

Indirect RBF for High-Order Integro-Differential Equations

Abstract

Two different approaches are applied to solve high-order integro-differential equations (IDEs), based on Radial Basis Functions (RBF). The first approach, which is called the direct approach (DRBF) is based on differentiation, and considers the solution as a finite linear combination of RBFs. While the second, the indirect approach (IRBF) is based on integration and considers the highest order derivative of the solution as a finite linear combination of RBFs. While the results of both approaches are accurate enough to be applied for low-order IDEs, IRBF leads to solutions, with high accuracy and advantageous for high-order IDEs will be clear. Illustrative examples are included to demonstrate the validity and applicability of the presented techniques.

Keywords: High order Integro-Differential Equations (IDEs), Direct Radial Basis Functions (DRBF), Indirect Radial Basis Functions (IRBF), Legendre-Gauss-Lobatto quadrature, Iterated integrals.

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1 Introduction

Integral and integro-differential equations arise from various applications, like physics, engineering, biology, medicine, economics, potential theory and many others (See [Wazwaz (2011); Delves and Mohamed (1985); Kress (1989)] and references therein). Since many of these equations can not be solved explicitly, it is often necessary to use numerical techniques. In recent years, a lot of attention has been devoted to the study of high-order integro-differential equations (IDEs) such as the Bernoulli matrix method [Bhrawy et al. (2012)], the collocation method [Yuzbasi et al. (2012)], Chebyshev polynomials [Dascioglu and Sezer (2005); Akyuz-Dascioglu and Sezer (2005); El-gendi (1969)], Legendre polynomials [Yalcinbas et al. (2009); Sohrabi (2012); Saadatmandi and Dehghan (2010)],

Homotopy Analysis Method [Jaradat et al. (2009)], Homotopy Perturbation Method [Aminikhah and Biazar (2009)], Adomian Decomposition Method [Wazwaz (2001)], Variational iteration method [Shang and Han (2010); Noor and Mohyud-Din (2008); Wang and He (2007)], Taylor polynomials [Yalcinbas and Sezer (2000); Nas et al. (2000); Maleknejad and Mahmoudi (2003)], and Compact Finite Difference Method [Zhao (2013)].

In recent years, meshless methods as a class of numerical methods are used for solving functional equations. Meshless methods just use a scattered set of collocation points, regardless any relationship information between the collocation points. This property is the main advantage of these techniques over the mesh dependent methods such as finite difference methods and finite element methods. Since 1990, radial basis function method [Hardy (1971)] are used as a well-known family of meshless method to approximate the solutions of various types of linear and nonlinear functional equations such as Partial Differential Equations (PDEs), Ordinary Differential Equations (ODEs), Integral Equations (IEs), and Integro-Differential Equation (IDEs) [Hardy (1971); Kansa (1990); Platte and Driscoll, (2004); Larsson and Fornberg (2003); Dehghan and Shokri (2007, 2008); Franke and Schaback (1997)].

In this paper, a modification of radial basis functions (RBF) is applied for numerical solution of high-order integro-differential equations which is based on integration approach (IRBF) and first introduced and used by Mai-Duy and Tran-Cong [Mai-Duy and Tran-Cong (2001, 2003); Mai-Duy (2005)]. Iterated integrals appeared in this approach are converted to one integral by using a formula of iterated integrals. A comparison is made between this approach and the differentiation approach of RBF (DRBF). We investigate these two approaches on the high order integro-differential equations and results demonstrate the good accuracy and efficiency of the presented technique for higher order IDEs than low order IDEs.

The paper is organized as follows. In Section 2, the radial basis functions are introduced. Section 3, reviews the Legendre-Gauss-Lobatto integration process. Section 4, as the main part, presents the solution of high-order integro-differential equations by integration process of radial basis functions. Numerical illustrative examples are included in Section 5. A conclusion is drawn in the Section 6.

2 Radial Basis Functions

Let's define the main features of the method.

2.1 Definition of Radial Basis Functions

Radial basis functions usually approximate a function as the following [Buhmann (2003)]

$$s(x) = \sum_{i=0}^N \lambda_i \phi(\|x - x_i\|), \quad x \in \mathbb{R}.$$

Where $\phi : [0, \infty) \rightarrow \mathbb{R}$ is a fixed univariate function, the coefficients $(\lambda_i)_{i=0}^N$ are real numbers, $(x_i)_{i=0}^N$, are finite number of distinct points (centers) in $\mathbb{R}$ and $\|\cdot\|$, denotes the Euclidean norm.

