1).

(b) Insert Theorem 1.1 (b) into (2.3).

- (c) Insert Theorem 1.1 (c) into (2.5).
- (d) Insert Theorem 1.1 (d) into (2.7).
- (e) Insert Theorem 1.1 (e) into (2.9).
- (f) Insert Theorem 1.1 (f) into (2.11).

□

Now let us consider the convolution sums composing with three divisor functions. In advance, we can see the following proposition in ([5], Theorem 3.11) by the definition of $\sigma_{s,r}(n; m)$ in (1.1) :

Proposition 3.1. *Let $n \in \mathbb{N}$. Then we have*

(a)

$$\begin{aligned} & \sum_{m=1}^{n-1} m\sigma_{1,1}(m; 2)\sigma_{3,1}(n-m; 2) \\ &= \frac{1}{240} \left[n \left\{ \sigma_5(n) + 48\sigma_5\left(\frac{n}{2}\right) + 7\sigma_1(n) - 14\sigma_1\left(\frac{n}{2}\right) \right\} - 8b(n) \right], \end{aligned}$$

(b)

$$\begin{aligned} & \sum_{m=1}^{n-1} m\sigma_{1,1}(m; 2)\sigma_{3,0}(n-m; 2) \\ &= \frac{1}{90} \left[n \left\{ \sigma_5(n) - 22\sigma_5\left(\frac{n}{2}\right) - 3\sigma_1(n) + 6\sigma_1\left(\frac{n}{2}\right) \right\} + 2b(n) \right], \end{aligned}$$

(c)

$$\sum_{m=1}^{n-1} m\sigma_{1,0}(m; 2)\sigma_{3,0}(n-m; 2) = \frac{1}{15} n \left\{ 7\sigma_5\left(\frac{n}{2}\right) - 3n\sigma_3\left(\frac{n}{2}\right) - \sigma_1\left(\frac{n}{2}\right) \right\},$$

(d)

$$\begin{aligned} & \sum_{m=1}^{n-1} m\sigma_{1,0}(m; 2)\sigma_{3,1}(n-m; 2) \\ &= \frac{1}{360} \left[n \left\{ 5\sigma_5(n) - 152\sigma_5\left(\frac{n}{2}\right) - 9n\sigma_{3,1}(n; 2) + 21\sigma_1\left(\frac{n}{2}\right) \right\} + 4b(n) \right]. \end{aligned}$$

Moreover we note that

$$\sum_{m=1}^{n-1} \sigma_1(m)b(n-m) = \frac{1}{48} \{ 32d(n) + c(n) - (3n-2)b(n) \} \quad (3.1)$$

in ([7], Lemma 3.1 (a)). We construct the convolutions sums by multiplying $\sigma_1(n)$ into the given convolution sums in Proposition 3.1 and obtain Theorem 1.3.

Proof of Theorem 1.3. (a) By Proposition 3.1 (a), we can expand

$$\begin{aligned}
 & \sum_{\substack{(a,b,c) \in \mathbb{N}^3 \\ a+b+c=n}} a\sigma_{1,1}(a;2)\sigma_{3,1}(b;2)\sigma_1(c) \\
 &= \sum_{m=1}^{n-1} \sum_{k=1}^{m-1} k\sigma_{1,1}(k;2)\sigma_{3,1}(m-k;2)\sigma_1(n-m) \\
 &= \sum_{m=1}^{n-1} \left[\frac{1}{240} \left\{ m \left(\sigma_5(m) + 48\sigma_5\left(\frac{m}{2}\right) + 7\sigma_1(m) - 14\sigma_1\left(\frac{m}{2}\right) \right) - 8b(m) \right\} \right] \sigma_1(n-m) \\
 &= \frac{1}{240} \left\{ \sum_{m=1}^{n-1} m\sigma_5(m)\sigma_1(n-m) + 48 \sum_{m < \frac{n}{2}} 2m\sigma_5(m)\sigma_1(n-2m) \right. \\
 &\quad \left. + 7 \sum_{m=1}^{n-1} m\sigma_1(m)\sigma_1(n-m) - 14 \sum_{m < \frac{n}{2}} 2m\sigma_1(m)\sigma_1(n-2m) \right. \\
 &\quad \left. - 8 \sum_{m=1}^{n-1} b(m)\sigma_1(n-m) \right\} \\
 &= \frac{1}{240} \{ I_{m,5,1}(n) + 48 \cdot 2 \cdot T_{m,5,1}(n) + 7 \cdot I_{m,1,1}(n) - 14 \cdot 2 \cdot T_{m,1,1}(n) \\
 &\quad - 8 \sum_{m=1}^{n-1} b(m)\sigma_1(n-m) \}.
 \end{aligned}$$

So we use Proposition 1.1 (a), (c), (1.16), Proposition 1.3 (b), and (3.1).

