

Short Research  
Article**Solving interval linear equations with modified interval arithmetic****Abstract**

In this paper, we use modified interval arithmetic for solving interval system of linear algebraic equations. If we solve interval linear equations system by existing interval arithmetic, then resulted solution does not satisfy equations of system, i.e. by replacing obtained solution in interval equations of system, resulted interval width will be more than the right hand side interval width. Since interval width resulted from solution obtained through modified interval arithmetic is less than interval width resulted by existing interval arithmetic, therefore, if for decision maker closeness of obtained interval to the right hand side interval is important, then we can solve interval linear equations system with modified interval arithmetic.

*Keywords:* Existing interval arithmetic; Interval linear equations; Modified interval arithmetic.

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**1 Introduction**

Solution of interval linear equations is a challenging problem in interval analysis. This problem was first considered by Oettli and Prager [Oettli and Prager (1964)]. Some researchers worked on interval linear system. They proposed several methods for solving interval linear equations [1-16]. Some of them focused on fuzzy linear equations [Nayak and Chakraverty (2013)]. Some others found an enclosure of solutions of interval linear system [Neumaier (1988); Polyak and Nazin (2004); Rohn (1997)].

An interval number  $X$  is generally represented as  $[\underline{X}, \overline{X}]$  where  $\underline{X} \leq \overline{X}$ . If  $\underline{X} = \overline{X}$ , then  $X$  will be degenerate.

If  $\underline{A}$  and  $\overline{A}$  are two matrices in  $\mathbb{R}^{m \times n}$  and  $\underline{A} \leq \overline{A}$ , then the set of matrices

$$\mathbf{A}^I = [\underline{A}, \overline{A}] = \{A \mid \underline{A} \leq A \leq \overline{A}\},$$

is called an interval matrix, and the matrices  $\underline{A}$  and  $\overline{A}$  are called its bounds. Center and radius matrices have been defined as

$$A^c = \frac{1}{2}(\underline{A} + \overline{A}), \quad \Delta_A = \frac{1}{2}(\overline{A} - \underline{A}).$$

A square interval matrix  $\mathbf{A}^I$  is called regular if each  $A \in \mathbf{A}^I$  is nonsingular.

A special case of an interval matrix is an interval vector which is a one-column interval matrix

$$\mathbf{x}^I = \{\mathbf{x} \mid \underline{\mathbf{x}} \leq \mathbf{x} \leq \overline{\mathbf{x}}\},$$

where  $\underline{\mathbf{x}}, \overline{\mathbf{x}} \in \mathbb{R}^n$ .

Interval arithmetic is defined in [Alefeld and Herzberger (1983)].

Let  $\mathbf{A}^I$  be a  $n \times n$  interval matrix and  $\mathbf{b}^I$  is an interval vector. For an interval system of linear algebraic equations  $\mathbf{A}^I \mathbf{x} = \mathbf{b}^I$ , the solution set is defined as

$$S = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = b, A \in \mathbf{A}^I, b \in \mathbf{b}^I\}.$$

If  $\mathbf{A}^I$  is regular, then for a matrix  $A \in \mathbf{A}^I$  and any vector  $b \in \mathbf{b}^I$  an ordinary linear system  $A\mathbf{x} = b$  has the unique solution.

The solution set  $S$  is a non-convex polyhedral set [Oettli and Prager (1964)]. The main objective is to find interval solution of linear interval system that is to determine The narrowest interval vector containing the set  $S$  is called the interval hull of  $S$ . This set is generally not an interval vector. It is usually difficult to describe  $S$ . Because  $S$  is generally so complicated in shape [Hansen et al. (2004)], it is usually impractical to use it. Instead, it is a common practice to seek the interval vector  $\mathbf{x}^I$  containing  $S$  that the narrowest possible interval components. This interval vector is called the hull of the solution set or simply the hull. In other words, we need to imbed the solution set  $S$  into the minimal box in  $\mathbb{R}^n$ . This problem is known to be NP-hard [Kreinovich et al. (1993)] and complicated from computational viewpoint for large-scale systems. Therefore, many researchers generally compute only outer bounds for the hull [Alefeld and Herzberger (1983); Hansen (2000); Hladik (2007); Kearfott (1996); Neumaier (1990, 1986, 1988); Polyak and Nazin (2004); Shary (1995)]. Some iterative approaches were established at this context as well as direct numerical methods that provide over bounding of  $\mathbf{x}^*$ . [Rohn (1997); Shary (1995)].

