

Original Research Article

Geometrical Methods and Numerical Computations for Prey-Predator Systems

ABSTRACT

The goal of this paper is the geometrical and numerical study of the main sizes of the mathematical models of prey-predator interactions which are important in determining long-time dynamics, based on the application of various notions from the theory of dynamical systems to the numerical approximation of initial value problems over long-time intervals. Between dynamical systems theory and computation analysis of dynamical systems there is a strong interplay. The numerical methods are widely used for the study of complicated temporal behavior of dynamical systems, in order to approximate different types of invariants sets or invariant manifolds and also to extract statistical information on the dynamical behavior in the computation of natural invariant measures or almost invariants sets. The present study is a interplay between dynamical systems geometrical theory and computational calculus of dynamical systems, knowing that the theory provides a framework for interpreting numerical observations and foundations for efficient numerical algorithms.

Keywords: Hamilton-Poisson realization; conservation law; Volterra-Lotka equations; Runge-Kutta methods.

1. INTRODUCTION

This paper is devoted to studying conservation laws for Volterra-Lotka type systems arising from biology and relationship between this in the geometric framework of Classical Mechanics. The Lotka-Volterra model indeed may be the simplest possible predator-prey model. Nevertheless, it is a useful tool containing the basic properties of the real predator-prey systems, and serves as a robust basis from which it is possible to develop more sophisticated models. The Lotka-Volterra model is very important in population modeling.

The viewpoint is geometric and we also compute and characterize objects of dynamical significance, in order to understanding the mathematical properties observed in numerical computation for dynamical systems.

We will present two very important examples. First example represent so called *variational dynamical systems*, that is dynamical systems described by a system of ordinary differential equations which can be written as the Euler-Lagrange equations associated to Lagrangian L ,

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{y}^i} \right) - \frac{\partial L}{\partial x^i} = 0 \quad (1)$$

This example is the 2D prey-predator Lotka-Volterra system ([9], [13], [18], [19]). This dynamical systems are included in the presymplectic case because the 2-form ω_L associated to the corresponding Lagrangian is degenerate ([9]). Finally, we present different versions of the well-known prey-predator 3D Lotka-Volterra system. This system is not a variational dynamical system. However, we can give more Hamilton-Poisson realizations of this bi-Hamiltonian system like in the 2D case.

34 2. THE 2D LOTKA-VOLTERRA SYSTEM

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36 Let us consider the system of ordinary differential equations ([11]):

$$37 \quad \begin{cases} \dot{x} = ax - bxy \\ \dot{y} = -cy + dxy \end{cases}, a, b, c, d > 0 \quad (2)$$

38 This system is called *Lotka-Volterra system* and represents a complex biological system
39 model, in which two species x and y live in a limited area, so that individuals of the
40 species y (predator) feed only individuals of species x (prey) and they feed only resources
41 of the area in which they live. Proportionality factors a and c are respectively increasing
42 and decreasing prey and predator populations. If we assume that the two populations come
43 into interaction, then the factor b is decreasing prey population x caused by this predator
44 population y and the factor d is population growth due to this population x . The evolution
45 system (2) can be written in the form of Euler-Lagrange equations (1), where the
46 Lagrangian L is

$$47 \quad L = \frac{1}{2} \left(\frac{\ln y}{x} \dot{x} - \frac{\ln x}{y} \dot{y} \right) + c \ln x - a \ln y - dx + by$$

48 and the corresponding Hamiltonian H is

$$49 \quad H = \frac{\partial L}{\partial \dot{x}} \dot{x} - \frac{\partial L}{\partial \dot{y}} \dot{y} - L = -c \ln x + a \ln y + dx - by$$

50 Let us remark that the total energy $E_L = H$ is a *conservation law* for prey-predator
51 system (2) and the Lagrangian L is singular.

52 According to [12] and [14] the Lotka-Volterra equations (2) has the following *Hamilton-*
53 *Poisson realization*

$$54 \quad \dot{x}^i = J \nabla H, \text{ where } H = c \ln x + a \ln y - dx - by \text{ is the Hamiltonian and } J = \begin{pmatrix} 0 & xy \\ -xy & 0 \end{pmatrix}$$

55 is the Poisson bracket.

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57 3. THE 3D LOTKA-VOLTERRA SYSTEM

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59 In [5] was discussed the next three-dimensional Lotka-Volterra system which models the
60 evolution of competition between three species:

$$61 \quad \begin{cases} \dot{x} = x(cy + z + \lambda) \\ \dot{y} = y(x + az + \mu) \\ \dot{z} = z(bz + y + \nu) \end{cases} \quad (3)$$

62 where $a, b, c \in R, \lambda, \mu, \nu > 0$.

