

# Linear metrics and effective separating sequences

## Abstract

On the real line equipped with the Euclidean metric every two effective separating sequences which have a common computable point are equivalent. We prove that the same result holds for every linear metric on the real line.

*Keywords:* Effective separating sequence; Computable metric space; Linear metric

2010 Mathematics Subject Classification: 03D78

## 1 Introduction

A dense sequence  $\alpha = (\alpha_i)$  in a metric space  $(X, d)$  is said to be effective separating if we can effectively compute the distance  $d(\alpha_i, \alpha_j)$  for all  $i, j \in \mathbb{N}$ . Two effective separating sequences  $\alpha$  and  $\beta$  in  $(X, d)$  are equivalent if  $\alpha$  can be computed from  $\beta$  in certain sense and conversely, if  $\beta$  can be computed from  $\alpha$ . Furthermore, we say that  $\alpha$  and  $\beta$  are equivalent up to isometry if there exists an isometry  $f : X \rightarrow X$  such that the  $\alpha$  and  $f \circ \beta$  are equivalent.

In general, effective separating sequences in a metric space  $(X, d)$  need not be equivalent, moreover they need not be equivalent up to isometry. However, it is known that in the metric space  $(\mathbb{R}, d)$ , where  $d$  is the Euclidean metric on  $\mathbb{R}$ , every two effective separating sequences are equivalent up to isometry (although not necessarily equivalent), see [Iljazović (2010); Melnikov (2013)]. It is easy to conclude from this result the following: if two effective separating sequences on  $(\mathbb{R}, d)$  have a common computable point, then they are equivalent.

In this paper we generalize the previous fact by taking  $d$  to be any linear metric on  $\mathbb{R}$ . By a linear metric  $d$  on  $\mathbb{R}$  we mean a metric such that for any real numbers  $x \leq y \leq z$  we have  $d(x, z) = d(x, y) + d(y, z)$ .

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## 2 Effective separating sequences

Let  $k \in \mathbb{N} \setminus \{0\}$ . A function  $F : \mathbb{N}^k \rightarrow \mathbb{Q}$  is called **computable** if there exist computable (recursive) functions  $a, b, c : \mathbb{N}^k \rightarrow \mathbb{N}$  such that

$$F(x) = (-1)^{c(x)} \frac{a(x)}{b(x) + 1}$$

for each  $x \in \mathbb{N}^k$ . A function  $f : \mathbb{N}^k \rightarrow \mathbb{R}$  is called **computable** if there exists a computable function  $F : \mathbb{N}^{k+1} \rightarrow \mathbb{Q}$  such that

$$|f(x) - F(x, i)| < 2^{-i}$$

for all  $x \in \mathbb{N}^k$  and  $i \in \mathbb{N}$  [Pour-El, Richards (1989); Weihrauch (2000)].

A number  $x \in \mathbb{R}$  is said to be **computable** if there exists a computable function  $g : \mathbb{N} \rightarrow \mathbb{Q}$  such that

$$|x - g(i)| < 2^{-i}$$

for each  $i \in \mathbb{N}$ .

In the following proposition we state some basic facts about computable functions  $\mathbb{N}^k \rightarrow \mathbb{R}$ .

**Proposition 2.1.** (i) If  $f, g : \mathbb{N}^k \rightarrow \mathbb{R}$  are computable, then  $f + g, f - g, |f| : \mathbb{N}^k \rightarrow \mathbb{R}$  are computable.

(ii) If  $F : \mathbb{N}^{k+1} \rightarrow \mathbb{R}$  is a computable function and  $f : \mathbb{N}^k \rightarrow \mathbb{R}$  such that  $|f(x) - F(x, i)| < 2^{-i}$  for all  $x \in \mathbb{N}^k$  and  $i \in \mathbb{N}$ , then  $f$  is computable.

(iii) If  $f, g : \mathbb{N}^k \rightarrow \mathbb{R}$  is a computable function, then the set  $\{x \in \mathbb{N}^k \mid f(x) > g(x)\}$  is c.e.