Suppose we solve interval linear equations system by gaussian elimination method with existing interval arithmetic. If we replace the obtained solution in interval equations of system, then the width of obtained interval is more than the width of the right hand side interval. Therefore, if closeness of obtained interval to the right hand side interval is important, then we can use modified interval arithmetic for solving interval linear equations system. In this paper, we solve interval linear equations system by gaussian elimination with modified interval arithmetic and we show that resulted interval of substituting solution obtained through modified interval arithmetic cannot give rise to excess width than the existing interval arithmetic.

## 2 Review of Interval Arithmetic

In this section, we recall the existing and modified interval arithmetic.

## 2.1 Existing interval arithmetic

In this section, we review the existing interval arithmetic.

If  $\bullet \in \{+, -, \times, \div\}$  denotes any one of these operations for arithmetic on real numbers  $x$  and  $y$ , then the corresponding operation for existing arithmetic on interval numbers  $X$  and  $Y$  is

$$X \bullet Y = \{x \bullet y | x \in X, y \in Y\}.$$

Therefore, this definition produces the following rules for generating the endpoint of  $X \bullet Y$ .

$$\begin{aligned} X + Y &= [\underline{X} + \underline{Y}, \overline{X} + \overline{Y}], \\ X - Y &= [\underline{X} - \overline{Y}, \overline{X} - \underline{Y}], \\ X \times Y &= [\min(\underline{X}\underline{Y}, \underline{X}\overline{Y}, \overline{X}\underline{Y}, \overline{X}\overline{Y}), \max(\underline{X}\underline{Y}, \underline{X}\overline{Y}, \overline{X}\underline{Y}, \overline{X}\overline{Y})], \\ X \div Y &= X \times (\frac{1}{Y}) = X \times [\frac{1}{\overline{Y}}, \frac{1}{\underline{Y}}], \end{aligned}$$

If we exclude division by an interval containing 0.

## 2.2 Modified interval arithmetic

In this section, we recall the modified interval arithmetic [Ganesan and Veermani (2005)].

Let  $\ast \in \{\oplus, \ominus, \otimes, \oslash\}$  be any one of the operations for arithmetic on real numbers  $x$  and  $y$ , then the corresponding operation for modified arithmetic on interval numbers  $X$  and  $Y$  is

$$X \ast Y = [m(X) \ast m(Y) - k, m(X) \ast m(Y) + k],$$

where  $k = \min\{(m(X) \ast m(Y)) - \alpha, \beta - (m(X) \ast m(Y))\}$  and  $m(\cdot)$  is the midpoint of interval.  $\alpha$  and  $\beta$  are the endpoint of the interval  $X \bullet Y$ .

$$\begin{aligned} X \oplus Y &= [m(X) + m(Y) - k, m(X) + m(Y) + k], \\ k &= \frac{(\overline{X} + \overline{Y}) - (\underline{X} + \underline{Y})}{2}, \\ X \ominus Y &= [m(X) - m(Y) - k, m(X) - m(Y) + k], \\ k &= \frac{(\overline{X} + \overline{Y}) - (\underline{X} + \underline{Y})}{2}, \\ X \otimes Y &= [m(X)m(Y) - k, m(X)m(Y) + k], \\ k &= \min\{(m(X)m(Y)) - \alpha, \beta - (m(X)m(Y))\}, \\ \alpha &= \min(\underline{X}\underline{Y}, \underline{X}\overline{Y}, \overline{X}\underline{Y}, \overline{X}\overline{Y}), \\ \beta &= \max(\underline{X}\underline{Y}, \underline{X}\overline{Y}, \overline{X}\underline{Y}, \overline{X}\overline{Y}). \\ X^{-1} &= [\frac{1}{m(X)} - k, \frac{1}{m(X)} + k], \\ k &= \min\{\frac{1}{\overline{X}}(\frac{\overline{X} - \underline{X}}{\overline{X} + \underline{X}}), \frac{1}{\underline{X}}(\frac{\overline{X} - \underline{X}}{\overline{X} + \underline{X}})\}, 0 \notin X. \end{aligned}$$

