

A Mathematical Model and Simulation of Chlamydia Trachomatis in a Human carrier

Abstract

A mathematical model describing the dynamics of the Chlamydia Trachomatis disease was presented. This is an extension of Wilson's model. The prove of the existence and uniqueness of solution of the model was done and a linear stability analysis was conducted. Also, the Analytical solution and the results are presented graphically. The results indicated which is depending on initial data shows that the parameters involved played a crucial role in the increase and decrease of number of infected, uninfected and recovered epithelial cells.

Keywords: Chlamydia Trachomatis, Mathematical Model, Dynamics, Stability Analysis, epithelial cells and Analytical Solution

1. Introduction

Chlamydia is an obligate intracellular bacterial pathogen that infects the genital and ocular mucosa of humans causing sexually transmitted disease (STI) and Trachoma. It is estimated that 70-75% of endocervical infections in women caused by bacterium "Chlamydia trachomatis" are asymptomatic and may persist for months to years (Dixon, 2009).

According to the centers for Disease control and preventions (CDC), Chlamydia is a group of bacteria that causes various infectious diseases in humans and animals. They consist of three species which are Chlamydia Trachomatis, Chlamydia Pneumoniae, and Chlamydia Psittaci.

But the most common Chlamydia illnesses are a disease that is caused by many strains of the Trachomatis species. One of the strains of Chlamydia Trachomatis is conjunctivitis, an eye

infection marked by redness and swelling, sensitivity to light, and pus discharge. A strain of fly-borne Chlamydia Trachomatis causes a severe form of conjunctivitis called trachoma.

This is more common in developing Countries, which when not treated may result in blindness.

In the United States each year, 3 million cases of sexually transmitted infections (STIs) in both men and women is found, including Nongonococcal Urethritis (NGU), an infection of the urinary tract. Lymphogranuloma Venereum, a lymphatic tissue disease that is rare in the United States, occurs more commonly in tropical regions.

In both men and women symptoms of sexually transmitted, Chlamydia Trachomatis may include a watery discharge and pain when urinating; in women, Chlamydia Trachomatis also causes inflammation of the vagina, cervix, uterus, fallopian tubes, and ovaries. Women may also experience vaginal discharge, fever, abdominal pain, and pain in the genital area. In as many as 50 to 70 percent of Chlamydia Trachomatis infections, however, women experience no symptoms. As a result, the infection remains untreated and may develop into more serious conditions, including pelvic inflammatory disease, ectopic pregnancy, and infertility. Pregnant women with Chlamydia infections can also pass it to their babies during birth.

Also, it is also known as a “silent” disease because about three quarters of infected women and about half of infected men have no noticeable symptoms. If symptoms do occur, they appear usually within 1 to 3 weeks after exposure (Fitch and Stine, 2002). The sequelae of Chlamydia Trachomatis genital tract infections in women, namely chronic pain, pelvic inflammatory disease (PID), infertility and ectopic pregnancy are the most costly outcomes of any sexually transmitted infection except human immunodeficiency virus (HIV/AIDS), (Hickey, 2009).

Turner (2004), some new data were introduced and existing data were reanalysed in novel ways in order to investigate the epidemiology of a Chlamydia Trachomatis disease. The results show that realistic outputs can be achieved when empirical data are used to parameterise mathematical models. Also, Wilson (2004) extended the current description of the Chlamydial developmental cycle and reviewed the models developed and their predictions about the Chlamydial pathogenesis. A simple mathematical description of the change in the interacting ‘species’ was given by the following equations:

$$\dot{C} = P\beta I(t) - \mu C(t) - \kappa C(t)E(t)$$

$$\dot{E} = P_E - \delta_E E(t) - \kappa C(t)E(t)$$

$$\dot{I} = \kappa C(t)E(t) - \gamma I(t) - \beta I(t)$$

Hence, the three equations shows the rate of Chlamydia particles, the rate of change of the free epithelial cells, and the rate of change of the Chlamydia infections respectively in the Mucosal lining of the infected uterus with the following parameters

$$p, \kappa, \beta, \gamma, \mu, p_E \text{ and } \delta_E .$$

In this work, we extend the model of Wilson (2004) by incorporating relevant features such as the recovery class and drug administration with the following assumption to be taken into consideration which are:

There are some free Chlamydial particles at the lower region of the genital tract.