2.2 Radial Basis Functions Interpolation

The radial basis functions approximation of a real function, say $u(x)$, is given by [Buhmann (2003)]

$$u(x) \approx u_N(x) = \sum_{i=0}^N \lambda_i \phi(\|x - x_i\|) = \sum_{i=0}^N \lambda_i \phi_i(x) = \Phi^T(x) \mathbf{\Lambda},$$

where

$$\phi_i(x) = \phi(\|x - x_i\|),$$

$$\Phi^T(x) = [\phi_0(x), \phi_1(x), \dots, \phi_N(x)],$$

$$\Lambda = [\lambda_0, \lambda_1, \dots, \lambda_N]^T,$$

and distinct points $(x_i)_{i=0}^N$ are in $\mathbb{R}$. Consider N distinct support points $(x_j, u(x_j))$, $j = 0, 1, 2, \dots, N$. One can find λ_i s by solving the following linear system

$$\mathbf{A}\Lambda = \mathbf{u}.$$

Where

$$\mathbf{A} = [\phi(\|x_j - x_i\|)]_{i,j=0}^N,$$

$\Lambda = [\lambda_0, \lambda_1, \dots, \lambda_N]^T$, and $\mathbf{u} = [u_0, u_1, \dots, u_N]^T$. Some well-known RBFs are listed in Table 1, where the Euclidian distance r is real and non-negative, and c is a positive scalar, called shape parameter.

Table 1: Some well-known RBFs

Name of the function	Definition
Gaussian	$\phi(r) = e^{-cr^2}$
Inverse Quadric	$\phi(r) = \frac{1}{r^2 + c^2}$
Hardy Multiquadric	$\phi(r) = \sqrt{r^2 + c^2}$
Inverse Multiquadric	$\phi(r) = \frac{1}{\sqrt{r^2 + c^2}}$
Cubic	$\phi(r) = r^3$
Thin plate splines	$\phi(r) = r^2 \log(r)$
Hyperbolic secant	$\phi(r) = \text{sech}(cr)$

3 Legendre-Gauss-Lobatto integration nodes and weights

Let L_N be the well-known Legendre polynomial of order N , on the interval $[-1, 1]$. Then the Legendre-Gauss-Lobatto nodes are

$$(1 - x_j^2)L'_N(x_j) = 0,$$

$$-1 = x_0 < x_1 < \dots < x_N = 1, \quad (3.1)$$

where x_m , $1 \leq m \leq N - 1$ are the zeros of L'_N , where L'_N is the derivative of L_N with respect to $x \in [-1, 1]$. No explicit formula for the nodes (3.1) is known, and so they are computed numerically using sub-routines [Canuto et al. (1988); Elnagar and Kazemi (1997); Elnagar and Razzaghi (1998)]. Now we approximate the integral of f on $[-1, 1]$ as

$$\int_{-1}^1 f(x) dx = \sum_{i=0}^N w_i f(x_i), \quad (3.2)$$

where x_j in Eq. (3.1) are Legendre-Gauss-Lobatto nodes and w_j are the weights given in [Shen and Tang (2005)]

$$w_j = \frac{2}{N(N+1)} \frac{1}{(L_N(x_j))^2}, \quad j = 0, 1, \dots, N. \quad (3.3)$$

Note that the integration in Eq.(3.2) is exact whenever $f(x)$ is a polynomial of degree $\leq 2N + 1$.

4 Application of RBF method

In this paper Radial Basis Functions are used to approximate solution of high order integro-differential equations of the second kind in the following general form

$$y^{(m)}(x) + p(x)y^{(r)}(x) = f(x) + \int_a^b K(x,t)y^{(s)}(t)dt, \quad a \leq x \leq b, \quad (4.1)$$

$$y(a) = \alpha_0, \quad y'(a) = \alpha_1, \quad \dots, \quad y^{(m-1)}(a) = \alpha_{m-1} \quad (4.2)$$

where $y(x)$ is an unknown real function defined on $[a, b]$, $r, s \leq m$, and $p(x)$ and $f(x)$ are analytic known functions. Moreover the kernel $K(x, t)$ is defined on the interval $a \leq x, t \leq b$.

4.1 Direct Radial Basis Functions

In the direct method, the unknown function of Eq. (4.1) approximates by a closed form of radial basis functions, and its derivatives of any order, e.g. n -th order, can then be calculated by differentiating such a closed form.

Let's approximate the function $y(x)$ in terms of radial basis functions, $\phi_i(x)$, as follows

$$y(x) \approx y_N(x) = \sum_{i=0}^N \lambda_i \phi_i(x) = \Phi^T(x)\Lambda, \quad (4.3)$$

where

$$\phi_i(x) = \phi(\|x - x_i\|),$$

$x_i, i = 0, 1, \dots, N$ are shifted nodes of Legendre-Gauss-Lobatto quadrature,

$$\Phi^T(x) = [\phi_0(x), \phi_1(x), \dots, \phi_N(x)]$$

and

$$\Lambda = [\lambda_0, \lambda_1, \dots, \lambda_N]^T$$

is an unknown vector.