63 Following [5], if $abc = -1$ and $\nu = \mu b - \lambda ab$, then the 3D Lotka-Volterra system (3) admit

64 two conservation laws $H_1 = ab \ln x - b \ln y + \ln z$ and

65 $H_2 = abx + y - az + \nu \ln y - \mu \ln z$, because (3) is a particular case of a *bi-Hamiltonian*

66 system. The dynamics of (3) has two distinct Hamilton-Poisson realizations
 67 $\dot{x}^i = J_1 \nabla H_2$ and $\dot{x}^i = J_2 \nabla H_1$, where

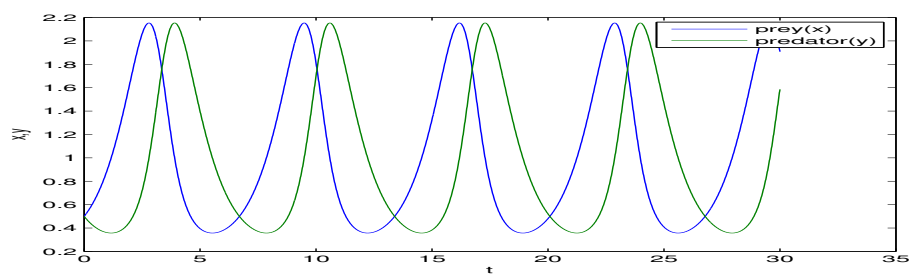
$$68 \quad J_1 = \begin{pmatrix} 0 & cxy & bcxz \\ -cxy & 0 & -yz \\ -bcxz & yz & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & cxy(az + \mu) & cxz(y + \nu) \\ -cxy(az + \mu) & 0 & xyz \\ -cxz(y + \nu) & -xyz & 0 \end{pmatrix}.$$

69 From $J_1 \nabla H_1 = 0$ and $J_2 \nabla H_2 = 0$ we have that H_1, H_2 are Casimir functions of $J_1,$
 70 J_2 , ([12]).

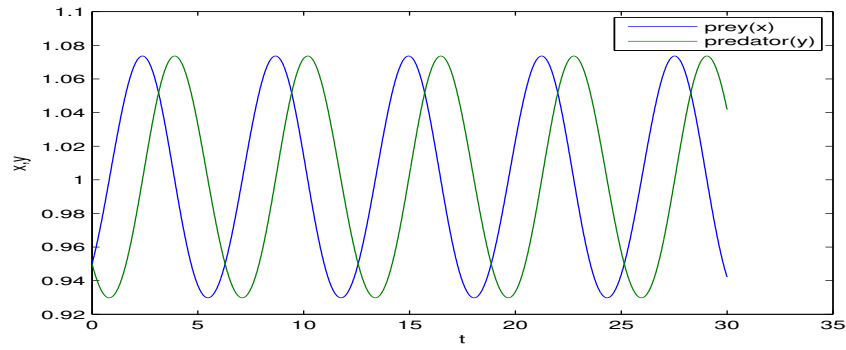
71 4. A NUMERICAL STUDY

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 73 Numerical integration is a important, actual and active subject due to the today high
 74 computer efficiency (speed and memory), being analysed by extensive theory and a vast
 75 range of software, platforms or libraries, [1], [3], [10]. Taking into account that even for the
 76 simplest 2D Lotka-Volterra system, the analytical solution is useless: root of a polynomial
 77 with an integral plus the special function Lambert, we must resort to numerical methods in
 78 order to have information about the trajectories. Thus, constructing a Matlab-based
 79 numerical code, we approximate and characterize different types of invariants and also
 80 extract informations on the dynamical behavior and perform comparisons for both different
 81 initial conditions associated to the considered problem and for different values of the
 82 parameters. In the first stage we focus on the numerical solving of the initial value problems
 83 by appropriate numerical methods, such as Runge-Kutta methods (for the 2D case we use a
 84 fourth order Runge-Kutta method, [4], and for the 3D case we used a fifth order Runge-Kutta
 85 method, [20]). We obtain the numerical solution represented by the approximate values of
 86 the solution function for a discrete set of data points. Using this approach we perform a
 87 numerical analysis of the conservation laws and main sizes.