There are no infected cells (all Chlamydial Particles are initially outside of the host cells).

Epithelial cells are at their normal concentration levels throughout the domain.

2. Model Formulation

With the newly incorporated parameters, our modified model is:

where

$$\left. \begin{aligned} \frac{dC}{dt} &= P_C + K_2 I - \mu_C C, \\ \frac{dE}{dt} &= P_E E - \mu_E E - K_1 CE + \omega R, \\ \frac{dI}{dt} &= -K_2 I - \gamma I + K_1 CE - \mu_E I - \delta I, \\ \frac{dR}{dt} &= (\delta + \gamma) I - \mu_E R - \omega R \end{aligned} \right\} \begin{aligned} C(0) &= C_0 \\ E(0) &= E_0 \\ I(0) &= I_0 \\ R(0) &= R_0 \end{aligned} \quad (2.1)$$

Here, as in Wilson [9]

$C(t)$ be the concentration of free extracellular Chlamydia particles,

$E(t)$ be the number of uninfected mucosal epithelial cells (main host cell for Chlamydia),

$I(t)$ be the number of Chlamydia-infected epithelial cell at time t,

$R(t)$ be the number of epithelial cells which recovered from Chlamydia-infection.

K_2 is the rate at which Chlamydia particles are released from infected cells.

K_1 is the rate of Epithelial cell infection which may be influenced by antibodies.

P_E is the Reproduction rate of mucosal epithelial cells.

μ_E is the natural death rate of epithelial cells.

μ_c is the natural death rate of Concentrated Chlamydia particles

γ is the rate of clearance (recovery) of infected cells due to cell-mediated immunity.

δ is the recovery rate due to drug administration.

P_c is the recruitment rate of the Chlamydia particles due to external interaction.

ω is the waning off Immunity.

3. Method of Solution

3.1 Stability Analysis

We have to distinguish between two types of equilibrium of (2.1).

Here, we shall let $C=x, E=y, I=z, R=v$

Then equations (2.1) becomes

$$\left. \begin{aligned} \frac{dx}{dt} &= P_c + K_2 z - \mu_c x, \\ \frac{dy}{dt} &= P_e y - \mu_e y - K_1 xy + \omega v, \\ \frac{dz}{dt} &= -K_2 z - \gamma z + K_1 xy - \mu_e z - \delta z, \\ \frac{dv}{dt} &= (\delta + \gamma)z - \mu_e v - \omega v \end{aligned} \right\} \begin{aligned} x(0) &= x_0 \\ y(0) &= y_0 \\ z(0) &= z_0 \\ v(0) &= v_0 \end{aligned} \quad (3.1)$$

and

At equilibrium state

$$\frac{d\phi(t)}{dt} = 0, \quad (3.2)$$

where $\phi(t) = \{x(t), y(t), z(t), v(t)\}$

Then, the steady states of (3.1) satisfy the following algebraic system:

$$P_c + K_2 z - \mu_c x = 0 \quad (3.3)$$

$$(P_e - \mu_e)y - K_1 xy + \omega v = 0 \quad (3.4)$$

$$-\alpha z + K_1 x y = 0 \quad (3.5)$$

$$(\delta + \gamma)z - (\mu_e + w)v = 0 \quad (3.6)$$

where

$$\alpha = (K_2 + \gamma + \mu_E + \delta)$$

$$\beta = \delta + \gamma$$

$$\sigma = \mu_E + \omega$$

$$\tau = (P_E - \mu_E)$$

when $x = \frac{P_C}{\mu_C}, y=0, z=0, v=0$, we have the disease free equilibrium

$$P_1 = (x, y, z, v) = \left(\frac{P_C}{\mu_C}, 0, 0, 0 \right)$$

and the endemic equilibrium i.e the infected equilibrium

$$P_2 = (x, y, z, v) = (\phi, \phi_1, \phi_2, \phi_3)$$

where

$$\phi = \frac{\alpha \sigma \tau}{\alpha \sigma K_1 - \beta K_1 \omega}, \quad \phi_1 = \frac{(\alpha \sigma \tau \mu_C - P_C \alpha \sigma K_1 + \beta P_C K_1 \omega)}{K_1 K_2 \sigma \tau}$$