Let  $(X, d)$  be a metric space and let  $\alpha = (\alpha_i)$  be a sequence in  $X$ . We say that  $\alpha$  is an **effective separating sequence** in  $(X, d)$  if the set  $\{\alpha_i \mid i \in \mathbb{N}\}$  is dense in  $(X, d)$  and the function  $\mathbb{N}^2 \rightarrow \mathbb{R}$ ,

$$(i, j) \mapsto d(\alpha_i, \alpha_j)$$

is computable [Yasugi, Mori, Tsujii (1999)]. If  $\alpha$  is an effective separating sequence in  $(X, d)$ , then the triple  $(X, d, \alpha)$  is called a **computable metric space**.

If  $(X, d, \alpha)$  is a computable metric space and  $(x_i)$  a sequence in  $X$ , then  $(x_i)$  is said to be a **computable sequence** in  $(X, d, \alpha)$  if there exists a computable function  $F : \mathbb{N}^2 \rightarrow \mathbb{N}$  such that

$$d(x_i, \alpha_{F(i,k)}) < 2^{-k}$$

for all  $i, k \in \mathbb{N}$ . A point  $x \in X$  is said to be **computable** in  $(X, d, \alpha)$  if there exists a computable function  $f : \mathbb{N} \rightarrow \mathbb{N}$  such that  $d(x, \alpha_{f(k)}) < 2^{-k}$  for each  $k \in \mathbb{N}$ . Note that  $x$  is a computable point in  $(X, d, \alpha)$  if and only if the constant sequence  $(x, x, x, \dots)$  is computable in  $(X, d, \alpha)$ .

**Proposition 2.2.** Let  $(X, d, \alpha)$  be a computable metric space and let  $x_0$  be a computable point in this space. Then the function  $\mathbb{N} \rightarrow \mathbb{R}$ ,  $i \mapsto d(x_0, \alpha_i)$  is computable.

*Proof.* Let  $f : \mathbb{N} \rightarrow \mathbb{N}$  be a computable function such that  $d(x_0, \alpha_{f(k)}) < 2^{-k}$  for each  $k \in \mathbb{N}$ . In general, for all  $a, b, c \in X$  we have  $|d(a, c) - d(b, c)| \leq d(a, b)$ , so  $|d(x_0, \alpha_i) - d(\alpha_{f(k)}, \alpha_i)| \leq d(x_0, \alpha_{f(k)})$  and therefore

$$|d(x_0, \alpha_i) - d(\alpha_{f(k)}, \alpha_i)| < 2^{-k}$$

for all  $i, k \in \mathbb{N}$ . Now the claim follows from Proposition 2.1(ii).  $\square$

If  $(X, d, \alpha)$  is a computable metric space, then we denote by  $S_\alpha$  the set of all sequences which are computable in  $(X, d, \alpha)$ .

Let  $(X, d)$  be a metric space and let  $\alpha$  and  $\beta$  be effective separating sequences in this space. We say that  $\alpha$  and  $\beta$  are **equivalent** and write  $\alpha \sim \beta$  if  $\beta$  is a computable sequence in  $(X, d, \alpha)$ . It is not hard to check that effective separating sequences  $\alpha$  and  $\beta$  in  $(X, d)$  are equivalent if and only if  $S_\alpha = S_\beta$  [Iljazović (2010)]. It follows that in this case the computable metric spaces  $(X, d, \alpha)$  and  $(X, d, \beta)$  have the same computable points.

**Example 2.1.** Let  $d$  be the Euclidean metric on  $\mathbf{R}$ . Let  $\alpha : \mathbf{N} \rightarrow \mathbf{R}$  be a computable function whose range is dense in  $(\mathbf{R}, d)$ . Then  $\alpha$  is an effective separating sequence in  $(\mathbf{R}, d)$ , which follows easily from Proposition 2.1(i).

