

A Fuzzy approach for solving three level quadratic programming problem with stochastic parameters in the objective function

Abstract

In this paper a fuzzy approach is presented for solving a three-decision maker's model. This approach solves a three level quadratic programming problem with stochastic parameters in the objective function. After converting probabilistic nature of the objective functions to equivalence deterministic objective functions each level attempts to optimize its problem separately using fuzzy programming technique, in this method the tolerance and membership function concepts are used to develop Tchebycheff problem for generating Pareto optimal solution for this problem. Finally, a numerical example is given to clarify the main results developed in this paper.

Keywords: stochastic programming; three level; fuzzy programming

1 introduction

In the field of mathematical programming (MP), multilevel programming (MLP) was developed to solve decentralized planning problems with multiple DMs in a hierarchical decision making organization. In MLPP, the execution of the decision power is sequential from a higher level to lower level, and each DM tries to optimize his own benefit under a conflicting environment in the hierarchical levels [1].

Multi-level programming techniques were developed to solve decentralized problems with multiple decisionmakers in hierarchical organization, where each unit or department independently seek its own interest, but is affected by the actions of other units through externalities[2,3,4,5].

DMs in a hierarchical organization, where each unit or department independently seeks its own interests, but is affected by the actions of other units through externalities. The MLPP can be encountered in almost any hierarchical organization such as government agencies, profit or nonprofit organizations, manufacturing plants, and logistic companies [6].

The basic concept of the fuzzy programming approaches implies that the LLDM optimizes his/her objective function, taking a goal or preference of the ULDM into consideration. In the decision process, considering the membership functions of the fuzzy goals for the decision variables of the ULDM, the LLDM solves a FP problem with the set of constraints on an overall satisfactory degree of the ULDM. If the proposed solution is not satisfactory to the ULDM, the solution search is continued by redefining the elicited membership functions until a satisfactory solution is reached [2,6,7].

Fuzzy approach uses the concept of tolerance membership to develop a fuzzy max-min decision model for generating Pareto optimal (satisfactory) solution for three level programming problem; the first level decision maker (FLDM) specifies his objective functions and decisions with possible tolerances which are described by membership functions of fuzzy set theory. Then, the second level decision maker (SLDM) specifies his objective functions and decisions, in the view of FLDM, with possible tolerances which are described by membership functions of fuzzy set theory. Finally, the

third level decision maker (TLDM) uses the preference information for upper levels decision –maker subject to the upper levels decision [2,8,9,10,11].

Decision problems of chance-constrained or stochastic optimization arise when certain coefficients of an optimization model are not fixed or known but instead, to some extent, are of probabilistic quantities. In most real life problems in mathematical programming, the parameters are considered as random variables[12,13].

Actually, stochastic optimization deals with the decision situations where some or all of the parameters of optimization problems are defined by stochastic (random / probabilistic) variables rather than deterministic quantities .In recent years, the methods of multiobjective stochastic optimization problems have become increasingly important in searching solutions of practical decision problems like economics , water resource management , healthcare , transportation , agriculture , energy systems , and other real-life problems.[14, 15]

In [5] Pramanik et al presented a fuzzy goal programming approach to solve chance constrained linear plus linear fractional bi-level programming problem. The chance constraints with right hand parameters as random variables of prescribed probability distribution functions are transformed into equivalent deterministic system constraints. They construct nonlinear membership functions based on deterministic system constraints. The nonlinear membership functions are transformed into linear membership functions by using first order Taylor’s series approximation. In the bi-level decision making context. In this paper some simple and easy techniques are used to solve the problem like the technique which used in [16] to convert the probabilistic nature of the constraints to equivalence deterministic constraints , simplex method to give the best and the worst individual solution for each decision maker’s problem, the fuzzy set theory to formulate the previous results to membership function and use the concept of of tolerance to develop Tchebycheff problem for generating Pareto optimal solution for our problem.