88 4.1 2D Lotka-Volterra System



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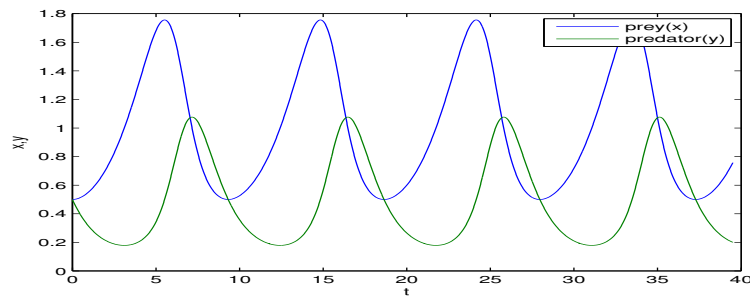


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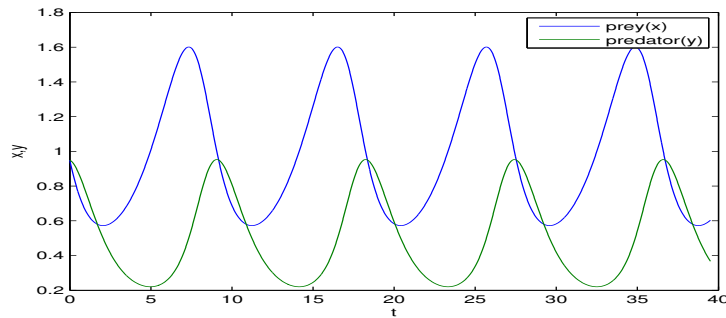
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Fig. 1. Profile of the numerical solution $(x(t), y(t))$, for $a = b = c = d = 1$ and initial conditions $(x_0 = 0.5, y_0 = 0.5)$ and $(x_0 = 0.95, y_0 = 0.95)$



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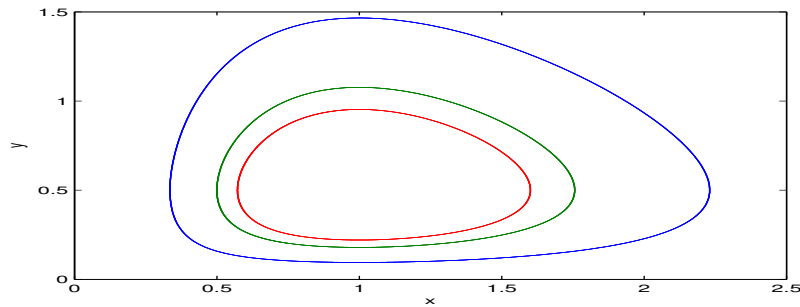


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Fig. 2. Profile of the numerical solution $(x(t), y(t))$, for $a = 0.5, b = c = d = 1$ and initial conditions $(x_0 = 0.5, y_0 = 0.5)$, $(x_0 = 0.95, y_0 = 0.95)$

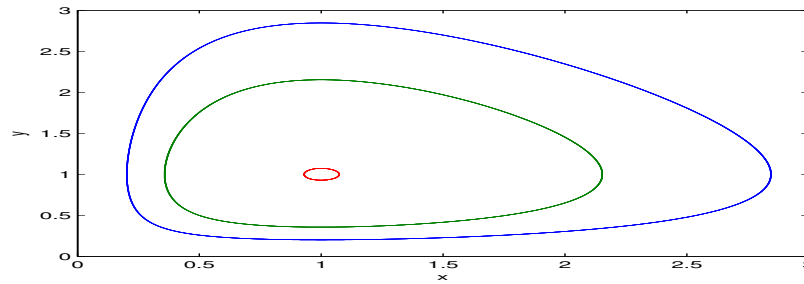


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Fig. 3. Phase space portrait for $a = 0.5, b = c = d = 1$ and initial conditions $(x_0 = 0.35, y_0 = 0.35)$, $(x_0 = 0.5, y_0 = 0.5)$, $(x_0 = 0.95, y_0 = 0.95)$ listed in order from outermost trajectory to innermost trajectory.

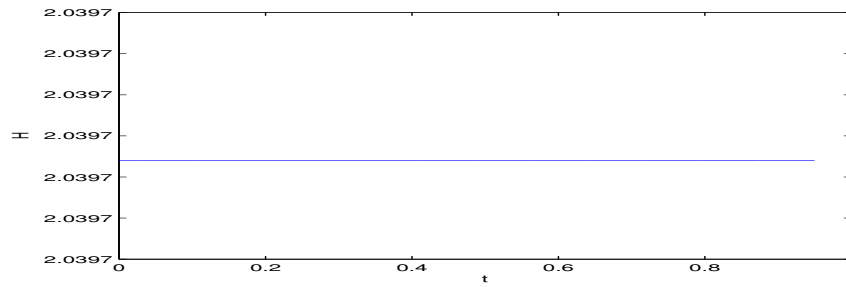


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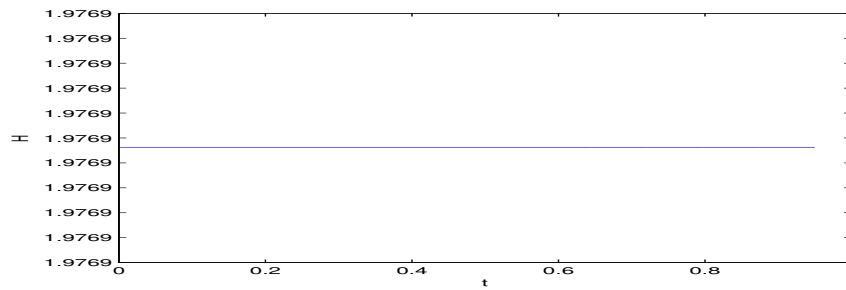
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Fig. 4. Phase space portrait for $a = 0.5$, $b = c = d = 1$ and initial conditions $(x_0 = 0.35, y_0 = 0.35)$, $(x_0 = 0.5, y_0 = 0.5)$, $(x_0 = 0.95, y_0 = 0.95)$ listed in order from outermost trajectory to innermost trajectory.