It is to be noted that  $X \ast Y \subseteq X \bullet Y$  [Ganesan and Veermani (2005)]. For example if we take  $X = [-1, 2]$  and  $Y = [3, 5]$ , then  $X \times Y = [-5, 10]$  whereas  $X \otimes Y = [-5, 9]$ .

## 3 Gaussian elimination method

There are several methods for solving linear equations that can be labeled as Gaussian elimination. An interval version of any of them can be obtained from one using ordinary real arithmetic by simple

replacing each real arithmetic step by the corresponding interval arithmetic step. One standard method involves converting the coefficient matrix to the upper triangular matrix and hence the interval solution vector computed using Gaussian elimination contains the set  $S$ . Suppose the elimination procedure does not fail because of division by an interval containing zero. The solution  $X^I$  computed by using interval Gaussian elimination contain  $S$ . [Hansen et al. (2004)].

Unfortunately, Using an interval version of classic gaussian elimination generally does not yield a suitable algorithm because of

- rounding errors in interval computations.
- division by an interval containing zero.
- dependence among generated intervals.

Now, Let  $X^{EI}$  and  $X^{MI}$  be solution obtained through solving interval linear equations  $A^I x = b^I$  with the existing and modified interval arithmetic, respectively. Also, let  $b^{EI}$  and  $b^{MI}$  be interval resulted from replacing  $X^{EI}$  and  $X^{MI}$  in equations of system, respectively. We define

$$\varepsilon^{EI} = b^I - b^{EI}, \quad \varepsilon^{MI} = b^I - b^{MI}.$$

For example, consider the equation  $[1, 2]X = [4, 5]$ .

$$X^{EI} = [4, 5] \div [1, 2] = [2, 5], \quad X^{MI} = [4, 5] \odot [1, 2] = [2, 4],$$

therefore

$$\begin{aligned} [1, 2]X^{EI} &= [2, 10] = b^{EI}, \quad [1, 2]X^{MI} = [2, 7] = b^{MI}, \\ \varepsilon^{EI} &= [4, 5] - [2, 10] = [-6, 3], \quad \varepsilon^{MI} = [4, 5] \ominus [2, 7] = [-3, 3], \end{aligned}$$

and hence

$$[1, 2]X^{EI} \supseteq [1, 2]X^{MI} \supseteq [4, 5].$$

## 4 Numerical Examples

In this section, we solve two interval systems of linear equations.

**Example 4.1.** Consider an interval linear equations system

$$\begin{cases} [2, 3]x_1 + [0, 1]x_2 = [0, 120] \\ [1, 2]x_1 + [2, 3]x_2 = [60, 240] \end{cases}$$

Solutions of this system with existing and modified interval arithmetic are given in Table 1 on page 104.

From Table 1, we conclude the solution obtained through modified interval arithmetic is closer to the right hand side intervals of system than existing interval arithmetic and hence the norm of error will be smaller than that one. Exact solution and the solution resulted from the existing and modified interval arithmetic is shown in Figure 1 on page 104.