$$\phi_2 = \frac{\alpha \sigma \tau \mu_C - P_C \alpha \sigma K_1 + \beta P_C K_1 \omega}{K_2 (\alpha \sigma K_1 - \beta K_1 \omega)}, \quad \phi_3 = \frac{\beta (\alpha \sigma \tau \mu_C - P_C \alpha \sigma K_1 + \beta P_C K_1 \omega)}{\sigma K_2 (\alpha \sigma K_1 - \beta K_1 \omega)}$$

The Jacobian of (2.1) is

$$Df(x, y, z, v) = \begin{pmatrix} -\mu_C & 0 & K_2 & 0 \\ -K_1 y & \tau - K_1 x & 0 & \omega \\ K_1 y & K_1 x & -\alpha & 0 \\ 0 & 0 & \beta & -\sigma \end{pmatrix} \quad (3.7)$$

The Linearization of (3.7) at P_1 is

$$Df\left(\frac{P_C}{\mu_C}, 0, 0, 0\right) = \begin{pmatrix} -\mu_C & 0 & K_2 & 0 \\ 0 & -m & 0 & \omega \\ 0 & n & -\alpha & 0 \\ 0 & 0 & \beta & -\sigma \end{pmatrix}, \quad (3.8)$$

where

$$m = \frac{K_1 P_C}{\mu_C} - \tau$$

$$n = \frac{K_1 P_C}{\mu_C}$$

$$\sigma = K_1 \frac{P_C}{\mu_C} + \mu_E - P_E$$

$$\tau = K_1 \frac{P_C}{\mu_C}$$

With eigenvalues,

$$\lambda_1 = -\mu_C, \lambda_2 = -\sigma, \lambda_3 = -m, \lambda_4 = -\alpha \quad (3.9)$$

By definition, all the parameters are non-negative.

1. If $\lambda_1 > \lambda_2$ distinct, non – zero and positive, (x_0, y_0) is an unstable node;
2. If $\lambda_1 > \lambda_2$ distinct, non – zero and negative, (x_0, y_0) is a stable node; and if
3. $\lambda_1 > \lambda_2$ distinct, non – zero and have opposite signs, (x_0, y_0) is a saddle;
4. If $\lambda_1 = \lambda_2 = \alpha + \beta i$, $\beta \neq 0$, are complex, then (x_0, y_0) is a stable spiral if $\alpha < 0$, an unstable spiral if $\alpha > 0$, and a centre if $\alpha = 0$.

Here, with four eigenvalues λ_i , $i = 1, \dots, 4$, with distinct, non – zero and negative roots. Then,

the equilibrium point $\left(\frac{P_C}{\mu_C}, 0, 0, 0 \right)$ is said to asymptotically stable.

The Linearization of (3.7) at P_2 is

$$Df(\phi_1, \phi, \phi_2, \phi_3) = \begin{pmatrix} -\mu_C & 0 & K_2 & 0 \\ -q & -r & 0 & 0 \\ q & s & -\alpha & 0 \\ 0 & 0 & \beta & -\sigma \end{pmatrix}, \quad (3.10)$$

where

$$q = K_1 \phi_1$$

$$r = K_1 \phi - \tau$$

$$s = K_1 \phi$$

Thus,

$$|A - \lambda I| = 0$$

Implies

$$\lambda_1 = -\sigma$$

$$\begin{aligned} P(\lambda) &= (\mu_C + \lambda)(r + \lambda)(\alpha + \lambda) + K_2 q(r + \lambda) + K_2 q s \\ &= \lambda^3 + (\mu_C + \sigma + \alpha)\lambda^2 + (\mu_C r + \mu_C \alpha + r\alpha + K_2 q)\lambda + (\mu_C r\alpha + K_2 q(s + r)) = 0 \end{aligned} \quad (3.11)$$

Hence

Using the Descartes rule of signs:

The number of positive zeros of a polynomial with real coefficients is either equal to the number of variations in sign of the polynomial or less than this by an even number and Routh-Hurwitz criteria (Wang and Li, 2006) shows that :

All zeros of $\lambda^3 + \alpha\lambda^2 + \beta\lambda + \gamma = 0$ have negative real parts if and only if $\alpha\beta - \gamma > 0$.