Let  $(x_i)$  be a sequence in  $\mathbf{R}$  and  $a \in \mathbf{R}$ . Then it is easy to prove that  $(x_i)$  is a computable sequence in  $(\mathbf{R}, d, \alpha)$  if and only if  $(x_i)$  is a computable sequence in  $\mathbf{R}$  (i.e. computable as a function  $\mathbf{N} \rightarrow \mathbf{R}$ ) and  $a$  is a computable point in  $(\mathbf{R}, d, \alpha)$  if and only if  $a$  is a computable number [Iljazović (2010)]. Hence  $S_\alpha$  is the set of all sequences in  $\mathbf{R}$  which are computable (as functions  $\mathbf{N} \rightarrow \mathbf{R}$ ).

Suppose  $d$  is the Euclidean metric on  $\mathbf{R}$  and  $\alpha$  and  $\beta$  are effective separating sequences in  $(\mathbf{R}, d)$ . Suppose  $\alpha$  is a computable sequence in  $\mathbf{R}$ .

If  $\alpha \sim \beta$ , then by Example 2.1  $\beta$  is a computable sequence in  $\mathbf{R}$ . Conversely, if  $\beta$  is a computable sequence in  $\mathbf{R}$ , then by Example 2.1 we have  $S_\alpha = S_\beta$  and consequently  $\alpha \sim \beta$ . Hence  $\alpha \sim \beta$  if and only if  $\beta$  is a computable sequence in  $\mathbf{R}$ .

Let  $(X, d)$  be a metric space,  $\alpha$  an effective separating sequence in  $(X, d)$  and  $f : X \rightarrow X$  an isometry (i.e. a surjective function such that  $d(f(x), f(y)) = d(x, y)$  for all  $x, y \in X$ ). Then  $f \circ \alpha$ , i.e.  $(f(\alpha_i))_{i \in \mathbf{N}}$  is clearly an effective separating sequence in  $(X, d)$ . Note the following: if  $x_0$  is computable in  $(X, d, \alpha)$ , then  $f(x_0)$  is computable in  $(X, d, f \circ \alpha)$ .

**Example 2.2.** Let  $d$  be the Euclidean metric on  $\mathbf{R}$ , let  $\alpha : \mathbf{N} \rightarrow \mathbf{R}$  be a computable function whose range is dense in  $(\mathbf{R}, d)$  and let  $c \in \mathbf{R}$ . We define  $\beta : \mathbf{N} \rightarrow \mathbf{R}$  by  $\beta_i = \alpha_i + c$ ,  $i \in \mathbf{N}$ . Then  $\beta = f \circ \alpha$ , where  $f : \mathbf{R} \rightarrow \mathbf{R}$  is the isometry given by  $f(x) = x + c$ .

Suppose  $c$  is not a computable number. Then  $\beta$  is not a computable sequence in  $\mathbf{R}$ . Namely, if  $(x_i)$  is a computable sequence in  $\mathbf{R}$ , then  $x_i$  is clearly a computable number for each  $i \in \mathbf{N}$ . Therefore, if  $\beta$  were computable,  $\beta_0$  would be a computable number, i.e.  $\alpha_0 + c$  would be a computable number and this would imply that  $c$  is computable (in general, the difference of two computable numbers is a computable number).

Since  $\beta$  is not a computable sequence in  $\mathbf{R}$ , we have that  $\alpha$  and  $\beta$  are not equivalent.

Let  $\alpha$  and  $\beta$  be effective separating sequences in a metric space  $(X, d)$ . We say that  $\alpha$  and  $\beta$  are **equivalent up to isometry** if there exists an isometry  $f : X \rightarrow X$  such that  $\alpha \sim f \circ \beta$ .

Example 2.2 shows that two effective separating sequences in  $(\mathbf{R}, d)$ , where  $d$  is the Euclidean metric on  $\mathbf{R}$ , need not be equivalent. However, every two effective separating sequences in  $(\mathbf{R}, d)$  are equivalent up to isometry, see [Iljazović (2010); Melnikov (2013)].

