

**Original Research Article****On a New Technique for the Solution of the Black-Scholes Partial Differential Equation for Vanilla Put Options****Abstract**

This paper presents a new technique for the solution of the Black-Scholes partial differential equation for vanilla putoptions using the Mellin transforms.

**Keywords:** Black-Scholes Partial Differential Equation, Mellin transform, Vanilla Option

**2010 Mathematics Subject Classification:** 44A15,91G20,91G30

**1.0 Introduction**

Vanilla option is a contract that gives the holder the right, but not the obligation, to buy or sell an instrument at a specified strike price on or before a specified date. Examples of vanilla options are American and European options.

American options allow option holders to exercise the option at any time prior to and including its maturity date, thus increasing the value of the option to the holder relative to European options, which can only be exercised at maturity. The majority of exchange-traded options are American.

European options are options that can only be exercised at the end of its life, at its maturity. They tend to sometimes trade at a discount to its comparable American option. This is because American options allow investors more opportunities to exercise the contract.

Nowadays, the Black-Scholes equation derived by Black-Scholes [2] is widely used in the field of financial mathematics. Despite the success of the Black-Scholes model on hedging and pricing contingent claims, Merton [4] noted early that options quoted on the markets differ systematically from their predicted values, which led up to questioning the distributional assumptions based on geometric wiener process.

The Mellin transforms in the theory of option pricing was introduced by Panini and Srivastav [5]. They derived the expression for the free boundary and price of an American perpetual put as the limit of finite lived options. In this paper we shall derive a new technique which does not require transformation of variablesfor the solution of the Black-Scholes partial differential equation for

vanilla put options using the Mellin transforms. See (F. AlAzemi et al [1], Black and Scholes [2], Jódar et al [3]) for other approaches for the solution of Black-Scholes equation.

The paper is structured as follows. Section 2 gives brief introduction of the Mellin transforms. Section 3 presents a new technique for the solution of the Black-Scholes partial differential equation for vanilla put options such as European put options. Section 4 concludes the paper.

## 2.0 Mellin Transforms

Let  $f(y)$  be a function defined on the positive real axis  $t \in (0, \infty)$ . The Mellin transformation denoted by  $M$  is the operation mapping the function  $f$  into the function  $F$  defined on the complex plane by the relation

$$M(f(y), v) \equiv F(v) = \int_0^{\infty} f(y) y^{v-1} dy \quad (1)$$

The function  $F(v)$  is called the Mellin transform of  $f$ . In general, the integral does exist only for complex values of  $v = a + jb$  such that  $a \in (a_1, a_2)$ , where  $a_1$  and  $a_2$  depend on the function  $f(y)$  to transform. This introduces what is called the strip of definition of the Mellin transform that will be denoted by  $V(a_1, a_2)$ . In some cases, this strip may extend to half-plane ( $a_1 = -\infty$ ) or ( $a_2 = \infty$ ) or to the complex  $v$ -plane ( $a_1 = -\infty$ ) and ( $a_2 = \infty$ ).

Conversely, the inversion formula of (1) is defined as

$$M^{-1}(F(v)) \equiv f(y, v) = \frac{1}{2\pi j} \int_{a-j\infty}^{a+j\infty} F(v) y^{-v} dv \quad (2)$$

where the integration is along a vertical line through  $\text{Re}(v) = a$ .

Some of the basic fundamental properties of the Mellin transforms are detailed below.

If  $f(x)$  is defined on the positive real axis  $t \in (0, \infty)$ , then the following properties hold.

(a) Shifting Property

$$M(y^a f(y); v) = \int_0^{\infty} y^a f(y) y^{v-1} dy = F(v + a) \quad (3)$$

(b) Scaling Property

$$M(f(ay;v)) = \int_0^{\infty} f(ay)y^{v-1}dy = a^{-v}F(v) \quad (4)$$

(c) The Mellin Transform of Derivatives

$$M\left(\frac{d^k}{dy^k}f(y);v\right) = (-1)^k(v-k)_k F(v-k), (v-k) \in V_f, k \in \mathbb{Z}^+ \quad (5)$$

where the symbol  $(v-k)_k$  is defined for  $k$  integer by;

$$(v-k)_k = (v-k)(v-k+1)\dots(v-1)$$

Equations (3) and (6) can be used in various ways to find the effect of linear combination of differential operator such that  $v^k\left(\frac{d}{dv}\right)^m$ ,  $k, m$  integers. The most remarkable results are

$$\begin{cases} M\left(\left(v\frac{d}{dy}\right)^k f(y);v\right) = (-1)^k v^k F(v) \\ M\left(\frac{d^k}{dv^k} v^k f(y);v\right) = (-1)^k (v-k)_k F(v) \\ M\left(v^k \frac{d^k}{dv^k} f(y);v\right) = (-1)^k (v)_k F(v) \end{cases} \quad (6)$$

where  $v \in V_f$ ,  $k$  a positive integer and  $v_k = v(v+1)\dots(v+k-1)$ .