Therefore, all zeros of (3.11) have negative real parts if and only if

$$(\mu_C + r + \alpha)(\mu_C r + \mu_C \alpha + r\alpha + K_2 q) - (\mu_C r\alpha + K_2 q(s + r)) > 0$$

That is

$$\mu_C r(\mu_C + r) + \mu_C \alpha(\mu_C + \alpha) + r\alpha(r + \alpha) + 2\mu_C r\alpha - K_2 q(\mu_C + \alpha + s) > 0$$

It has been shown by Descartes rule of signs. [] that (3.11) has three negative roots or one negative root and two complex roots. Hence, the critical point $(\phi, \phi_1, \phi_2, \phi_3)$ is globally asymptotically stable.

3.5 Analytical Solution

We first reduce the system (2.1) to origin by let

$$\alpha = K_2 + \gamma + \mu_E + \delta,$$

$$\beta = \delta + \gamma,$$

$$\sigma = \mu_E + \omega,$$

$$\tau = P_E - \mu_E$$

in equation (3.3) – (3.6)

Then equation (3.3) – (3.6) becomes

$$\left. \begin{aligned} \frac{dx}{dt} &= P_C + K_2 z - \mu_C x \\ \frac{dy}{dt} &= \tau y - K_1 xy + \omega v \\ \frac{dz}{dt} &= -\alpha z + K_1 xy \\ \frac{dv}{dt} &= \beta z - \sigma v \end{aligned} \right\} \quad (3.12)$$

Suppose the solution $x(t), y(t), z(t)$ and $v(t)$ in (3.12) can be expressed as:

$$\left. \begin{aligned} x &= x_0 + K_1 x_1 + K_1^2 x_2 + h.o.t \\ y &= y_0 + K_1 y_1 + K_1^2 y_2 + h.o.t \\ z &= z_0 + K_1 z_1 + K_1^2 z_2 + h.o.t \\ v &= v_0 + K_1 v_1 + K_1^2 v_2 + h.o.t \end{aligned} \right\}, \quad (3.13)$$

Where h.o.t reads “higher order terms in K_1 ”. In our analysis, we are interested only in the first two terms.

Then, Substitute (3.12) into (3.13),

we obtain

$$\begin{aligned} \frac{d}{dt}(x_0 + K_1 x_1) &= P_C + K_2(z_0 + K_1 z_1) - \mu_C(x_0 + K_1 x_1) \\ \frac{d}{dt}(y_0 + K_1 y_1) &= \tau(y_0 + K_1 y_1) - K_1(x_0 + K_1 x_1)(y_0 + K_1 y_1) + \omega(v_0 + K_1 v_1) \\ \frac{d}{dt}(z_0 + K_1 z_1) &= -\alpha(z_0 + K_1 z_1) + K_1(x_0 + K_1 x_1)(y_0 + K_1 y_1) \\ \frac{d}{dt}(v_0 + K_1 v_1) &= \beta(z_0 + K_1 z_1) - \sigma(v_0 + K_1 v_1) \end{aligned}$$

Collecting like power of order of K_1 ,

we have

K_1^0 :

$$\frac{dx_0}{dt} = P_C + K_2 z_0 - \mu_C x_0, \quad x_0(0) = x_0 \quad (3.14)$$

$$\frac{dy_0}{dt} = \tau y_0 + \omega v_0, \quad y_0(0) = y_0 \quad (3.15)$$

$$\frac{dz_0}{dt} = -\alpha z_0, \quad z_0(0) = z_0 \quad (3.16)$$

$$\frac{dv_0}{dt} = \beta z_0 - \sigma v_0, \quad v_0(0) = v_0 \quad (3.17)$$

K_1^1 :

$$\frac{dx_1}{dt} = K_2 z_1 - \mu_C x_1, \quad x_1(0) = 0 \quad (3.18)$$

$$\frac{dy_1}{dt} = \tau y_1 - x_0 y_0 - \omega v_1, \quad y_1(0) = 0 \quad (3.19)$$

$$\frac{dz_1}{dt} = -\alpha z_1 + x_0 y_0, \quad z_1(0) = 0 \quad (3.20)$$

$$\frac{dv_1}{dt} = \beta z_1 - \sigma v_1, \quad v_1(0) = 0 \quad (3.21)$$

Hence,

We solve equations (3.14) – (3.21) as follows:

Consider (3.17), that is

$$\frac{dz_0}{dt} = -\alpha z_0$$

$$\frac{dz_0}{z_0} = -\alpha dt$$

Integrating both side,

We obtain

$$\ln z_0 = -\alpha t + \text{Constant}$$

i.e.

$$z_0(t) = e^c \cdot e^{-\alpha t}$$

i.e.

$$z_0(t) = a e^{-\alpha t}$$

$$z_0(0) = a = z_0$$

Therefore,

$$z_0(t) = z_0 e^{-\alpha t} \quad (3.22)$$

Then equation (3.18) becomes

$$\frac{dv_0}{dt} = \beta z_0 e^{-\alpha t} - \sigma v_0 \quad (3.23)$$

Compare (3.23) with (Boyce and Diprima 1986, pg.16).

$$y' + ay = g(t) \quad (3.24)$$

$$a = \sigma, \quad g(t) = \beta z_0 e^{-\alpha t}$$

The solution of (3.24) is

$$y(t) = e^{-at} \int^t e^{am} g(m) dm + c e^{-at} \quad (3.25)$$

Thus, the solution of (3.23) can be obtained as:

$$v_0(t) = e^{-\sigma t} \int^t \beta z_0 e^{\sigma m} e^{-\alpha m} dm + c e^{-\sigma t}$$

i.e.

$$v_0(t) = e^{-\sigma t} \beta z_0 \int^t e^{(\sigma-\alpha)m} dm + c e^{-\sigma t}$$

$$= \beta z_0 e^{-\sigma t} \left. \frac{e^{(\sigma-\alpha)m}}{(\sigma-\alpha)} \right| + c e^{-\sigma t}$$

$$= \frac{\beta z_0 e^{-\sigma t} \cdot e^{(\sigma-\alpha)t}}{(\sigma-\alpha)} + c e^{-\sigma t}$$

$$= \frac{\beta z_0}{(\sigma-\alpha)} e^{-\alpha t} + c e^{-\sigma t}$$

$$v_0(0) = \frac{\beta z_0}{\sigma-\alpha} + c = v_0$$

$$\Rightarrow c = \left(v_0 - \frac{\beta z_0}{\sigma-\alpha} \right)$$

Therefore,

$$v_0(t) = \frac{\beta z_0}{(\sigma-\alpha)} e^{-\alpha t} + \left(v_0 - \frac{\beta z_0}{\sigma-\alpha} \right) e^{-\sigma t} \quad (3.26)$$

Then equation (3.15) becomes

$$\frac{dy_0}{dt} = \tau y_0 + \omega \left(\frac{\beta z_0}{(\sigma-\alpha)} e^{-\alpha t} + \left(v_0 - \frac{\beta z_0}{\sigma-\alpha} \right) e^{-\sigma t} \right) \quad (3.27)$$

Compare (3.27) with (3.24), we have

$$a = -\tau, \quad g(t) = \omega \left(\frac{\beta z_0}{(\sigma-\alpha)} (e^{-\alpha t} - e^{-\sigma t}) + v_0 e^{-\sigma t} \right)$$