### 3 Effective separating sequences with a common computable point

There are metric spaces in which every two effective separating sequences are equivalent, see [Iljazović (2010)]. As we saw,  $\mathbf{R}$  with the Euclidean metric is not such a space, but the fact that every two effective separating sequences in this metric space are equivalent up to isometry has the following consequence.

**Proposition 3.1.** Let  $d$  be the Euclidean metric on  $\mathbf{R}$ . Suppose  $\alpha$  and  $\beta$  are effective separating sequences in  $(\mathbf{R}, d)$  such that the computable metric spaces  $(\mathbf{R}, d, \alpha)$  and  $(\mathbf{R}, d, \beta)$  have a common computable point. Then  $\alpha \sim \beta$ .

*Proof.* Let  $x_0$  be a common computable point of  $(\mathbf{R}, d, \alpha)$  and  $(\mathbf{R}, d, \beta)$ .

Let us first assume that  $x_0 = 0$  and that  $\beta$  is a computable sequence in  $\mathbf{R}$ .

There exists an isometry  $f : \mathbf{R} \rightarrow \mathbf{R}$  such that  $\alpha \sim f \circ \beta$ . Since 0 is a computable point in  $(X, d, \beta)$ ,  $f(0)$  is a computable point in  $(X, d, f \circ \beta)$ . Therefore  $f(0)$  is a computable point in  $(X, d, \alpha)$ .

In general, if  $y$  and  $z$  are computable points in  $(X, d, \alpha)$ , then  $d(y, z)$  is a computable number. Therefore,  $d(f(0), 0)$  is a computable number.

Since  $f$  is an isometry, there exists  $l \in \mathbf{R}$  such that  $f(x) = x + l$  for each  $x \in \mathbf{R}$  or  $f(x) = -x + l$  for each  $x \in \mathbf{R}$ . In either case  $f(0) = l$  and therefore  $d(f(0), 0) = |l|$ . Hence  $l$  is a computable number. It follows from Proposition 2.1(i) that the sequences  $(\beta_i + l)$  and  $(-\beta_i + l)$  are computable in  $\mathbf{R}$ .

We have  $(\alpha_i) \sim (\beta_i + l)$  or  $(\alpha_i) \sim (-\beta_i + l)$ . Therefore  $\alpha$  is computable sequence in  $\mathbf{R}$  and it follows  $\alpha \sim \beta$ .

In general, let  $g : \mathbf{R} \rightarrow \mathbf{R}$  be the function defined by  $g(x) = x - x_0$ . Then  $g$  is an isometry and we have that  $g(x_0)$  is a computable point in  $(X, d, g \circ \alpha)$  and in  $(X, d, g \circ \beta)$ . Let  $\gamma : \mathbf{N} \rightarrow \mathbf{R}$  be some computable function whose range is dense in  $(\mathbf{R}, d)$  (for example, we can take any computable surjection  $\mathbf{N} \rightarrow \mathbf{Q}$ ). Then 0 is a computable point in  $(X, d, \gamma)$  and by the first case we have  $g \circ \alpha \sim \gamma$  and  $g \circ \beta \sim \gamma$ . It follows  $g \circ \alpha \sim g \circ \beta$ . Therefore  $g^{-1} \circ (g \circ \alpha) \sim g^{-1} \circ (g \circ \beta)$ , hence  $\alpha \sim \beta$ .  $\square$

Let  $d$  be a metric on  $\mathbf{R}$ . We say that  $d$  is a **linear metric** if for all  $x, y, z \in \mathbf{R}$  such that  $x \leq y \leq z$  we have  $d(x, z) = d(x, y) + d(y, z)$ .

Clearly, the Euclidean metric on  $\mathbf{R}$  is a linear metric. However, it is not the only linear metric as the following proposition implies.

**Proposition 3.2.** *Let  $d : \mathbf{R} \times \mathbf{R} \rightarrow \mathbf{R}$  be a function. Then  $d$  is a linear metric if and only if there exists a strictly increasing function  $f : \mathbf{R} \rightarrow \mathbf{R}$  such that*

$$d(x, y) = |f(x) - f(y)| \quad (3.1)$$

for all  $x, y \in \mathbf{R}$ .