In [14]Kumar M and Pal BB presented Fuzzy Goal Programming Approach to Chance Constrained Multilevel Programming Problems.first the chance constraints of a problem are converted into their respective deterministic equivalent in the decision making context. Then, the objective functions of decision makers (DMs) located at different hierarchical levels are converted into fuzzy goals by introducing an imprecise aspiration level to each of them to make decision in an uncertain environment. In the model formulation, the concept of tolerance membership functions in fuzzy sets for measuring the degree of satisfaction of DMs with regard to achieving the aspired levels of fuzzy goals as well as degree of optimality of the decision vectors controlled by upper-level DMs on the basis of their order of hierarchy in the organizational system. In the solution process, minimization of under deviational variables associated with membership goals defined for the membership functions are considered for achieving the highest membership value (unity) of each of the fuzzy goals to the extent possible on the basis of their weights of importance in the decision making horizon. To illustrate the effectiveness of the proposed approach, a numerical example is solved.

This paper is divided into the follow sections: section 2 presents the problem formulation and solution concept of the model of three level quadratic programming problem with stochastic parameters in the objective function, section 3 introduce how to convert the probabilistic nature of the stochastic nature in the objective functions to equivalence deterministic objective functions,

section 4 uses Fuzzy approach to solve the three level quadratic programming problem with stochastic parameters in the objective function, in section 5 a numerical example is provided to clarify the results. Finally, concluding remarks and future works are given in section 6.

2 Problem Formulation and Solution Concept

Let $x_j \in R^n, (j=1,2,3)$ be a vector variable indicating the first decision level choice, the second decision level choice, and the third decision level choice, $n = \sum_{j=1}^3 n_j$.

Let $F_i: R^n \rightarrow R^{N_i}, (i=1,2,3)$ be the first level objective function, the second level objective function, and the third level objective function respectively. Let the FLDM (First Level Decision Maker), SLDM (Second Level Decision Maker), and TLDM (Third Level Decision Maker) have N_1 , N_2 and N_3 objective function, respectively.

Therefore, the three-level quadratic programming problem with stochastic parameters in the objective (TLQPP) problem may be formulated as follows:

[First Level]

$$\max_{x_1} f_1(x) = sx + \frac{1}{2}x^T bx, \quad (1)$$

Where x_2, x_3 solves

[Second Level]

$$\max_{x_2} f_2(x) = sx + \frac{1}{2}x^T bx, \quad (2)$$

Where x_3 solves

[Third Level]

$$\max_{x_3} f_3(x) = sx + \frac{1}{2}x^T bx, \quad (3)$$

Subject to

$$(x_1, x_2, x_3) \in G. \quad (4)$$

Where

$$G = \{ x \in R^n : g_i(x) = \sum_{j=1}^n c_{ij}x_j \leq d_i, (i=1,2,\dots,n), x_1, x_2, x_3 \geq 0 \}. \quad (5)$$

Where f_1, f_2 and f_3 are the objective functions with stochastic parameters of the first level decision maker (FLDM), second level decision maker (SLDM), and third level decision maker (TLDM), b is

$n \times n$ are real matrices contain random stochastic coefficients and s is $(1 \times n)$ matrix, the vector of decision variables x is n -vector partitioned between the three planners.

Definition 1.

For any $(x_1 \in G_1 = \{x_1 | (x_1, x_2, x_3) \in G\})$ given by FLDM and $(x_2 \in G_2 = \{x_2 | (x_1, x_2, x_3) \in G\})$ given by SLDM, if the decision-making variable $(x_3 \in G_3 = \{x_3 | (x_1, x_2, x_3) \in G\})$ is the optimal solution of the TLDM, then (x_1, x_2, x_3) is the feasible solution of TLQP problem with stochastic parameters.

Definition 2.

If (x_1^*, x_2^*, x_3^*) is a feasible solution of the TLQP problem with stochastic parameters; no other feasible solution $(x_1, x_2, x_3) \in G$ exists ; so (x_1^*, x_2^*, x_3^*) is the optimal solution of the TLQP problem with stochastic parameters .