Thus, the solution of (3.27) can be obtained as

$$\begin{aligned} y_0(t) &= e^{\tau t} \int^t \omega e^{-\tau m} \left(\frac{\beta z_0}{(\sigma-\alpha)} (e^{\alpha m} - e^{-\sigma m}) + v_0 e^{-\sigma m} \right) dm + c_1 e^{\tau t} \\ &= \omega e^{\tau t} \int^t \left(\frac{\beta z_0}{(\sigma-\alpha)} e^{-(\tau+\alpha)m} - \frac{\beta z_0}{(\sigma-\alpha)} e^{-(\tau+\sigma)m} + v_0 e^{-(\tau+\sigma)m} \right) dm + c_1 e^{\tau t} \\ &= \omega e^{\tau t} \left(-\frac{\beta z_0}{(\sigma-\alpha)(\tau+\alpha)} e^{-(\tau+\alpha)m} \Big|_0^t + \frac{\beta z_0}{(\sigma-\alpha)(\tau+\alpha)} e^{-(\tau+\sigma)m} \Big|_0^t - \frac{v_0}{(\tau+\sigma)} e^{-(\tau+\sigma)m} \Big|_0^t \right) + c_1 e^{\tau t} \\ &= \omega e^{\tau t} \left(\frac{\beta z_0}{(\sigma-\alpha)(\tau+\alpha)} e^{-(\tau+\sigma)t} - \frac{\beta z_0}{(\sigma-\alpha)(\tau+\alpha)} e^{-(\tau+\alpha)t} - \frac{v_0}{(\tau+\sigma)} e^{-(\tau+\sigma)t} \right) + c_1 e^{\tau t} \\ &= \frac{\beta \omega z_0}{(\sigma-\alpha)(\tau+\alpha)} e^{-\sigma t} - \frac{\beta \omega z_0}{(\sigma-\alpha)(\tau+\alpha)} e^{-\alpha t} - \frac{v_0}{(\tau+\sigma)} e^{-\sigma t} + c_1 e^{\tau t} \end{aligned}$$

$$y_0(0) = \frac{\beta\omega z_0}{(\sigma - \alpha)(\tau + \alpha)} - \frac{\beta\omega z_0}{(\sigma - \alpha)(\tau + \alpha)} - \frac{v_0}{(\tau + \sigma)} + c_1 = y_0$$

$$\Rightarrow c_1 = \left(y_0 + \frac{\beta\omega z_0}{(\sigma - \alpha)(\tau + \alpha)} - \frac{\beta\omega z_0}{(\sigma - \alpha)(\tau + \alpha)} - \frac{v_0}{(\tau + \sigma)} \right)$$

Therefore,

$$y_0(t) = \frac{\beta\omega z_0}{(\sigma - \alpha)(\tau + \alpha)} e^{-\sigma t} - \frac{\beta\omega z_0}{(\sigma - \alpha)(\tau + \alpha)} e^{-\alpha t} - \frac{v_0}{(\tau + \sigma)} e^{-\sigma t} + \left(y_0 + \frac{\beta\omega z_0}{(\sigma - \alpha)(\tau + \alpha)} - \frac{\beta\omega z_0}{(\sigma - \alpha)(\tau + \alpha)} - \frac{v_0}{(\tau + \sigma)} \right) e^{\pi t} \quad (3.28)$$

Then,

Equation (3.18) becomes

$$\frac{dx_0}{dt} = P_C + K_2 z_0 e^{-\alpha t} - \mu_C x_0 \quad (3.29)$$

Compare (3.29) with (3.15), we have

$$a = \mu_C, \quad g(t) = P_C + K_2 z_0 e^{-\alpha t}$$

Thus, the solution of (3.29) can be obtained as:

$$x_0(t) = e^{-\mu_C t} \int^t e^{-\mu_C m} (P_C + K_2 z_0 e^{-\alpha m}) dm + c_2 e^{-\mu_C t}$$

i.e.

$$x_0(t) = e^{-\mu_C t} \int^t e^{-\mu_C m} (P_C e^{\mu_C m} + K_2 z_0 e^{(\mu_C - \alpha)m}) dm + c_2 e^{-\mu_C t}$$

$$= e^{-\mu_C t} \left(\frac{P_C}{\mu_C} e^{\mu_C m} \Big|_0^t + \frac{K_2 z_0}{\mu_C - \alpha} e^{(\mu_C - \alpha)m} \Big|_0^t \right) + c_2 e^{-\mu_C t}$$