By m times differentiating from Eq. (4.3), we obtain

$$y^{(m)}(x) \approx y_N^{(m)}(x) = \mathcal{D}^m \Phi^T(x)\Lambda \quad (4.4)$$

where

$$\mathcal{D}^m \Phi^T(x) = [\phi_0^{(m)}(x), \phi_1^{(m)}(x), \dots, \phi_N^{(m)}(x)]$$

Obviously $y^{(r)}$ and $y^{(s)}$ are calculated.

Substituting $y^{(m)}, y^{(r)}$, and $y^{(s)}$ in Eq. (4.1), leads to

$$\begin{aligned} & \mathcal{D}^m \Phi^T(x)\Lambda + p(x)\mathcal{D}^r \Phi^T(x)\Lambda \\ &= f(x) + \int_a^b K(x,t)\mathcal{D}^s \Phi^T(t)\Lambda dt, \quad a \leq x \leq b, \end{aligned}$$

or

$$\left(\mathcal{D}^m \Phi^T(x) + p(x)\mathcal{D}^r \Phi^T(x) - \int_a^b K(x,t)\mathcal{D}^s \Phi^T(t)dt \right) \Lambda = f(x).$$

For obtaining $\lambda_i, i = 0, 1, \dots, N$, by collocating at the points $x = x_j$ for $j = 0, 1, \dots, N$, we have

$$\left(\mathcal{D}^m \Phi^T(x_j) + p(x_j)\mathcal{D}^r \Phi^T(x_j) - \int_a^b K(x_j,t)\mathcal{D}^s \Phi^T(t)dt \right) \Lambda = f(x_j), \quad (4.5)$$

where $x_j, j = 0, 1, \dots, N$ are shifted zeros of the Legendre-Gauss-Lobatto integration nodes. By using the Legendre-Gauss-Lobatto integration formula, we can approximate the integral in Eq. (4.5) and hence these equations can be written as follows:

$$\left(\mathcal{D}^m \Phi^T(x_j) + p(x_j) \mathcal{D}^r \Phi^T(x_j) - \sum_{i=0}^N w_i K(x_j, t_i) \mathcal{D}^s \Phi^T(t_i) \right) \Lambda = f(x_j), \quad (4.6)$$

for $j = 0, 1, \dots, N$. Where $t_i \in [a, b]$ and w_i , for $i = 0, \dots, N$ are Legendre-Gauss-Lobatto integration nodes and weights, respectively. Eq. (4.6) generates a system of linear equations for the unknowns Λ .

Many researchers substitute initial conditions

$$\begin{aligned} \Phi^T(a) &= \alpha_0, \\ \mathcal{D}\Phi^T(a) &= \alpha_1, \\ &\vdots \\ \mathcal{D}^{m-1}\Phi^T(a) &= \alpha_{m-1}, \end{aligned}$$

for the same number of equation in the foregoing linear system. There is not any criterion for such substitution and seems it is the author's option. This freedom usually reduces the accuracy.

4.2 Indirect Radial Basis Functions

In order to apply indirect radial basis functions approach, let's approximate the highest order derivative in terms of radial basis functions $\phi_i(x)$ as follows

$$y^{(m)}(x) \approx \hat{y}_N = \sum_{i=0}^N \lambda_i \phi_i(x) = \Phi^T(x) \Lambda \quad (4.7)$$

Successive integrating the obtained expression yields expressions for lower order derivatives and finally for the original function itself. For example

$$\begin{aligned} \int_a^x y^{(m)}(t) dt &= y^{(m-1)}(x) - y^{(m-1)}(a) \\ &\approx \sum_{i=0}^N \lambda_i \int_a^x \phi_i(t) dt = \mathcal{I} \Phi^T(x) \Lambda \end{aligned}$$

or

$$y^{(m-1)}(x) \approx \mathcal{I} \Phi^T(x) \Lambda + y^{(m-1)}(a),$$

where

$$\mathcal{I} \Phi^T(x) = \left[\int_a^x \phi_0(t) dt, \int_a^x \phi_1(t) dt, \dots, \int_a^x \phi_N(t) dt \right].$$

Similarly

$$\begin{aligned} y^{(m-2)}(x) &\approx \sum_{i=0}^N \lambda_i \int_a^x \int_a^{t_2} \phi_i(t) dt dt_2 + y^{(m-1)}(a)x + y^{(m-2)}(a) \\ &= \mathcal{I}^2 \Phi^T(x) \Lambda + y^{(m-1)}(a)x + y^{(m-2)}(a), \end{aligned}$$

and finally

$$\begin{aligned} y &\approx \mathcal{I}^m \Phi^T(x) \Lambda + y^{(m-1)}(a) \frac{x^{m-1}}{(m-1)!} \\ &\quad + y^{(m-2)}(a) \frac{x^{m-2}}{(m-2)!} + \dots + y'(a)x + y(a) \end{aligned} \quad (4.8)$$

where the i -th component of $\mathcal{I}^m \Phi^T(x)$ is

$$\int_a^x \int_a^{t_m} \dots \int_a^{t_3} \int_a^{t_2} \phi_i(t) dt dt_2 \dots dt_{m-1} dt_m$$

and $y(a), y'(a), \dots, y^{(m-1)}(a)$ are known from initial conditions.