3 Converting Probabilistic Nature of the objective functions to Equivalence Deterministic objective functions

The basic idea in this section is to convert the probabilistic nature of the objective function to equivalence deterministic objective function. Using:

$$f_r(x) = \sum_{j=1}^n s_{rj} x_j + k_1^r \sum_{j=1}^n x_j^2 E(b_j^r) + k_2^r \sqrt{\sum_{j=1}^n \frac{1}{2} x_j^2 \sigma^2(b_j^r)} x, r = 1, 2, \dots, k. \quad [14].(6)$$

Where $E(b_j^r)$ = mean of b_j^r and $\sigma^2(b_j^r)$ = variance of, and k_1^r, k_2^r are non-negative constants whose values indicate the relative importance of the mean and the standard deviation of the variable b_j^r for maximization. If $k_1^r = k_2^r = 1$, it is an indication that equal importance is given to the maximization of the mean as well as the standard deviation of b_j^r .

So the TLCCQP problem can be written as:
[First Level]

$$\max_{x_1} f_1(x) = sx + \frac{1}{2} x^T b^r x, \quad (7)$$

Where x_2, x_3 solves
[Second Level]

$$\max_{x_2} f_2(x) = sx + \frac{1}{2} x^T b^r x, \quad (8)$$

Where x_3 solves

[Third Level]

$$\max_{x_3} f_3(x) = sx + \frac{1}{2}x^T b^x, \quad (9)$$

Subject to

$$(x_1, x_2, x_3) \in G. \quad (10)$$

Where

$$G = \{ x \in R^n : g_i(x) = \sum_{j=1}^n c_{ij}x_j \leq d_i, (i = 1, 2, \dots, n), x_1, x_2, x_3 \geq 0 \}. \quad (11)$$

4-Fuzzy Approach of the three level quadratic programming problem with stochastic parameters in the objective function

To solve the three-level quadratic programming problem with stochastic parameters in the objective function, one first convert the probabilistic nature of the objective functions to equivalence deterministic objective functions, after that gets the satisfactory solution that is acceptable to FLDM, and then gives the FLDM decision variable and goal with some leeway to the SLDM for him/her to seek the optimal solution, then the SLDM give the decision variables and goals with some leeway to the TLDM for him/her to seek the satisfactory solution and to arrive at the solution which is closer to the optimal solution of the FLDM. This due to, the TLDM who should not only optimize his/her objective function, but also try to satisfy the SLDM goals and preferences as much as possible, SLDM also do the same action to satisfy the FLDM goals and preferences as much as possible.

4.1 FLDM Problem

First, the FLDM solves (7) subject to $\bar{G}$, by applying the simplex method. The individual best solution (F_1^*) and individual worst solution (F_1^-) of (7) subject to $\bar{G}$ are:

$$F_1^* = \max_{x \in \bar{G}} F_1(x), \quad F_1^- = \min_{x \in \bar{G}} F_1(x) \quad (12)$$

Goals and tolerances can then be reasonably set for individual solution and the differences of the best and worst solutions, respectively. This data can then be formulated as the following membership function of fuzzy set theory [8]:

$$\mu [F_1(x)] = \begin{cases} 1 & \text{if } F_1(x) > F_1^*, \\ \frac{F_1(x) - F_1^-}{F_1^* - F_1^-} & \text{if } F_1^- \leq F_1(x) \leq F_1^*, \\ 0 & \text{if } F_1^- \geq F_1(x). \end{cases} \quad (13)$$

4.2 SLDM Problem

In the same way, the SLDM independently solves (8) subject to $\bar{G}$, by applying the simplex method. The individual best solution (F_2^*) and individual worst solution (F_1^-) of (8) subject to $\bar{G}$ are:

$$F_2^* = \underset{x \in \bar{G}}{\text{Max}} F_2(x), \quad F_2^-(x) = \underset{x \in \bar{G}}{\text{Min}} F_2(x) \quad (14)$$