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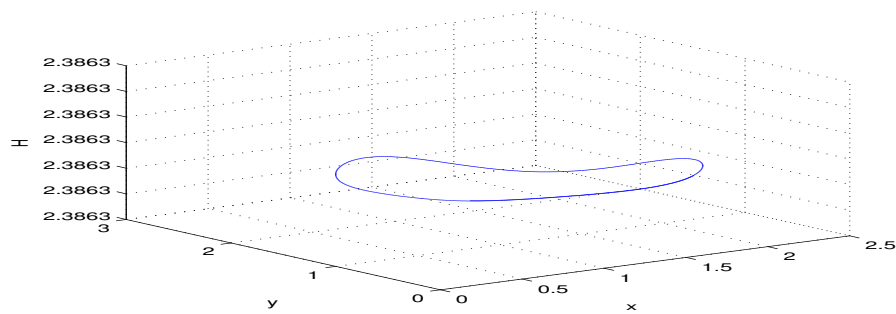


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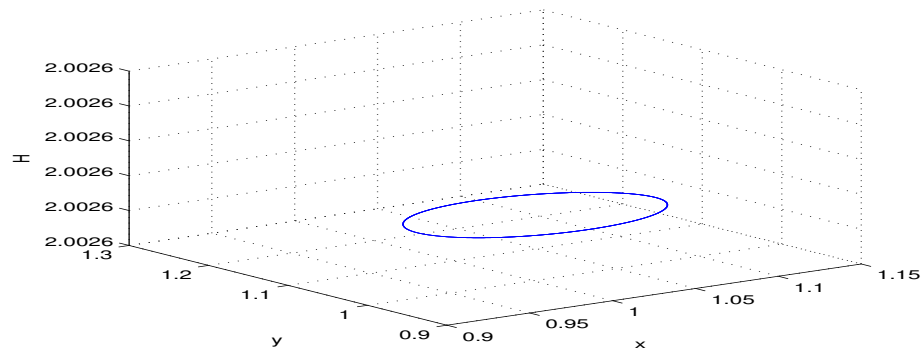
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Fig. 5. The profile of the Hamiltonian H as function of t , for $a = 0.5$, $b = c = d = 1$ and initial conditions $(x_0 = 0.5, y_0 = 0.5)$, and $(x_0 = 0.95, y_0 = 0.95)$ respectively.

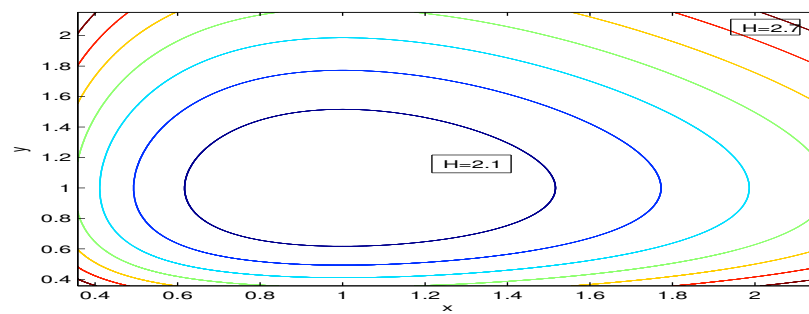


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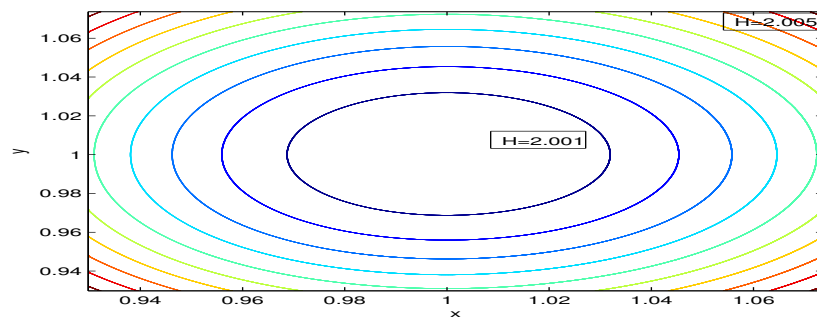


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109 Fig. 6. The profile of $H(x, y)$ for $a = 0.5$, $b = c = d = 1$ and initial conditions $(x_0 = 0.5, y_0 = 0.5)$, and $(x_0 = 0.95, y_0 =$
 110 $0.95)$ respectively.