Also we have

$$\begin{aligned} y^{(r)}(x) &\approx \mathcal{I}^{m-r} \Phi^T(x) \mathbf{\Lambda} + y^{(m-1)}(a) \frac{x^{m-r-1}}{(m-r-1)!} \\ &+ y^{(m-2)}(a) \frac{x^{m-r-2}}{(m-r-2)!} + \dots + y^{(r-1)}(a)x + y^{(r)}(a) \end{aligned} \quad (4.9)$$

and

$$\begin{aligned} y^{(s)}(x) &\approx \mathcal{I}^{m-s} \Phi^T(x) \mathbf{\Lambda} + y^{(m-1)}(a) \frac{x^{m-s-1}}{(m-s-1)!} \\ &+ y^{(m-2)}(a) \frac{x^{m-s-2}}{(m-s-2)!} + \dots + y^{(s-1)}(a)x + y^{(s)}(a) \end{aligned} \quad (4.10)$$

Substituting from Eqs. (4.7), (4.9), and (4.10) in Eq. (4.1), leads to

$$\begin{aligned} &\Phi^T(x) \mathbf{\Lambda} + p(x) \mathcal{I}^{m-r} \Phi^T(x) \mathbf{\Lambda} \\ &+ p(x) \left(y^{(m-1)}(a) \frac{x^{m-r-1}}{(m-r-1)!} + y^{(m-2)}(a) \frac{x^{m-r-2}}{(m-r-2)!} \right. \\ &\left. + \dots + y^{(r-1)}(a)x + y^{(r)}(a) \right) \\ &= f(x) + \int_a^b K(x, t) \mathcal{I}^{m-s} \Phi^T(t) \mathbf{\Lambda} dt \\ &+ \int_a^b K(x, t) \left(y^{(m-1)}(a) \frac{t^{m-s-1}}{(m-s-1)!} + y^{(m-2)}(a) \frac{t^{m-s-2}}{(m-s-2)!} \right. \\ &\left. + \dots + y^{(s-1)}(a)t + y^{(s)}(a) \right) dt, \end{aligned}$$

or

$$\left(\Phi^T(x) + p(x) \mathcal{I}^{m-r} \Phi^T(x) - \int_a^b K(x, t) \mathcal{I}^{m-s} \Phi^T(t) dt \right) \mathbf{\Lambda} = g(x), \quad (4.11)$$

in which

$$\begin{aligned} g(x) &= f(x) \\ &- p(x) \left(y^{(m-1)}(a) \frac{x^{m-r-1}}{(m-r-1)!} + y^{(m-2)}(a) \frac{x^{m-r-2}}{(m-r-2)!} \right. \\ &\left. + \dots + y^{(r-1)}(a)x + y^{(r)}(a) \right) \\ &+ \int_a^b K(x, t) \left(y^{(m-1)}(a) \frac{t^{m-s-1}}{(m-s-1)!} + y^{(m-2)}(a) \frac{t^{m-s-2}}{(m-s-2)!} \right. \\ &\left. + \dots + y^{(s-1)}(a)t + y^{(s)}(a) \right) dt, \end{aligned}$$

Fortunately, integrals on left hand side of (4.11), can be reduced to one dimensional integrals by using the formula of iterated integrals [Abramowitz and Stegun (1972)], for example

$$\int_a^x \int_a^{t_m} \dots \int_a^{t_3} \int_a^{t_2} \phi_i(t) dt dt_2 \dots dt_{m-1} dt_m = \frac{(x-a)^m}{(m-1)!} \int_0^1 t^{m-1} \phi_i(x - (x-a)t) dt \quad (4.12)$$

Now numerical quadratures can be applied.

For calculating λ_i , $i = 0, 1, \dots, N$, we apply collocation process similar to direct approach by using Legendre-Gauss-Lobatto nodes as collocation points. Integrals appeared in process, determined by Legendre-Gauss-Lobatto quadrature. Finally approximate solution of Eq. (4.1), is given by (4.8).

In the integration process all initial conditions are considered in the RBF expansion of solution and extra equations for initial conditions do not required. Furthermore, in the differentiation process, initial conditions substituted for the same number of equations and this substitution usually reduces the accuracy. Indeed it is expected that through the indirect process, the approximating functions are much smoother and therefore have higher approximation power.