**Example 4.2.** Consider an interval system of linear algebraic equations

$$\begin{cases} [0.95, 1.05]x_1 + [1.95, 2.05]x_2 + [2.95, 3.05]x_3 = [0.2, 1] \\ [1.95, 2.05]x_1 + [3.95, 4.05]x_2 + [6.95, 7.05]x_3 = [0.2, 1] \\ [1.95, 2.05]x_1 + [-0.05, 0.05]x_2 + [0.95, 1.05]x_3 = [0.2, 1]. \end{cases}$$

Solutions of this system with existing and modified interval arithmetic are given in Table 2 on page 105.

By considering Table 2, the solution resulted from modified interval arithmetic is closer to the right hand side intervals of system than the existing interval arithmetic and hence the norm of error will be smaller.

Table 1: Solutions and errors of Example 4.1

|                        |                                                                          |
|------------------------|--------------------------------------------------------------------------|
| $X^{EI}$               | $\begin{pmatrix} [-120, 90] \\ [-60, 240] \end{pmatrix}$                 |
| $X^{MI}$               | $\begin{pmatrix} [-27.406, 54.679] \\ [-6.909, 110.546] \end{pmatrix}$   |
| $b^{EI}$               | $\begin{pmatrix} [-420, 510] \\ [-420, 900] \end{pmatrix}$               |
| $b^{MI}$               | $\begin{pmatrix} [-89.127, 209.127] \\ [-75.539, 375.539] \end{pmatrix}$ |
| $\ \varepsilon^{EI}\ $ | $[0, 998.599]$                                                           |
| $\ \varepsilon^{MI}\ $ | $[0, 378.549]$                                                           |

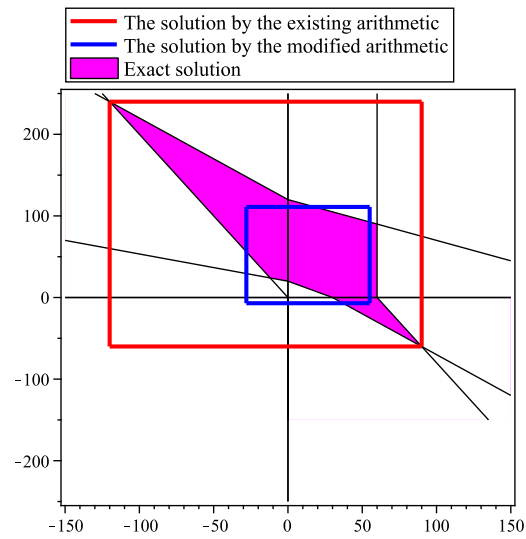


Figure 1: Solution of Example 4.1

Table 2: Solutions and errors of Example 4.2

|                        |                                                                                                 |
|------------------------|-------------------------------------------------------------------------------------------------|
| $X^{EI}$               | $\begin{pmatrix} [-28.045, 48.563] \\ [-2.141, 4.977] \\ [-12.041, 4.876] \end{pmatrix}$        |
| $X^{MI}$               | $\begin{pmatrix} [-5.513, 9.713] \\ [-0.612, 0.912] \\ [-2.231, 1.031] \end{pmatrix}$           |
| $b^{EI}$               | $\begin{pmatrix} [-71.727, 76.422] \\ [-152.215, 154.441] \\ [-131.612, 129.660] \end{pmatrix}$ |
| $b^{MI}$               | $\begin{pmatrix} [-13.847, 15.047] \\ [-29.508, 30.708] \\ [-24.828, 26.028] \end{pmatrix}$     |
| $\ \varepsilon^{EI}\ $ | $[0, 217.223]$                                                                                  |
| $\ \varepsilon^{MI}\ $ | $[0, 42.641]$                                                                                   |

## 5 Conclusion

Gaussian elimination method is one of the solving methods of interval system of linear algebraic equations. If we solve interval linear equations system by the existing interval arithmetic, then interval width resulted from replacing the obtained solution in interval equations of system is more than interval width resulted from the modified interval arithmetic. Therefore, if for decision maker closeness of the obtained interval to the right hand side interval is important, then we can solve interval linear equations system with the modified interval arithmetic.

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