*Proof.* Suppose  $d$  is a linear metric. Let  $f : \mathbf{R} \rightarrow \mathbf{R}$  be the function defined by

$$f(x) = \begin{cases} d(0, x), & x \geq 0 \\ -d(0, x), & x \leq 0. \end{cases}$$

Let  $x, y \in \mathbf{R}$ ,  $x < y$ . We claim that  $f(x) < f(y)$ .

If  $0 \leq x < y$ , then  $d(0, y) = d(0, x) + d(x, y)$  and it follows  $f(x) < f(y)$ . In a similar way we conclude  $f(x) < f(y)$  if  $x < y \leq 0$ . If  $x < 0 < y$ , then  $f(x) < 0 < f(y)$ .

Hence the function  $f$  is strictly increasing. We claim that (3.1) holds. It is enough to prove  $d(x, y) = f(y) - f(x)$  for all  $x, y \in \mathbf{R}$  such that  $x < y$ . However this follows easily from definition of  $f$  and the fact that  $d$  is a linear metric.

Conversely, if  $f : \mathbf{R} \rightarrow \mathbf{R}$  is a strictly increasing function such that (3.1) holds, then it is easy to see that  $d$  is a linear metric.  $\square$

For example, the function  $d : \mathbf{R} \times \mathbf{R} \rightarrow \mathbf{R}$  defined by  $d(x, y) = |\arctan x - \arctan y|$  is a linear metric. Note that  $d$  is a bounded metric.

In general, a linear metric need not be even topologically equivalent to the Euclidean metric as the following example shows.

**Example 3.1.** *Let  $f : \mathbf{R} \rightarrow \mathbf{R}$  be the function defined by*

$$f(x) = \begin{cases} x, & x \leq 0 \\ x + 1, & x > 0. \end{cases}$$

*Clearly,  $f$  is strictly increasing. Let  $d : \mathbf{R} \times \mathbf{R} \rightarrow \mathbf{R}$  be defined by  $d(x, y) = |f(x) - f(y)|$ . Then  $d$  is a linear metric. We have*

$$\{x \in \mathbf{R} \mid d(x, 0) < 1\} = \langle -1, 0],$$

*hence  $\langle -1, 0]$  is an open ball in the metric space  $(\mathbf{R}, d)$  and therefore it is an open set in  $(\mathbf{R}, d)$ . On the other hand, the set  $\langle -1, 0]$  is not open in  $\mathbf{R}$  with respect to the Euclidean metric, hence  $d$  and the Euclidean metric on  $\mathbf{R}$  are not topologically equivalent.*

**Lemma 3.2.** *Let  $d$  be a linear metric on  $\mathbf{R}$  and  $a, b \in \mathbf{R}$ ,  $a < b$ . Let  $x \in \mathbf{R}$ . Then*

$$a < x \iff d(x, b) < d(a, b) \text{ or } d(x, b) < d(x, a) \quad (3.2)$$

$$x < b \iff d(x, a) < d(a, b) \text{ or } d(x, a) < d(x, b) \quad (3.3)$$

*Proof.* Suppose  $a < x$ . If  $x \leq b$ , then  $d(a, b) = d(a, x) + d(x, b)$  and it follows  $d(x, b) < d(a, b)$ . If  $b < x$ , then  $d(a, x) = d(a, b) + d(b, x)$  and  $d(x, b) < d(x, a)$ . Hence the implication  $\Rightarrow$  in (3.2) holds. Conversely, suppose  $d(x, b) < d(a, b)$  or  $d(x, b) < d(x, a)$ . If  $x \leq a$ , then

$$d(x, b) = d(x, a) + d(a, b)$$

implying  $d(x, a) < d(x, b)$  and  $d(a, b) \leq d(x, b)$  which is impossible. Hence  $a < x$  and we conclude that (3.2) holds. In a similar way we get that (3.3) holds.  $\square$