### 3.0 A New Technique for the Solution of the Black-Scholes Partial Differential Equation for Vanilla Put Options

Let us consider the Black-Scholes partial differential equation for vanilla put options  $P(S_t, t)$  with the initial and boundary conditions given by

$$\left. \frac{\partial P(S_t, t)}{\partial t} + \frac{\sigma^2 S_t^2}{2} \frac{\partial^2 P(S_t, t)}{\partial S_t^2} + rS_t \frac{\partial P(S_t, t)}{\partial S_t} - rP(S_t, t) = 0, S_t \in (0, \infty), t \in [0, T] \right\} \quad (7a)$$

with the boundary conditions given by

$$\left. \begin{aligned} &P(S_t, T) = \max \left[ (K - S_t), 0 \right] \\ &\lim_{S_t \rightarrow 0} P(S_t, t) = K \exp(-r(T-t)) \\ &\lim_{S_t \rightarrow \infty} P(S_t, t) = 0 \end{aligned} \right\} \quad (7b)$$

where  $\sigma$  is the volatility,  $r$  is a risk-free interest rate,  $K$  is called the strike price and  $T$  is the maturity date. It is a known fact that the partial differential equation in (7a) has a closed form solution obtained after several change of variables and solving certain related diffusion equations. This procedure is not applicable in the vector framework where  $P(S_t, t)$  is a vector and  $\sigma$ ,  $r$  are matrices.

The Mellin transform for the price of vanilla put options is defined as

$$M(P(S_t, t), v) = p(v, t) = \int_0^{\infty} P(S_t, t) S_t^{-1+v} dS_t \quad (8)$$

and the inversion formula for the Mellin transform is given by

$$M^{-1}(p(v, t)) = P(S_t, t) = \frac{1}{2\pi j} \int_{a-j\infty}^{a+j\infty} p(v, t) S_t^{-v} dv \quad (9)$$

To use the Mellin transform and the conditions that guarantee its existence, we assume that  $p(S_t, t)$  is bounded of polynomial degree when  $S_t \rightarrow 0$  and  $S_t \rightarrow \infty$  i.e.

$$p(S_t, t) = \begin{cases} O(S_t^{a_1}), & S_t \rightarrow 0 \\ O(S_t^{a_2}), & S_t \rightarrow \infty \end{cases} \quad (10)$$

for any  $u \in C$  on  $-a_1 < R(u) < -a_2$  where  $(-a_1, -a_2)$  is called fundamental strip.

Taking the Mellin transform of the Black-Scholes partial differential equation for vanilla put option in (7a), we have that

$$M \left( \frac{\partial P(S_t, t)}{\partial t} + \frac{\sigma^2 S_t^2}{2} \frac{\partial^2 P(S_t, t)}{\partial S_t^2} + r S_t \frac{\partial P(S_t, t)}{\partial S_t} \right) = M(rP(S_t, t)) \quad (11)$$

Using the properties of the Mellin transforms, we have

$$\left. \begin{aligned} M\left(\frac{\partial P(S_t, t)}{\partial t}\right) &= \frac{d}{dt} p(v, t) \\ M\left(\frac{\sigma^2 S_t^2}{2} \frac{\partial^2 P(S_t, t)}{\partial S_t^2}\right) &= \frac{\sigma^2}{2} (v^2 + v) p(v, t) \\ M\left(r S_t \frac{\partial P(S_t, t)}{\partial S_t}\right) &= -rvp(v, t) \\ M(rP(S_t, t)) &= rp(v, t) \end{aligned} \right\} \quad (12)$$

Substituting (12) into (11) and simplifying further yields

$$\frac{dp(v, t)}{dt} = -\left(\frac{v^2 \sigma^2}{2} + \left(\frac{\sigma^2}{2} - r\right)v - r\right)p(v, t), \quad t \in [0, T] \quad (13)$$

Integrating (13) yields

$$p(v, t) = A(v) \exp\left(-\left(\frac{v^2 \sigma^2}{2} + \left(\frac{\sigma^2}{2} - r\right)v - r\right)t\right) \quad (14)$$