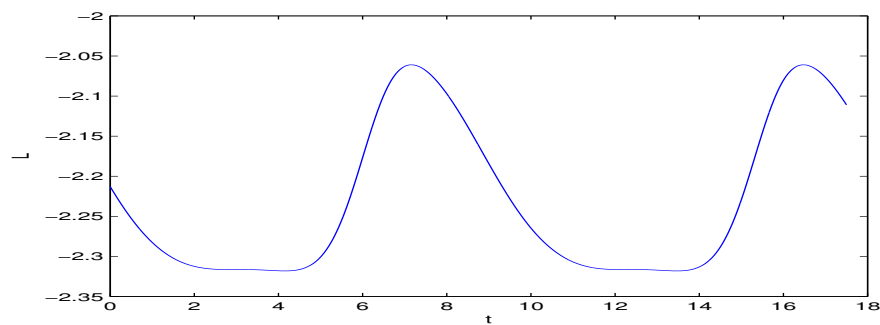


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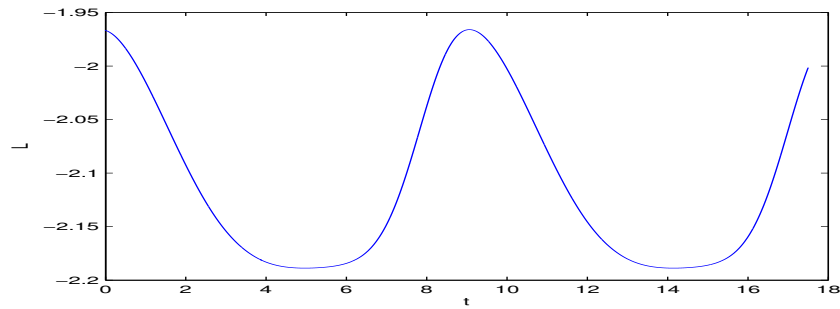


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113 Fig. 7. Some level curves of $H(x, y)$ for $a = b = c = d = 1$ and initial conditions $(x_0 = 0.5, y_0 = 0.5)$, and $(x_0 = 0.95,$
 114 $y_0 = 0.95)$ respectively.

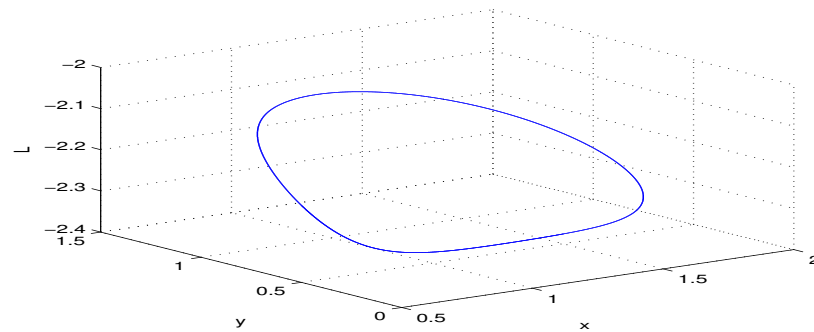


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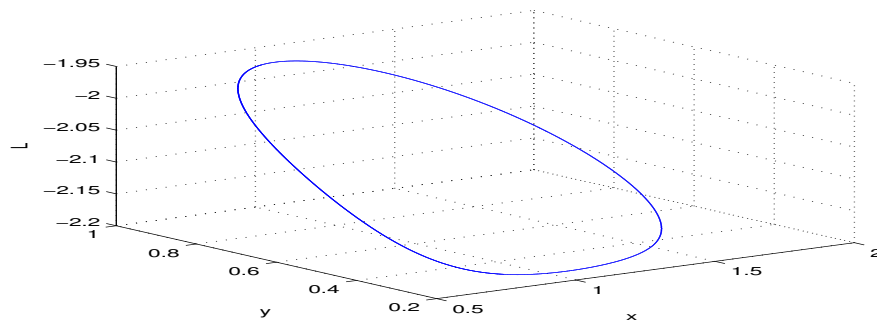


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117 Fig. 8. The profile of the Lagrangian L as function of t for $a = 0.5$, $b = c = d = 1$ and initial conditions $(x_0 = 0.5, y_0 =$
 118 $0.5)$, and $(x_0 = 0.95, y_0 = 0.95)$ respectively.