This information can then be formulated as the following membership function:

$$\mu [F_2(x)] = \begin{cases} 1 & \text{if } F_2(x) > F_2^*, \\ \frac{F_2(x) - F_2^-}{F_2^* - F_2^-} & \text{if } F_2^- \leq F_2(x) \leq F_2^*, \\ 0 & \text{if } F_2^- \geq F_2(x). \end{cases} \quad (15)$$

4.3 TLDM Problem

In the same way, the TLDM independently solves (9) subject to $\bar{G}$, by applying the simplex method. The individual best solution (F_3^*) and individual worst solution (F_3^-) of (9) subject to $\bar{G}$ are:

$$F_3^* = \underset{x \in \bar{G}}{\text{Max}} F_3(x), \quad F_3^-(x) = \underset{x \in \bar{G}}{\text{Min}} F_3(x) \quad (16)$$

This information can then be formulated as the following membership function:

$$\mu [F_3(x)] = \begin{cases} 1 & \text{if } F_3(x) > F_3^*, \\ \frac{F_3(x) - F_3^-}{F_3^* - F_3^-} & \text{if } F_3^- \leq F_3(x) \leq F_3^*, \\ 0 & \text{if } F_3^- \geq F_3(x). \end{cases} \quad (17)$$

Now the solution of the FLDM, SLDM and TLDM are disclosed. However, three solutions are usually different because of nature between three levels objective functions. The FLDM and SLDM

222 know that using the optimal decisions, x_2^S as a control factor for the TLDM are not practical. It is
 223 more reasonable to have some tolerance that gives the TLDM an extent feasible region to search for
 224 his/her optimal solution, and reduce searching time or interactions.

225 In this way, the range of decision variable x_1, x_2 should be around x_1^F, x_2^S with maximum tolerance
 226 t_1, t_2 and the following membership function specify x_1^F, x_2^S as:

$$227 \quad \mu_{x_1}(x_1) = \begin{cases} \frac{x_1 - (x_1^F - t_1)}{t_1} & x_1^F - t_1 \leq x_1 \leq x_1^F, \\ \frac{(x_1^F + t_1) - x_1}{t_1} & x_1^F \leq x_1 \leq x_1^F + t_1. \end{cases} \quad (18)$$

$$228 \quad \mu_{x_2}(x_2) = \begin{cases} \frac{x_2 - (x_2^S - t_2)}{t_2} & x_2^S - t_2 \leq x_2 \leq x_2^S, \\ \frac{(x_2^S + t_2) - x_2}{t_2} & x_2^S \leq x_2 \leq x_2^S + t_2. \end{cases} \quad (19)$$

230
 231 Finally, in order to generate the satisfactory solution, which is also the optimal solution with overall
 232 satisfaction for all decision - makers, the Tchebycheff problem can be solved as the following: [17]

$$233 \quad \text{Max } \delta, (20)$$

234 Subject to

$$235 \quad \frac{[(x_1^F + t_1) - x_1]}{t_1} \geq \delta I,$$

$$236 \quad \frac{[x_1 - (x_1^F - t_1)]}{t_1} \geq \delta I,$$

$$237 \quad \frac{[(x_2^S + t_2) - x_2]}{t_2} \geq \delta I,$$

$$238 \quad \frac{[x_2 - (x_2^S - t_2)]}{t_2} \geq \delta I,$$

$$\mu[F_1(x)] \geq \delta,$$

$$\mu[F_2(x)] \geq \delta,$$

$$\mu[F_3(x)] \geq \delta,$$

$$(x_1, x_2, x_3) \in G,$$

$$t_1, t_2 > 0,$$

$$\delta \in [0, 1]$$

Where δ is the overall satisfaction, and I is the column vector with all elements equal to 1s. By solving problem (20). If the FLDM is satisfied with the solution then optimal solution is reached. Otherwise, he/she should provide a new membership function for the control variable and objectives to the SLDM and TLDM, until an optimal solution is reached. It is easy to see that there is an inverse correlation between t_1, t_2 and δ .