Setting

$$\varphi(v) = \left(\frac{v^2 \sigma^2}{2} + \left(\frac{\sigma^2}{2} - r\right)v - r\right), \text{ then (14) becomes}$$

$$p(v, t) = A(v) \exp(-\varphi(v)t) \quad (15)$$

where  $A(v)$  is a constant of integration to be determined and it is defined as

$$A(v) = \psi(v, t) \exp(\varphi(v)T) \quad (16)$$

$\psi(v, t)$  can be obtained by taking the Mellin transform of the initial condition of the form

$$P(S_t, T) = \theta(S_t, v) = \max((K - S_t), 0) \quad (17)$$

then we have

$$\psi(v, t) = \int_0^\infty \max(K - S_t)^+ S_t^{v-1} dS_t = \frac{K^{1+v}}{v(v+1)} \quad (18)$$

Using equations (15), (16) and (18), we have that

$$p(v, t) = \frac{K^{1+v}}{v(v+1)} \exp(-\varphi(v)(t-T)) \quad (19)$$

The Mellin inversion of (19) is obtained as

$$M^{-1}(p(v, t)) = P(S_t, t) = \frac{1}{2\pi j} \int_{a-j\infty}^{a+j\infty} \frac{K^{1-v}}{v(v-1)} \exp(-\varphi(v)(t-T)) S_t^v dv \quad (20)$$

Next we want to prove that the expression (20) is a solution of the Black-Scholes partial differential equation for vanilla put options given by (7a). Let us assume that

$$v = m + jn \Rightarrow dv = jdn \quad (21)$$

Substituting (21) into (20) yields

$$P(S_t, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{K^{1+m+jn}}{(m+jn)(m+jn+1)} S_t^{-(m+jn)} \exp(-\varphi(m+jn)(t-T)) dn \quad (22)$$

Substituting  $v = m + jn$  into  $\varphi(v) = \left( \frac{v^2 \sigma^2}{2} + \left( \frac{\sigma^2}{2} - r \right) v - r \right)$ , then

$$\begin{aligned} -\varphi(m+jn) &= - \left( \frac{(m+jn)^2 \sigma^2}{2} + \left( \frac{\sigma^2}{2} - r \right) (m+jn) - r \right) \\ &= \left( -\frac{\sigma^2 m^2}{2} + \frac{\sigma^2 n^2}{2} - \frac{\sigma^2 m}{2} + rm + r \right) + j \left( -mn\sigma^2 - \frac{\sigma^2 n}{2} + rn \right) \end{aligned} \quad (23)$$

Since  $P(S_t, t)$  is Mellin transformable and continuous, setting  $t = T$  then (22) becomes

$$P(S_t, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{K^{1+m+jn}}{(m+jn)(m+jn+1)} S_t^{-(m+jn)} dn \quad (24)$$

Equation (22) is well defined and satisfies (24).

115 Using the definition of the Mellin transforms, then

$$116 \quad \left| \frac{K^{1+m+jn}}{(m+jn)(m+jn+1)} \right| \leq M(m) = \int_0^\infty |f(s)| s^{m-1} ds \quad \forall n \in R \quad (25)$$

117 and for  $t \in [0, T)$  we have that

$$118 \quad \left\{ \begin{aligned} & \int_{-\infty}^\infty \left| \frac{K^{1+m+jn}}{(m+jn)(m+jn+1)} \right| |S_t^{-(m+jn)}| \exp(\varphi(m+jn)(T-t)) dn \\ & \leq M(m) S_T^{-m} \exp\left(\left(-\frac{\sigma^2 m^2}{2} - \frac{\sigma^2 m}{2} + rm + r\right)(t-T)\right) \int_{-\infty}^\infty \exp\left(\left(-\frac{\sigma^2 n^2}{2}\right)(T-t)\right) dn \end{aligned} \right. \quad (26)$$

119 Using the differentiation theorem of parameter integrals and the fact that

$$120 \quad \int_{-\infty}^\infty n^j \exp\left(-\frac{\sigma^2}{2}\right)(T-t) dn < \infty, \quad j = 0, 1, 2, \dots, t \in [0, T) \quad (27)$$

121 Then it follows that upon differentiation of (22), we have that

$$122 \quad \left. \begin{aligned} \frac{\partial P(S_t, t)}{\partial t} &= -\frac{1}{2\pi} \int_{-\infty}^\infty \frac{K^{1+m+jn}}{(m+jn)(m+jn+1)} \varphi(m+jn) S_t^{-(m+jn)} \exp(-\varphi(m+jn)(t-T)) dn \\ \frac{\partial P(S_t, t)}{\partial S_t} &= \frac{1}{2\pi} \int_{-\infty}^\infty \frac{K^{1+m+jn}}{(m+jn)(m+jn+1)} (m+jn) S_t^{(m+jn-1)} \exp(\varphi(m+jn)(t-T)) dn \end{aligned} \right\} \quad (28)$$

$$123 \quad \frac{\partial^2 P(S_t, t)}{\partial S_t^2} = \frac{1}{2\pi} \int_{-\infty}^\infty \frac{K^{1+m+jn}}{(m+jn)(m+jn+1)} (m+jn)(m+jn+1) S_t^{-(m+jn+2)} \exp(-\varphi(m+jn)(t-T)) dn \quad (29)$$