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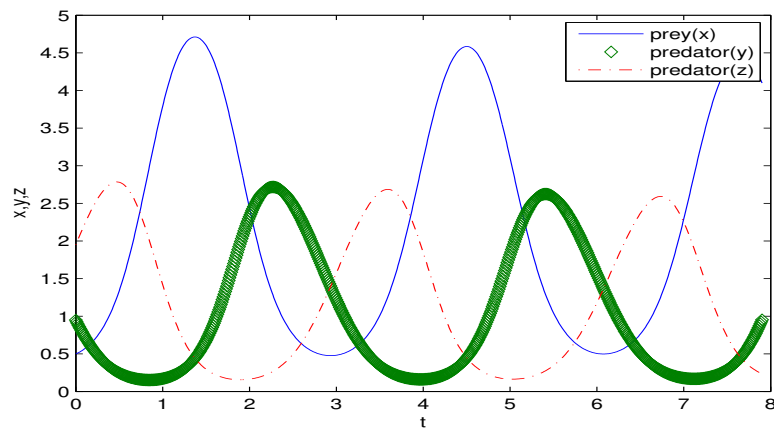


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121 Fig. 9. The profile of $L(x, y)$ for $a = 0.5$, $b = c = d = 1$ and initial conditions $(x_0 = 0.5, y_0 = 0.5)$ and $(x_0 = 0.95, y_0 =$
 122 $0.95)$ respectively.

123 4.2 3D Lotka-Volterra System

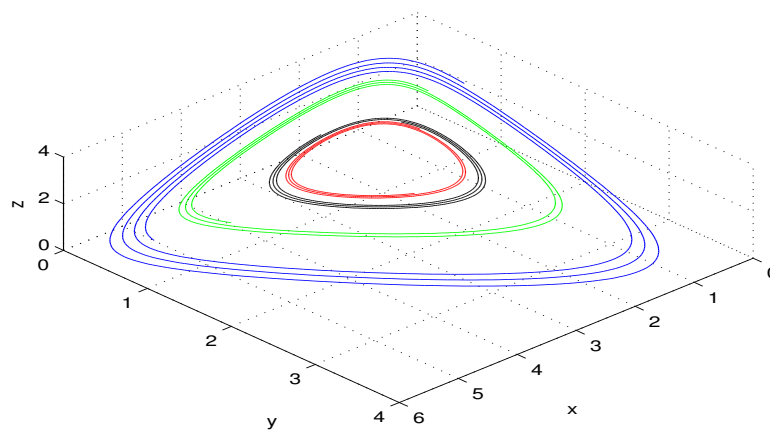
124 We consider the case $abc = -1$, $\nu = \mu b - \lambda ab$. We set $a = b = c = -1$ and $\mu = 1$,
 125 $\lambda = 0$, $\nu = -1$.



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Fig. 10. The profile of the numerical solution $(x(t), y(t), z(t))$, for the initial conditions $(x_0 = 0.5, y_0 = 1, z_0 = 2)$.



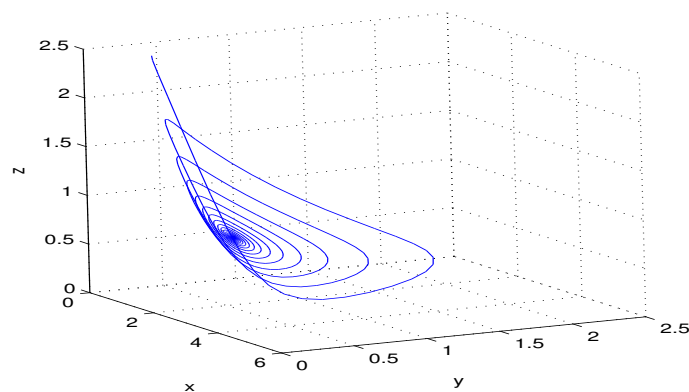
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Fig. 11. Phase space portrait for the initial conditions $(x_0 = 0.5, y_0 = 0.95, z_0 = 2.95)$, $(x_0 = 0.5, y_0 = 0.5, z_0 = 1.95)$, $(x_0 = 1, y_0 = 0.75, z_0 = 1.25)$, $(x_0 = 2.1, y_0 = 0.35, z_0 = 1.55)$ listed in order from outermost trajectory to innermost trajectory.