The convergence of radial basis function interpolation has been discussed by Buhmann [Buhmann (2003, 1989)] and other researchers [Jackson (1988); Schaback (1996); Wendland (1998)].

5 Numerical Examples

In this section, some examples are provided to illustrate the efficiency of this approach. For the sake of comparing purposes, we use the norm two of errors.

5.1 Example

Consider the following second order Fredholm IDE

$$y''(x) = -e^x + \frac{x}{2} + \int_0^1 xty(t), \quad y(0) = 0, \quad y'(0) = -1, \quad 0 \leq x \leq 1. \quad (5.1)$$

The exact solution is $y(x) = 1 - e^x$.

Errors of the numerical solutions for $N = 5, 10, 15$ and GA-RBF, MQ-RBF, and IMQ-RBF are shown in Table 2 and Fig. 1,2, and 3.

Table 2: Errors for Example 5.1

N	GA		MQ		IMQ	
	DRBF	IRBF	DRBF	IRBF	DRBF	IRBF
5	6.1815e-01	3.6710e-06	1.4967e+00	8.3951e-06	2.5269e+00	4.1628e-05
10	3.5582e-04	4.9120e-09	3.3468e-02	2.8375e-09	1.7722e-01	1.5954e-08
15	1.3948e-05	3.1498e-08	3.2377e-04	4.3236e-09	1.9503e-03	7.6985e-10

5.2 Example

Consider the following third order Fredholm IDE

$$y'''(x) = \sin(x) - x - \int_0^{\frac{\pi}{2}} xty'(t)dt, \quad 0 \leq x \leq \frac{\pi}{2}. \quad (5.2)$$

with the initial conditions $y(0) = 1$, $y'(0) = 0$, and $y''(0) = -1$. The exact solution is $y(x) = \cos(x)$. Errors of the numerical solutions for $N = 5, 10, 15$ and same RBFs as Example 5.1 are shown in Table 3 and Fig. 4,5, and 6.

Table 3: Errors for Example 5.2

N	GA		MQ		IMQ	
	DRBF	IRBF	DRBF	IRBF	DRBF	IRBF
5	5.0037e-01	9.2822e-06	5.9669e-01	1.8861e-05	6.8239e-01	1.0677e-04
10	5.6828e-02	1.1815e-10	2.2210e+00	1.0834e-08	1.1974e+00	6.2370e-08
15	5.9581e-02	3.4710e-10	8.0248e-01	8.3737e-11	1.3336e+00	4.4014e-10

5.3 Example

Now consider the following fourth-order Fredholm IDE

$$y^{(4)} = -1 + \sin(x) + \int_0^{\frac{\pi}{2}} ty(t)dt, \quad 0 \leq x \leq \frac{\pi}{2}. \quad (5.3)$$

with the initial conditions $y(0) = y''(0) = 0$, $y'(0) = 1$, and $y'''(0) = -1$. The exact solution is $y(x) = \sin(x)$.

Errors of the numerical solutions for $N = 5, 10, 15$ and different RBFs are shown in Table 4 and Fig. 7,8, and 9.

Table 4: Errors for Example 5.3

N	GA		MQ		IMQ	
	DRBF	IRBF	DRBF	IRBF	DRBF	IRBF
5	9.2288e-01	3.4373e-06	2.0403e+01	1.2329e-05	1.0582e+01	5.3253e-05
10	3.4521e-02	6.0687e-12	3.4740e-01	5.1621e-09	6.3145e+01	2.6527e-08
15	7.3037e-03	4.2092e-10	3.3438e-01	5.6452e-12	9.6491e-02	1.4849e-11

6 Conclusion

A modified radial basis function approach was used for solving high-order integro-differential equations. Some numerical examples are presented to demonstrate that the method is very effective and useful for finding approximate solutions of high order integro-differential equations. A comparison with direct process, for numerical solution of the high-order integro-differential equations, shows that this technique is accurate enough to be known as a powerful device. Especially, as the order of integro-differential equations is increased, the advantages of the indirect approach will be more clear and appeared in Figures and Tables.