124 Substituting (22), (28) and (29) into the Black-Scholes partial differential equation for vanilla put  
 125 options given by (7a), we have

$$\begin{aligned}
 & \left. \begin{aligned}
 & \frac{\partial P(S_t, t)}{\partial t} + \frac{\sigma^2 S_t^2}{2} \frac{\partial^2 P(S_t, t)}{\partial S_t^2} + r S_t \frac{\partial P(S_t, t)}{\partial S_t} - r E_c(S_t, t) = \\
 & -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{K^{1+m+jn}}{(m+jn)(m+jn+1)} \varphi(m+jn) S_t^{-(m+jn)} \exp(-\varphi(m+jn)(t-T)) dn + \\
 & \frac{\sigma^2 S_t^2}{2} \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{K^{1+m+jn}}{(m+jn)(m+jn+1)} (m+jn)(m+jn+1) S_t^{-(m+jn+2)} \exp(\varphi(m+jn)(t-T)) dn - \\
 & r S_t \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{K^{1+m+jn}}{(m+jn)(m+jn+1)} (m+jn) S_t^{-(m+jn-1)} \exp(-\varphi(m+jn)(t-T)) dn + \\
 & r \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{K^{1+m+jn}}{(m+jn)(m+jn+1)} S_t^{-(m+jn)} \exp(\varphi(m+jn)(t-T)) dn \\
 & = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left( -\varphi(m+jn) + \frac{\sigma^2 S_t^2}{2} (m+jn)(m+jn+1) S_t^{(-2)} - r S_t (m+jn) S_t^{(-1)} - r \right) \\
 & \quad \times \left( \frac{K^{1+m+jn}}{(m+jn)(m+jn+1)} S_t^{-(m+jn)} \exp(-\varphi(m+jn)(t-T)) \right) dn \\
 & = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left( -\varphi(m+jn) + \left( \left( -\frac{\sigma^2 m^2}{2} + \frac{\sigma^2 n^2}{2} - \frac{\sigma^2 m}{2} + rm + r \right) + j \left( -mn\sigma^2 - \frac{\sigma^2 n}{2} + rn \right) \right) \right) \\
 & \quad \times \frac{K^{1+m+jn}}{(m+jn)(m+jn+1)} S_t^{-(m+jn)} \exp(-\varphi(m+jn)(t-T)) dn \\
 & = \frac{1}{2\pi} \int_{-\infty}^{\infty} (-\varphi(m+jn) + \varphi(m+jn)) \frac{K^{1+m+jn}}{(m+jn)(m+jn+1)} S_t^{-(m+jn)} \exp(\varphi(m+jn)(t-T)) dn \\
 & = 0
 \end{aligned} \right\} \quad (30)
 \end{aligned}$$

127 Hence  $P(S_t, t)$  defined by (20) is a solution of (7a).

## 128 4.0 Conclusion

129 In this paper we developed a new technique for the solution of the homogeneous Black-Scholes  
 130 partial differential equation for vanilla put options using the Mellin transform method. The  
 131 approach used in this paper does not require variables transformation.

132

133

134 **References**

- 135 [1] F. AlAzemi, A. AlAzemi and I. Boyadjiev, Mellin transform method for solving the Black-  
 136 Scholes equation, *International Journal of Pure and Applied Mathematics*, **97**(2014), 287-  
 137 301
- 138 [2] F.Black and M. Scholes, The pricing of options and corporate liabilities, *Journal of Political*  
 139 *Economy*, 81(1973), 637-654.
- 140 [3] L. Jódar , P. Sevilla, J.C. Cortes, R. Sala, A new direct method for solving the Black-Scholes  
 141 equation, *Applied Mathematics and Letters*, **18**(2005), 29-32.
- 142 [4] R. Merton, Theory of rational option pricing, *Bell Journal of Econom. Management Science*,  
 143 **4**(1973), 141-183
- 144 [5] R. Panini and R.P.Srivastav, Option pricing with Mellin transforms, *Mathematical and*  
 145 *Computer Modelling*, 40(2004), 43-56, doi:10.1016/j.mcm.2004.07.008.

146

147

148

149