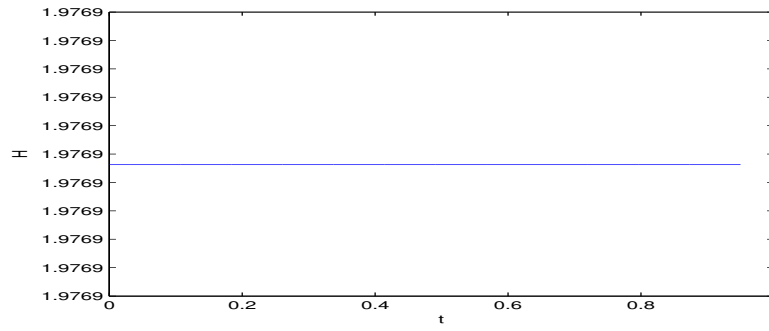


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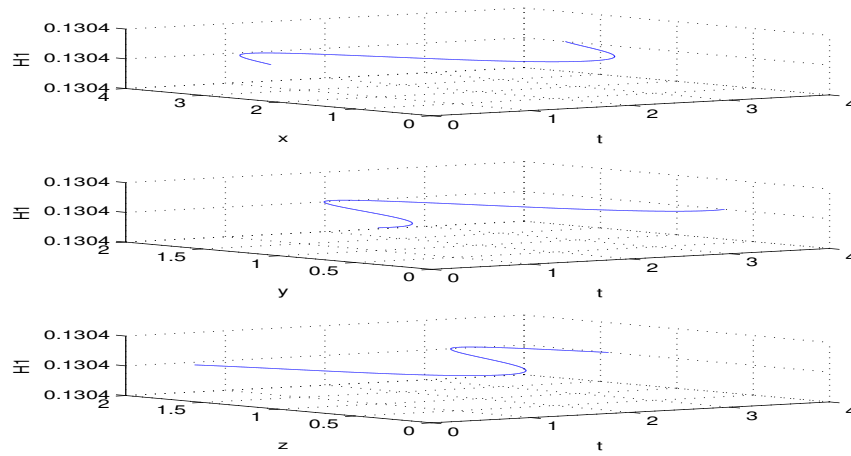
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Fig. 12. Phase space for the initial conditions $(x_0 = 1, y_0 = 0.25, z_0 = 2.5)$ The numerical solution presents a graphical profile given by downward spirals.



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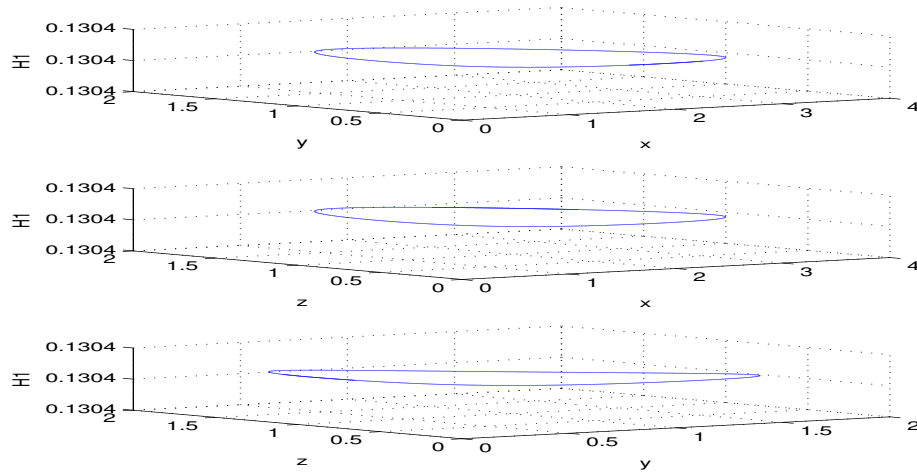
136 Fig. 13. The profile of the Hamiltonian $H1$ as a function of t , for the initial conditions $(x_0 = 2.1, y_0 = 0.35, z_0 = 1.55)$.



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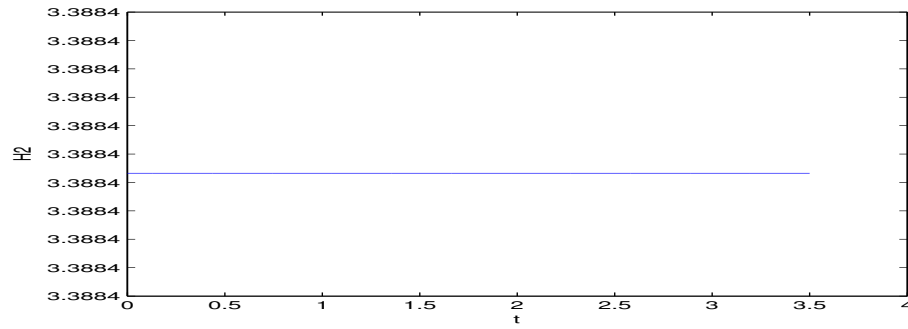
138 Fig. 14. The profile of the Hamiltonian $H1$ as a function of (t, x) , (t, y) and (t, z) , and for the initial conditions $(x_0 = 2.1,$
139 $y_0 = 0.35, z_0 = 1.55)$.