$$= e^{-\mu_C t} \left(\frac{P_C}{\mu_C} e^{\mu_C t} + \frac{K_2 z_0}{\mu_C - \alpha} e^{(\mu_C - \alpha)t} \right) + c_2 e^{-\mu_C t}$$

$$= \frac{P_C}{\mu_C} + \frac{K_2 z_0}{\mu_C - \alpha} e^{-\alpha t} + c_2 e^{-\mu_C t}$$

$$x_0(0) = \frac{P_C}{\mu_C} + \frac{K_2 z_0}{\mu_C - \alpha} + c_2 = x_0$$

$$\Rightarrow c_2 = x_0 - \frac{P_C}{\mu_C} - \frac{K_2 z_0}{\mu_C - \alpha}$$

$$x_0(t) = e^{-\mu_C t} \left(\frac{P_C}{\mu_C} e^{\mu_C t} + \frac{K_2 Z_0}{\mu_C - \alpha} e^{(\mu_C - \alpha)t} \right) + \left(x_0 - \frac{P_C}{\mu_C} + \frac{K_2 Z_0}{\mu_C - \alpha} \right) e^{-\mu_C t} \quad (3.30)$$

Then equation (3.23) becomes

$$\begin{aligned} \frac{dz_1}{dt} = & -\alpha z_1 + a_1 e^{-\sigma t} + a_2 e^{-(\sigma+\alpha)t} + a_3 e^{-(\sigma+\mu_C)t} + a_4 e^{\tau} - a_5 e^{-\alpha t} - a_6 e^{-2\alpha t} + a_7 e^{(\tau-\alpha)t} - \\ & a_8 e^{-(\alpha+\mu_C)t} + a_9 e^{(\tau-\mu_C)t} \end{aligned} \quad (3.31)$$

Compare (3.31) with (3.23), we have

$$a = \alpha, \quad g(t) = a_1 e^{-\sigma t} + a_2 e^{-(\sigma+\alpha)t} + a_3 e^{-(\sigma+\mu_C)t} + a_4 e^{\tau} - a_5 e^{-\alpha t} - a_6 e^{-2\alpha t} + a_7 e^{(\tau-\alpha)t} - a_8 e^{-(\alpha+\mu_C)t} + a_9 e^{(\tau-\mu_C)t}$$

Thus, the solution of (3.31) can be obtained as:

$$\begin{aligned} z_1(t) = & e^{-\alpha t} \int^t e^{\alpha m} \left(\frac{a_1 e^{-\sigma m} + a_2 e^{-(\sigma+\alpha)m} + a_3 e^{-(\sigma+\mu_C)m} + a_4 e^{\tau} - a_5 e^{-\alpha m} - a_6 e^{-2\alpha m} + a_7 e^{(\tau-\alpha)m} - a_8 e^{-(\alpha+\mu_C)m} + a_9 e^{(\tau-\mu_C)m}}{a_6 e^{-2\alpha m} + a_7 e^{(\tau-\alpha)m} - a_8 e^{-(\alpha+\mu_C)m} + a_9 e^{(\tau-\mu_C)m}} \right) dm + c_3 e^{-\alpha t} \\ = & e^{-\alpha t} \int^t \left(\frac{a_1 e^{(\alpha-\sigma)m} + a_2 e^{-\sigma m} + a_3 e^{(\alpha-\sigma-\mu_C)m} + a_4 e^{(\tau+\alpha)m} - a_5 e^0 - a_6 e^{-\alpha m} + a_7 e^{(\tau-\alpha)m} - a_8 e^{-(\alpha+\mu_C)m} + a_9 e^{(\tau-\mu_C)m}}{a_6 e^{-2\alpha m} + a_7 e^{(\tau-\alpha)m} - a_8 e^{-(\alpha+\mu_C)m} + a_9 e^{(\tau-\mu_C)m}} \right) dm + c_3 e^{-\alpha t} \\ = & e^{-\alpha t} \left(\frac{a_1 e^{(\alpha-\sigma)t}}{(\alpha-\sigma)} - \frac{a_2}{\sigma} e^{-\sigma t} + \frac{a_3 e^{(\alpha-\sigma-\mu_C)t}}{(\alpha-\sigma-\mu_C)} + \frac{a_4 e^{(\tau+\alpha)t}}{(\tau+\alpha)} - a_5 e^t + \frac{a_6}{\alpha} e^{-\alpha t} + \frac{a_7}{\tau} e^{\tau} + \frac{a_8}{\mu_C} e^{-\mu_C t} \right. \\ & \left. + \frac{a_9 e^{(\alpha+\tau-\mu_C)t}}{(\alpha+\tau-\mu_C)} \right) + c_3 e^{-\alpha t} \end{aligned}$$