5 Numerical Example

To demonstrate the solution for (TLQP) problem with stochastic parametrs , let us consider the following example:

[Upper Level]

$$\underset{x_1}{Max} F_1(x_1, b_1) = \underset{x_1}{Max} (2 + b_1)x_1^2 + 2x_2^2 + x_3^2,$$

Where x_2 solves

[Second Level]

$$\underset{x_2}{Max} F_2(x_2, b_2) = \underset{x_2}{Max} 2x_1 + (b_2 + 1)x_2^2 + x_3^2,$$

258

Where x_3 solves

[Third Level]

$$\underset{x_3}{Max} F_3(x_3, b_3) = \underset{x_3}{Max} x_1^2 + x_2 + (b_3 + 2)x_3^2,$$

262

Subject to

$$\{(x_1, x_2, x_3) | x_1 + x_2 + x_3 \leq 13, \\ x_1 + 2x_2 + x_3 \leq 17, \\ x_1 + x_2 + 2x_3 \leq 13.5, \\ x_1, x_2, x_3 \geq 0. \}.$$

Suppose that b_i , ($i=1,2,3$) is normally distributed random parameters with the following means and variances, $E(b_1)=3$, $E(b_2)=5$, $E(b_3)=3$, $VAR(b_1)=4$, $VAR(b_2)=16$, $VAR(b_3)=4$.

1. In the first the probability nature of the objective functions will be converted to equivalent deterministic nature.
2. Secondly, reformulate the problem in to deterministic form as following:

[Upper Level]

$$Max_{x_1} F_1(x_1, b_1) = Max_{x_1} 7x_1^2 + 2x_2^2 + x_3^2,$$

Where x_2 solves

[Second Level]

$$Max_{x_2} F_1(x_2, b_2) = Max_{x_2} 2x_1 + 10x_2^2 + x_3^2,$$

Where x_3 solves

[Third Level]

$$Max_{x_3} F_1(x_3, b_3) = Max_{x_3} x_1^2 + x_2 + 7x_3^2,$$

subject to

$$x_1 + x_2 + x_3 \leq 13,$$

$$x_1 + 2x_2 + x_3 \leq 17,$$

$$x_1 + x_2 + 2x_3 \leq 13.5,$$

$$x_1, x_2, x_3 \geq 0.$$

First, the FLDM solves his/her problem as follows:

287 Find individual best and worst solutions by solving the FLDM problem using simplex method as
 288 $F_1^* = 1183$, and $F_1^-(x) = 0$.

289 Second, the SLDM solves his/her problem as follows:

290 Find individual best and worst solutions by solving the SLDM problem using simplex method as
 291 $F_2^* = 722.5$, and $F_2^-(x) = 0$.

292 Third, the TLDM solves his/her problem as follows:

293 Find individual best and worst solutions by solving the TLDM problem using simplex method
 294 $F_3^* = 186.04$, and $F_3^-(x) = 0$.

295 Finally

- 296 1. Assume the FLDM control decision x_1^F is around (0) with tolerance 1 .
- 297 2. Assume the SLDM control decision x_2^F is around (8.97) with the tolerance 2 .

298

299 Max δ ,

300 Subject to

301 $5x_1^2 + 3x_2^2 + x_3^2 - 849\delta \geq 0$,

302 $x_1 + 5x_2^2 + x_3^2 - 402.30\delta \geq 0$,

303 $x_1^2 + x_2 + 4x_3^2 - 186.04\delta \geq 0$,

304 $x_1 + \delta \leq 14.03$,

305 $x_1 - \delta \geq 12.03$,

306 $x_2 + 2\delta \leq 10.970$,

307 $x_2 - 2\delta \geq 6.97$,

308 $\delta \in [0,1]$.

309 Whose compromise solution is $(x_1, x_2, x_3) = (13.02, 8.26, 3)$ and $\delta = 0.67$ (overall satisfaction for all
 310 decision maker's). and the objective function value are $F_1 = 927.829$, $F_2 = 90.247$, $F_3 = 213.78$.