**Proposition 3.3.** *Let  $d$  be a linear metric and let  $\alpha$  be an effective separating sequence in  $(\mathbf{R}, d)$ . Suppose  $x_0$  is a computable point in  $(\mathbf{R}, d, \alpha)$ . Then the sets*

$$S = \{i \in \mathbf{N} \mid x_0 < \alpha_i\} \text{ and } T = \{i \in \mathbf{N} \mid \alpha_i < x_0\}$$

*are computably enumerable.*

*Proof.* Let us choose  $y_0 \in \mathbf{R}$  such that  $x_0 < y_0$ . Let  $r = d(x_0, y_0)$ . Clearly  $r > 0$ . Since the set  $\{\alpha_i \mid i \in \mathbf{N}\}$  is dense in  $(\mathbf{R}, d)$ , there exists  $n \in \mathbf{N}$  such that  $d(\alpha_n, y_0) < r$ . We have  $d(\alpha_n, y_0) < d(x_0, y_0)$  and Lemma 3.2 implies that

$$x_0 < \alpha_n.$$

Let  $i \in \mathbf{N}$ . By Lemma 3.2 we have

$$x_0 < \alpha_i \iff d(\alpha_i, \alpha_n) < d(x_0, \alpha_n) \text{ or } d(\alpha_i, \alpha_n) < d(\alpha_i, x_0). \quad (3.4)$$

Let  $f, g : \mathbf{N}^2 \rightarrow \mathbf{R}$  be the functions defined by  $f(i) = d(\alpha_i, \alpha_n)$ ,  $g(i) = d(\alpha_i, x_0)$ . By Proposition 2.2  $f$  and  $g$  are computable functions. By (3.4) we have

$$i \in S \iff f(i) < g(n) \text{ or } f(i) < g(i).$$

It follows from this and Proposition 2.1(iii) that  $S$  is the union of two c.e. sets. Hence  $S$  is c.e. In the same way we get that  $T$  is c.e.  $\square$

**Lemma 3.3.** *Let  $d$  be a linear metric. Let  $x, a, b \in \mathbf{R}$ . If  $x \leq a$  and  $x \leq b$  or  $a \leq x$  and  $b \leq x$ , then*

$$|d(x, a) - d(x, b)| = d(a, b). \quad (3.5)$$

*Proof.* Suppose  $x \leq a$  and  $x \leq b$ . If  $a \leq b$ , then  $d(x, b) = d(x, a) + d(a, b)$  and (3.5) holds. The same conclusion we get if  $b \leq a$ . The inequalities  $a \leq x$  and  $b \leq x$  imply (3.5) in the same way.  $\square$

In the next theorem we show that the claim of Proposition 3.1 holds for every linear metric  $d$ .

**Theorem 3.4.** *Let  $d$  be a linear metric. Suppose  $\alpha$  and  $\beta$  are effective separating sequences in  $(\mathbf{R}, d)$  such that the computable metric spaces  $(\mathbf{R}, d, \alpha)$  and  $(\mathbf{R}, d, \beta)$  have a common computable point. Then  $\alpha \sim \beta$ .*

*Proof.* Let  $x_0$  be a common computable point of  $(\mathbf{R}, d, \alpha)$  and  $(\mathbf{R}, d, \beta)$ . Let

$$S^+ = \{i \in \mathbf{N} \mid x_0 < \alpha_i\}, \quad S^- = \{i \in \mathbf{N} \mid \alpha_i < x_0\}$$

and

$$T^+ = \{i \in \mathbf{N} \mid x_0 < \beta_i\}, \quad T^- = \{i \in \mathbf{N} \mid \beta_i < x_0\}.$$

Let  $i, k \in \mathbb{N}$ . We claim that there exists  $j \in \mathbb{N}$  such that

$$|d(x_0, \beta_i) - d(x_0, \alpha_j)| < 2^{-k} \quad (3.6)$$

and

$$(j \in S^+ \text{ and } i \in T^+) \text{ or } (j \in S^- \text{ and } i \in T^-) \text{ or } \left( d(x_0, \beta_i) < 2^{-(k+1)} \text{ and } d(x_0, \alpha_j) < 2^{-(k+1)} \right). \quad (3.7)$$