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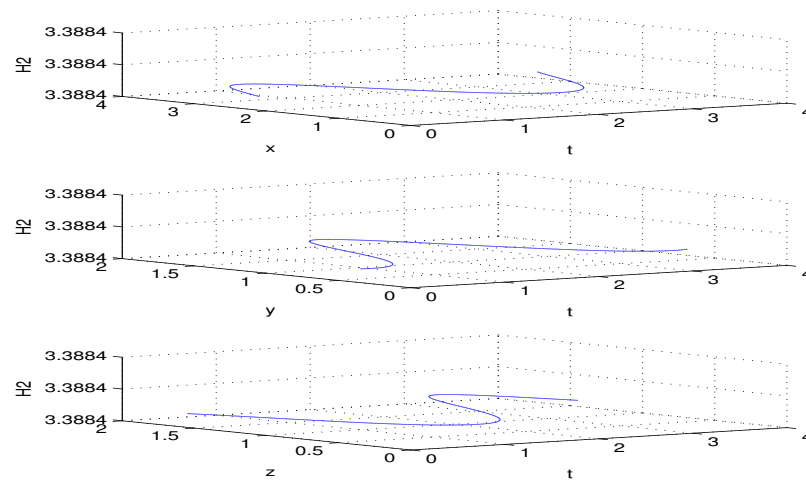
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142 Fig. 15. The profile of the Hamiltonian $H1$ as a function of (x, y) , (x, z) and (y, z) , for the initial conditions $(x_0 = 2.1,$
143 $y_0 = 0.35, z_0 = 1.55)$.



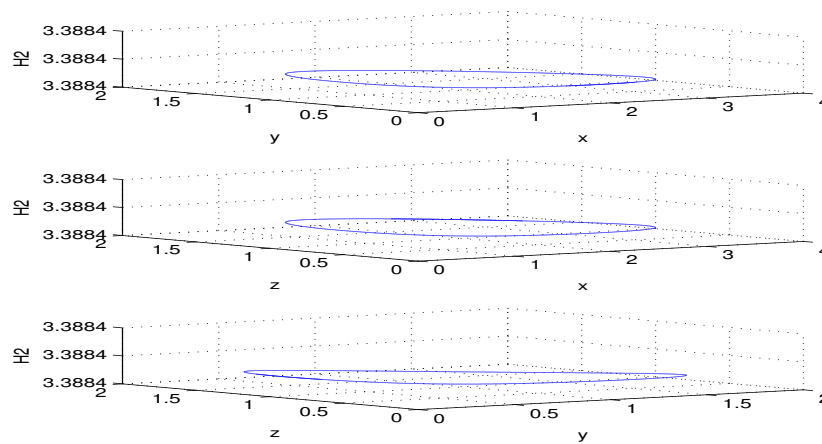
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145 Fig. 16. The profile of the Hamiltonian H_2 as a function of t , for the initial conditions ($x_0 = 2.1$, $y_0 = 0.35$, $z_0 = 1.55$).



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147 Fig. 17. The profile of the Hamiltonian H_2 as a function of (t, x) , (t, y) and (t, z) , for the initial conditions ($x_0 = 2.1$,
148 $y_0 = 0.35$, $z_0 = 1.55$).



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150 Fig. 18. The profile of the Hamiltonian H_2 as a function of (x, y) , (x, z) and (y, z) , for the initial conditions ($x_0 = 2.1$,
151 $y_0 = 0.35$, $z_0 = 1.55$).

4. RESULTS AND DISCUSSION

The predator population begins to decline shortly after the prey population starts to decrease. Then after the prey population begins to recover, the predator population also starts to recover, in agreement with [18]. They share a common period, figure 1 and figure 2.

The total energy H takes constant values for different values of parameters a, b, c, d and different initial conditions, as we presented in figure 5 and figure 6. Thus we are in agreement with the property to be a conservation law.

In the 3D case displaying the graph of x, y and z across time t , one observes the periodic behavior of the system. Each predator population also peaks and then begins to decrease shortly after its respective prey population peaks and begins to decrease, figure 10.

The two Hamiltonians H_1, H_2 associated to the 3D case of Lotka-Volterra system are characterized through our numerical study by constant values, for different initial conditions, as we presented in figures 13 - 18.

5. CONCLUSION

We perform a computational analysis of these mathematical models, in order to approximate different types of invariants and main sizes, through numerical codes based on appropriate numerical calculus techniques for numerical integration of these type problems. Thus, starting from certain initial value problems associated to our models, we obtain the numerical solution and we develop the numerical characterization of the main sizes previously analysed from the geometrical point of view. Thus we are able to make different comparisons between these studied quantities for different values of parameters, for different initial conditions etc. Taking into considerations that the predator and prey populations display a variety of dynamic patterns, we emphasize that the present study can be useful also to make certain adjustments for the parameters involved in the differential equations in order to calibrate more precisely the models to better predict some realistic situations and take account for the differences from one example to the next.

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