311

6 Summary and Concluding Remarks

In this paper three-level quadratic programming problem with stochastic parameters in the objective function (TLQP) problem with stochastic parameters is presented by using fuzzy approach which is easy to apply and simple to be understood. After converting probabilistic nature of the objective functions to equivalence deterministic constraints the FLDM solved his/her problem and give the individual best solution and individual worst solution by applying the simplex method, the SLDM solved his/her problem and give the individual best solution and individual worst solution by applying the simplex method, TLDM solved his/her problem and give the individual best solution and individual worst solution by applying the simplex method, after that the membership functions were created using the tolerance concept to develop Tchebycheff problem for generating Pareto optimal solution for this problem. However, there are many other aspects, which should be explored and studied in the area of stochastic multi-level optimization such as:

- 1- A three-level integer programming problem with stochastic parameters in objective Function and constraints .
- 2- A three-level multi objective chance constraints quadratic mixed integer programming problem with stochastic parameters in objective function.

References

- [1] Biswas P, Pal BB. Fuzzy Goal Programming Approach to Solve Linear Multilevel Programming Problems using Genetic Algorithm. International Journal of Computer Applications. 2015;115(3): 0975 – 8887
- [2] Emam OE, Abdel-Fattah MA and Ahmed AS. A fuzzy approach for solving Three-level chance constrained quadratic programming problem. British journal of Mathematics & computer science. 2015;103:2231-0851.
- [3] Emam OE. Interactive approach to bi-level integer multi objective fractional programming problem. Applied Mathematics and Computation, 2013; 223: 17–24.
- [4] Osman MS, Abo-Sinna MA, Amer AH. , and Emam OE. A multi-level non-linear multi-objective decision making under fuzziness. Applied Mathematics and Computation. 2004;153: 239–252.
- [5] Pramanik S, Banerjee D, Giri B. Chance constrained linear plus linear fractional bi-level programming problem. International Journal of Computer Applications. 2012;56:34-39.
- [6] HSU SS. An Interactive Approach for Integrated Multilevel Systems in a Fuzzy Environment. Mathematical and Computer Modelling. 2002 ;36: 569-585.
- [7] Saraj M, Safaei N. Fuzzy linear fractional bi-level multi-objective programming problems.

- International Journal of Applied Mathematical Research. 2012;4:643-658.
- [8] Pal BB, Moitra BN. A Fuzzy goal programming procedure for solving quadratic bi-level programming problems. International Journal of Intelligent Systems. 2003;18:529-540.
- [9] Osman MS, Abo-Sinna MA, Amer AH, Emam OE. A multi- level non-linear multi objective decision-making under fuzziness. Applied Mathematics and Computation. 2004;153:239-252.
- [10] Sultan TI, Emam OE, Abohany AA. A Fuzzy approach for solving a three–level large scale linear programming problem. International Journal of Pure and Applied Sciences and Technology. 2013;19(2):22-34.
- [11] Sakawa M, Matsui T. Fuzzy random non-cooperative two-level linear programming through Fractile models with possibility and necessity. Engineering Optimization. 2012;44:1-23.
- [12] S. Rao, Engineering Optimization: Theory and Practice, Fourth Edition, 2009.
- [13] Emam OE, Kholeif SA and Azzam SM. A decomposition Algorithm for Solving Stochastic Multi-Level Large Scale Quadratic Programming Problem. Applied Mathematics & Information Sciences. 2015; 9(40):1817-1822.
- [14] Kumar M, Pal BB. Fuzzy Goal Programming Approach to Chance Constrained Multilevel Programming Problems. International Journal of Advanced Computer Research. 2013;13: 2249-2277.
- [15]M. Bravo, and I. Ganzalez, “Applying stochastic goal programming: A case study on water use planning”, European Journal of Operational Research, 196: 1123 – 1129. (2009).
- [16] Emam OE. A Parametric study on multi-objective integer quadratic programming problems under uncertainty. Gen Math Notes. 2011;6(1):49-60.
- [17] Ali I, Hasan SS. Bi-criteria optimization technique in stochastic system maintenance allocation problem. American Journal of Operation Research. 2013;3:17-29.