In order to prove this, let us first assume that  $x_0 < \beta_i$ . Let  $r = d(x_0, \beta_i)$ . Since  $\alpha$  is a dense sequence, there exists  $j \in \mathbb{N}$  such that

$$d(\alpha_j, \beta_i) < \min\{2^{-k}, r\}. \quad (3.8)$$

We have  $d(\alpha_j, \beta_i) < r = d(x_0, \beta_i)$  and Lemma 3.2 implies that  $x_0 < \alpha_j$ . So  $i \in T^+$  and  $j \in S^+$ . On the other hand,  $|d(x_0, \beta_i) - d(x_0, \alpha_j)| \leq d(\beta_i, \alpha_j)$  and  $d(\beta_i, \alpha_j) < 2^{-k}$  by (3.8), so  $|d(x_0, \beta_i) - d(x_0, \alpha_j)| < 2^{-k}$ . Hence (3.6) and (3.7) hold.

In the same way we conclude that (3.6) and (3.7) hold if  $\beta_i < x_0$ .

If  $x_0 = \beta_i$ , then we choose  $j \in \mathbb{N}$  such that  $d(x_0, \alpha_j) < 2^{-(k+1)}$ . Then (3.6) and (3.7) clearly hold.

Let  $\Omega$  be the set of all  $(i, k, j) \in \mathbb{N}^3$  such that (3.6) and (3.7) hold. Hence for all  $i, k \in \mathbb{N}$  there exists  $j \in \mathbb{N}$  such that  $(i, k, j) \in \Omega$ . By Proposition 2.2 and Proposition 2.1 the set of all  $(i, k, j) \in \mathbb{N}^3$  such that (3.6) holds is c.e. Similarly, using Proposition 3.3 we conclude that the set of all  $(i, k, j) \in \mathbb{N}^3$  such that (3.7) holds is c.e. Therefore  $\Omega$  is c.e. as the intersection of two c.e. sets. Since for all  $i, k \in \mathbb{N}$  there exists  $j \in \mathbb{N}$  such that  $(i, k, j) \in \Omega$ , there exists a computable function  $\varphi : \mathbb{N}^2 \rightarrow \mathbb{N}$  such that

$$(i, k, \varphi(i, k)) \in \Omega \quad (3.9)$$

for all  $i, k \in \mathbb{N}$  (Single Valuedness Theorem).

Suppose that  $(i, k, j) \in \Omega$ . We claim that  $d(\beta_i, \alpha_j) < 2^{-k}$ . Since (3.7) holds, we have three cases.

If  $j \in S^+$  and  $i \in T^+$ , then  $x_0 < \alpha_j$  and  $x_0 < \beta_i$  and it follows from Lemma 3.3 that

$$|d(x_0, \beta_i) - d(x_0, \alpha_j)| = d(\beta_i, \alpha_j).$$

Now (3.6) implies  $d(\beta_i, \alpha_j) < 2^{-k}$ .

The same conclusion we get if  $j \in S^-$  and  $i \in T^-$ . If  $d(x_0, \beta_i) < 2^{-(k+1)}$  and  $d(x_0, \alpha_j) < 2^{-(k+1)}$ , then clearly  $d(\beta_i, \alpha_j) < 2^{-k}$ .

Hence  $(i, k, j) \in \Omega$  implies  $d(\beta_i, \alpha_j) < 2^{-k}$ . We conclude from (3.9) that

$$d(\beta_i, \alpha_{\varphi(i, k)}) < 2^{-k}$$

for all  $i, k \in \mathbb{N}$ . Hence  $\beta$  is a computable sequence in  $(\mathbb{R}, d, \alpha)$ , i.e.  $\alpha \sim \beta$ .  $\square$

## References

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