(b) By Proposition 3.1 (b), we have

$$\begin{aligned}
 & \sum_{\substack{(a,b,c) \in \mathbb{N}^3 \\ a+b+c=n}} a\sigma_{1,1}(a;2)\sigma_{3,0}(b;2)\sigma_1(c) \\
 &= \sum_{m=1}^{n-1} \sum_{k=1}^{m-1} k\sigma_{1,1}(k;2)\sigma_{3,0}(m-k;2)\sigma_1(n-m) \\
 &= \sum_{m=1}^{n-1} \left[\frac{1}{90} \left\{ m \left(\sigma_5(m) - 22\sigma_5\left(\frac{m}{2}\right) - 3\sigma_1(m) + 6\sigma_1\left(\frac{m}{2}\right) \right) + 2b(m) \right\} \right] \sigma_1(n-m) \\
 &= \frac{1}{90} \left\{ \sum_{m=1}^{n-1} m\sigma_5(m)\sigma_1(n-m) - 22 \sum_{m < \frac{n}{2}} 2m\sigma_5(m)\sigma_1(n-2m) \right. \\
 &\quad \left. - 3 \sum_{m=1}^{n-1} m\sigma_1(m)\sigma_1(n-m) + 6 \sum_{m < \frac{n}{2}} 2m\sigma_1(m)\sigma_1(n-2m) \right. \\
 &\quad \left. + 2 \sum_{m=1}^{n-1} b(m)\sigma_1(n-m) \right\} \\
 &= \frac{1}{90} \{ I_{m,5,1}(n) - 22 \cdot 2 \cdot T_{m,5,1}(n) - 3 \cdot I_{m,1,1}(n) + 6 \cdot 2 \cdot T_{m,1,1}(n) \\
 &\quad + 2 \sum_{m=1}^{n-1} b(m)\sigma_1(n-m) \}.
 \end{aligned}$$

Thus we use Proposition 1.1 (a), (c), (1.16), Proposition 1.3 (b), and (3.1).

(c) By Proposition 3.1 (c), we obtain

$$\begin{aligned}
 & \sum_{\substack{(a,b,c) \in \mathbb{N}^3 \\ a+b+c=n}} a\sigma_{1,0}(a;2)\sigma_{3,0}(b;2)\sigma_1(c) \\
 &= \sum_{m=1}^{n-1} \sum_{k=1}^{m-1} k\sigma_{1,0}(k;2)\sigma_{3,0}(m-k;2)\sigma_1(n-m) \\
 &= \sum_{m=1}^{n-1} \left\{ \frac{1}{15}m \left(7\sigma_5\left(\frac{m}{2}\right) - 3m\sigma_3\left(\frac{m}{2}\right) - \sigma_1\left(\frac{m}{2}\right) \right) \right\} \sigma_1(n-m) \\
 &= \frac{1}{15} \left\{ 7 \sum_{m < \frac{n}{2}} 2m\sigma_5(m)\sigma_1(n-2m) - 3 \sum_{m < \frac{n}{2}} (2m)^2\sigma_3(m)\sigma_1(n-2m) \right. \\
 &\quad \left. - \sum_{m < \frac{n}{2}} 2m\sigma_1(m)\sigma_1(n-2m) \right\} \\
 &= \frac{1}{15} \{ 7 \cdot 2 \cdot T_{m,5,1}(n) - 3 \cdot 4 \cdot T_{m,3,1}^2(n) - 2 \cdot T_{m,1,1}(n) \}.
 \end{aligned}$$

Thus we refer to (1.16), Proposition 1.3 (b), and Theorem 1.1 (c).

(d) By Proposition 3.1 (d), we have

$$\begin{aligned}
 & \sum_{\substack{(a,b,c) \in \mathbb{N}^3 \\ a+b+c=n}} a\sigma_{1,0}(a;2)\sigma_{3,1}(b;2)\sigma_1(c) \\
 &= \sum_{m=1}^{n-1} \sum_{k=1}^{m-1} k\sigma_{1,0}(k;2)\sigma_{3,1}(m-k;2)\sigma_1(n-m) \\
 &= \sum_{m=1}^{n-1} \frac{1}{360} \left[m \left\{ 5\sigma_5(m) - 152\sigma_5\left(\frac{m}{2}\right) - 9m\sigma_{3,1}(m;2) + 21\sigma_1\left(\frac{m}{2}\right) \right\} + 4b(m) \right] \quad (3.2) \\
 &\quad \times \sigma_1(n-m) \\
 &= \sum_{m=1}^{n-1} \frac{1}{360} \left[m \left\{ 5\sigma_5(m) - 152\sigma_5\left(\frac{m}{2}\right) - 9m \left(\sigma_3(m) - 8\sigma_3\left(\frac{m}{2}\right) \right) + 21\sigma_1\left(\frac{m}{2}\right) \right\} \right. \\
 &\quad \left. + 4b(m) \right] \sigma_1(n-m),
 \end{aligned}$$