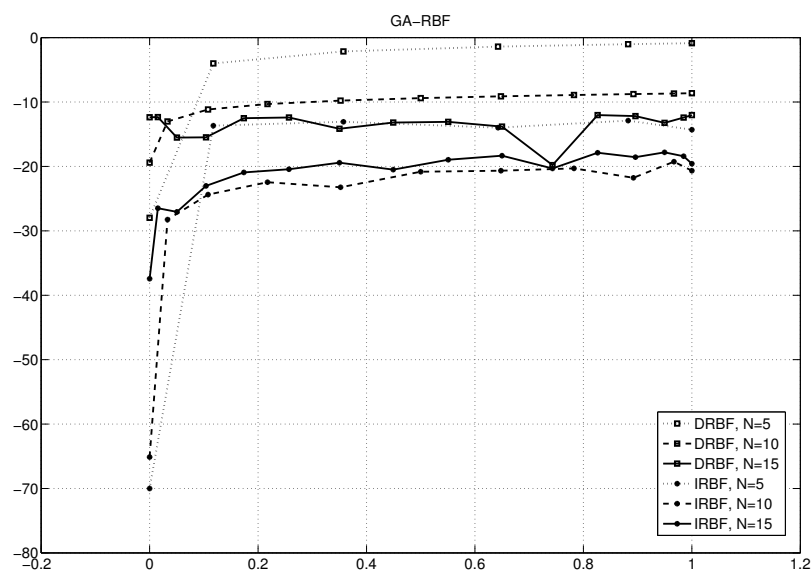


Figure 1: Ga-RBF: $\log |y_{Exact} - y_{Approx}|$ for Example 5.1.

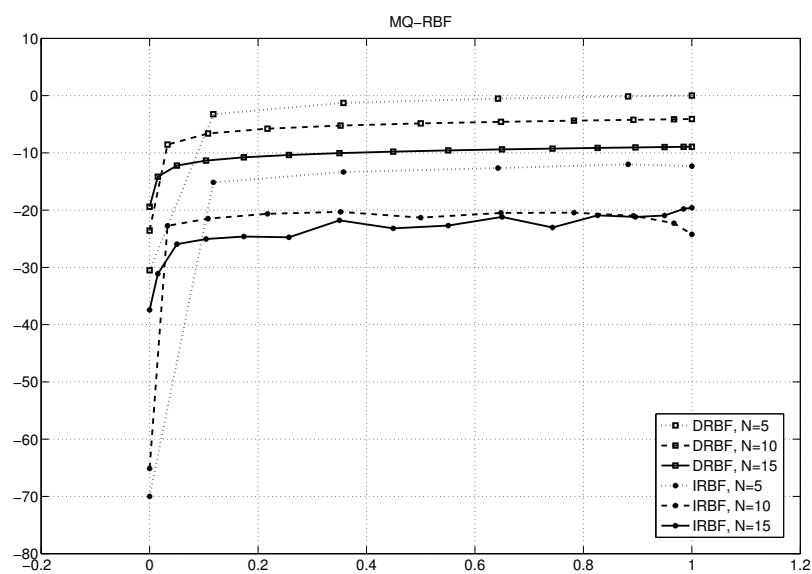


Figure 2: MQ-RBF: $\log |y_{Exact} - y_{Approx}|$ for Example 5.1.

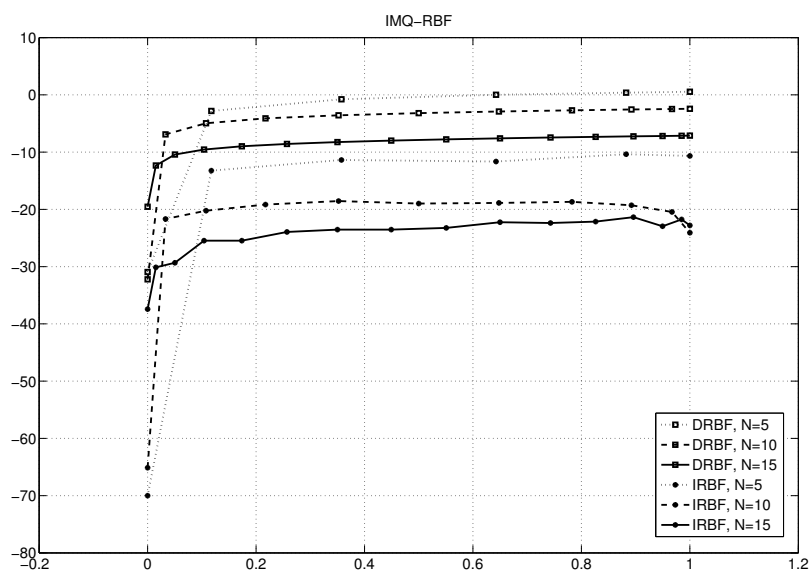


Figure 3: IMQ-RBF: $\log |y_{Exact} - y_{Approx}|$ for Example 5.1.

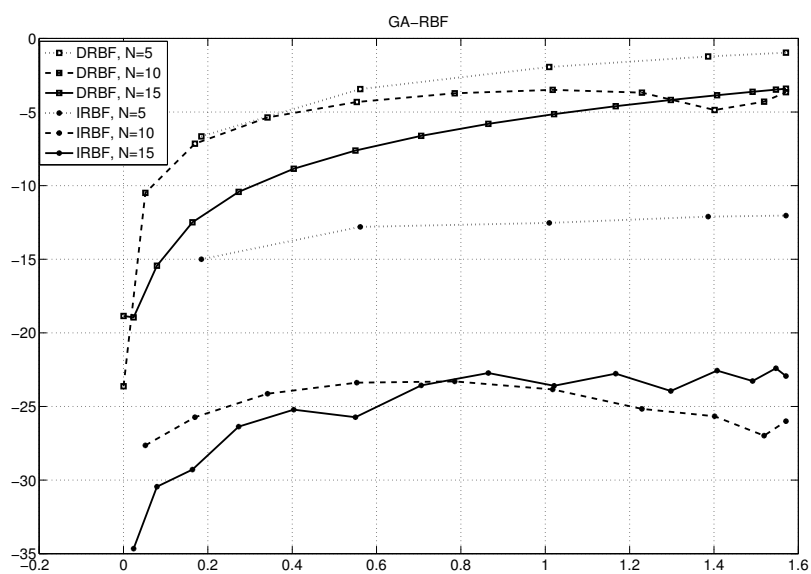


Figure 4: Ga-RBF: $\log |y_{Exact} - y_{Approx}|$ for Example 5.2.

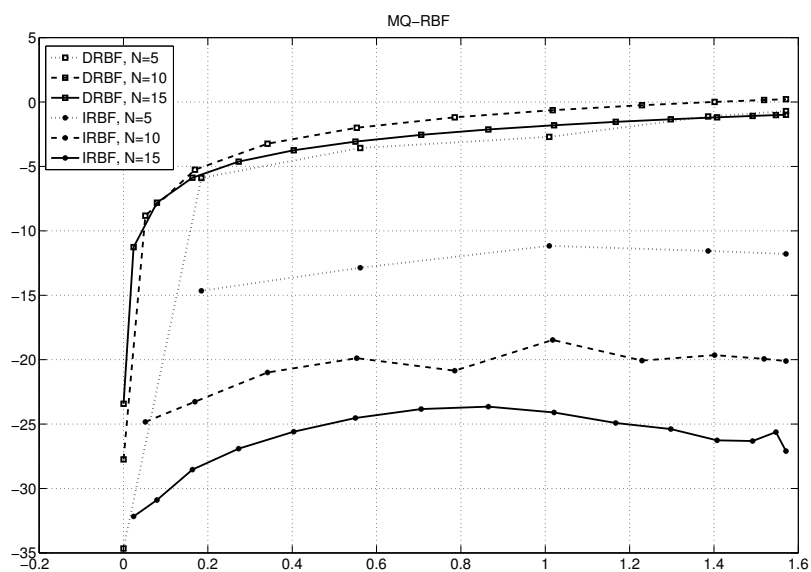


Figure 5: MQ-RBF: $\log |y_{Exact} - y_{Approx}|$ for Example 5.2.

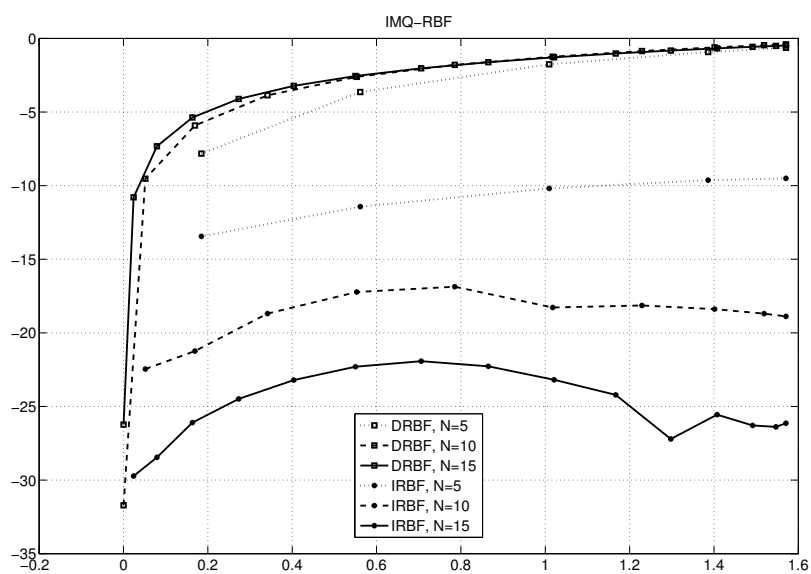


Figure 6: IMQ-RBF: $\log |y_{Exact} - y_{Approx}|$ for Example 5.2.