where we used the property of (1.3) and (1.4) for $\sigma_{3,1}(m;2)$. So (3.2) can be written as

$$\begin{aligned}
& \sum_{\substack{(a,b,c) \in \mathbb{N}^3 \\ a+b+c=n}} a\sigma_{1,0}(a;2)\sigma_{3,1}(b;2)\sigma_1(c) \\
&= \frac{1}{360} \left\{ 5 \sum_{m=1}^{n-1} m\sigma_5(m)\sigma_1(n-m) - 152 \sum_{m < \frac{n}{2}} 2m\sigma_5(m)\sigma_1(n-2m) \right. \\
&\quad - 9 \sum_{m=1}^{n-1} m^2\sigma_3(m)\sigma_1(n-m) + 9 \cdot 8 \sum_{m < \frac{n}{2}} (2m)^2\sigma_3(m)\sigma_1(n-2m) \\
&\quad \left. + 21 \sum_{m < \frac{n}{2}} 2m\sigma_1(m)\sigma_1(n-2m) + 4 \sum_{m=1}^{n-1} b(m)\sigma_1(n-m) \right\} \\
&= \frac{1}{360} \left\{ 5 \cdot I_{m,5,1}(n) - 152 \cdot 2 \cdot T_{m,5,1}(n) - 9 \cdot I_{m,3,1}^2(n) + 9 \cdot 8 \cdot 4 \cdot T_{m,3,1}^2(n) \right. \\
&\quad \left. + 21 \cdot 2 \cdot T_{m,1,1}(n) + 4 \sum_{m=1}^{n-1} b(m)\sigma_1(n-m) \right\}.
\end{aligned}$$

Thus we use Proposition 1.1 (c), Proposition 1.2 (b), (1.16), Proposition 1.3 (b), Theorem 1.1 (c), and (3.1). □

4 Conclusions

In this paper we obtain the various convolution sum formulae of

$$T_{m,e,f}^2(n) = \sum_{m < \frac{n}{2}} m^2 \sigma_e(m) \sigma_f(n-2m)$$

for $n \in \mathbb{N}$ and an odd positive integer e and f . Furthermore we have some identities induced from $T_{m,e,f}^2(n)$.

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5 Appendix

The first twenty values of $\tau(n)$ are given in the Table 1,

n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$
1	1	6	–6048	11	534612	16	987136
2	–24	7	–16744	12	–370944	17	–6905934
3	252	8	84480	13	–577738	18	2727432
4	–1472	9	–113643	14	401856	19	10661420
5	4830	10	–115920	15	1217160	20	–7109760

TABLE 1. $\tau(n)$ for n ($1 \leq n \leq 20$)

similarly the first twenty values of $b(n)$, $c(n)$ and $d(n)$ are listed in the following tables.

n	$b(n)$	n	$b(n)$	n	$b(n)$	n	$b(n)$
1	1	6	–96	11	1092	16	4096
2	–8	7	1016	12	768	17	14706
3	12	8	–512	13	1382	18	16344
4	64	9	–2043	14	–8128	19	–39940
5	–210	10	1680	15	–2520	20	–13440

TABLE 2. $b(n)$ for n ($1 \leq n \leq 20$)

n	$c(n)$	n	$c(n)$	n	$c(n)$	n	$c(n)$
1	1	6	2496	11	-38996	16	-65536
2	-16	7	-4536	12	39936	17	311442
3	100	8	-4096	13	37806	18	-74448
4	-256	9	23085	14	15232	19	128244
5	-154	10	-13920	15	-146472	20	-222720

TABLE 3. $c(n)$ for n ($1 \leq n \leq 20$)

n	$d(n)$	n	$d(n)$	n	$d(n)$	n	$d(n)$
1	0	6	-156	11	-536	16	4096
2	1	7	112	12	-2496	17	-17472
3	-8	8	256	13	4384	18	4653
4	16	9	-576	14	-952	19	5848
5	32	10	870	15	336	20	13920

TABLE 4. $d(n)$ for n ($1 \leq n \leq 20$)

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