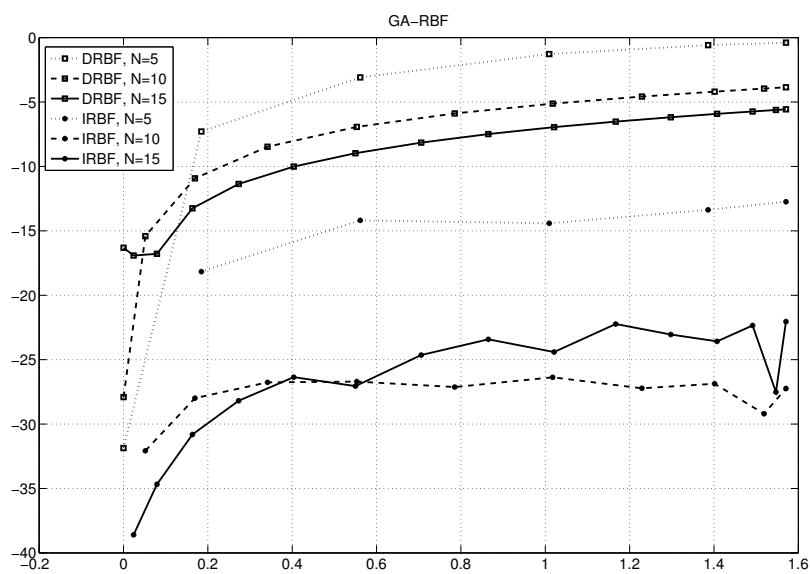


Figure 7: Ga-RBF: $\log |y_{Exact} - y_{Approx}|$ for Example 5.3.

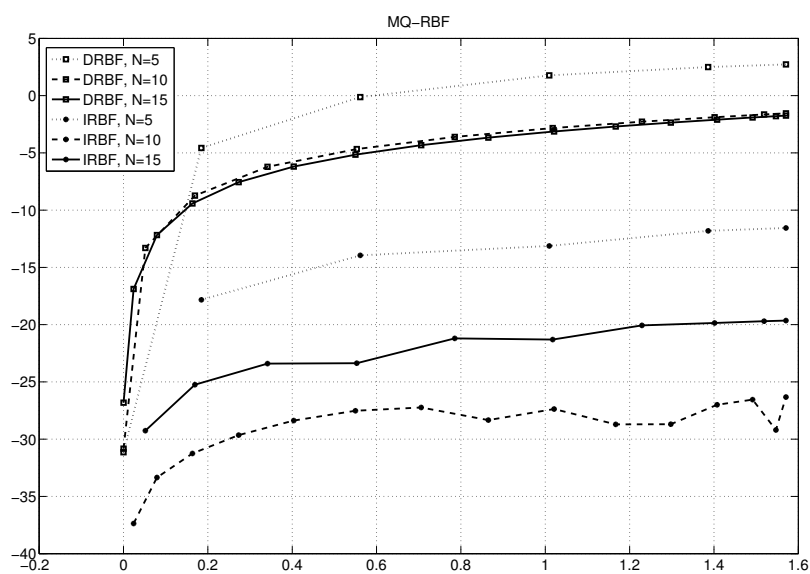


Figure 8: MQ-RBF: $\log |y_{Exact} - y_{Approx}|$ for Example 5.3.

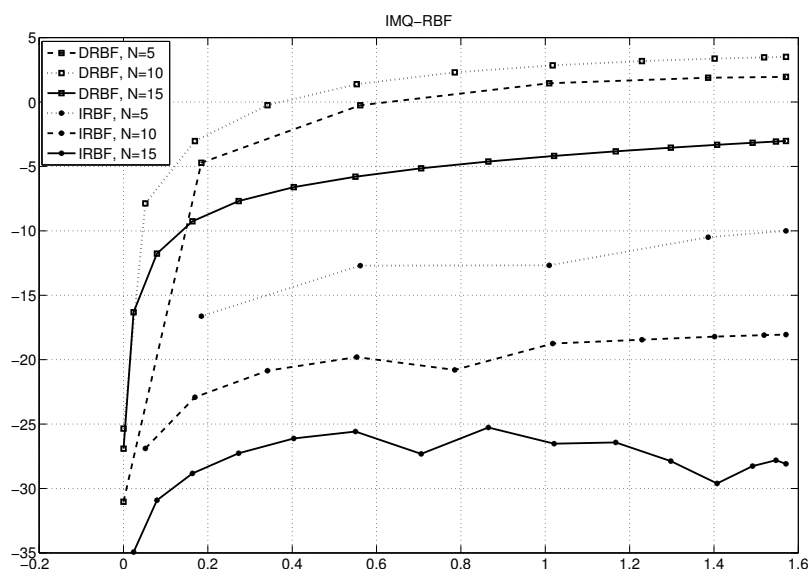


Figure 9: IMQ-RBF: $\log |y_{Exact} - y_{Approx}|$ for Example 5.3.

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