$$z_1(0) = \frac{a_1}{(\alpha-\sigma)} - \frac{a_2}{\sigma} + \frac{a_3}{(\alpha-\sigma-\mu_C)} + \frac{a_4}{(\tau+\alpha)} + \frac{a_6}{\alpha} + \frac{a_7}{\tau} + \frac{a_8}{\mu_C} + \frac{a_9}{(\alpha+\tau-\mu_C)} + c_3 = 0$$

$$\Rightarrow c_3 = \left(\frac{a_2}{\sigma} - \frac{a_1}{(\alpha-\sigma)} - \frac{a_3}{(\alpha-\sigma-\mu_C)} - \frac{a_4}{(\tau+\alpha)} - \frac{a_6}{\alpha} - \frac{a_7}{\tau} - \frac{a_8}{\mu_C} - \frac{a_9}{(\alpha+\tau-\mu_C)} \right)$$

Therefore,

$$\begin{aligned} z_1(t) = & \frac{a_1}{(\alpha-\sigma)} e^{-\sigma t} - \frac{a_2}{\sigma} e^{-(\alpha+\sigma)t} + \frac{a_3}{(\alpha-\sigma-\mu_C)} e^{-(\sigma+\mu_C)t} + \frac{a_4}{(\tau+\alpha)} e^{\tau} - a_5 t e^{-\alpha t} + \frac{a_6}{\alpha} e^{-2\alpha t} + \\ & \frac{a_7}{\tau} e^{(\tau-\alpha)t} + \frac{a_8}{\mu_C} e^{-(\alpha+\mu_C)t} + \frac{a_9}{(\alpha+\tau-\mu_C)} e^{(\tau-\mu_C)t} + \left(\frac{a_2}{\sigma} - \frac{a_1}{(\alpha-\sigma)} - \frac{a_3}{(\alpha-\sigma-\mu_C)} - \frac{a_4}{(\tau+\alpha)} - \frac{a_6}{\alpha} - \frac{a_7}{\tau} - \frac{a_8}{\mu_C} - \frac{a_9}{(\alpha+\tau-\mu_C)} \right) e^{-\alpha t} \end{aligned} \quad (3.32)$$

Where

$$b = \frac{P_C}{\mu_C}, b_1 = \frac{K_2 z_0}{\mu_C - \alpha}, b_2 = x_0 - \frac{P_C}{\mu_C} - \frac{K_2 z_0}{\mu_C - \alpha},$$

$$b_3 = \frac{\beta \omega z_0}{(\sigma - \alpha)(\tau + \sigma)}, b_4 = \frac{\beta \omega z_0}{(\sigma - \alpha)(\tau + \sigma)}, b_5 = \frac{v_0}{\tau + \sigma},$$

$$b_6 = y_0 + \frac{\beta \omega z_0}{(\sigma - \alpha)(\tau + \sigma)} - \frac{\beta \omega z_0}{(\sigma - \alpha)(\tau + \sigma)} + \frac{v_0}{\tau + \sigma}$$

Also,

$$a_1 = (bb_3 - bb_5) = b(b_3 - b_5), a_2 = b_1(b_3 - b),$$

$$a_3 = a_3 = b_2(b_3 - b_5), a_4 = bb_6, a_5 = bb_4, a_6 = b_1b_4, a_7 = b_1b_6,$$

$$a_8 = b_2b_4, a_9 = b_2b_6$$

Then equation (3.20) becomes

$$\frac{dv_1}{dt} + \sigma v_1 = \beta z_1 \quad (3.33)$$

Compare (3.33) with (3.23), we have

$$a = \sigma, g(t) = \beta z_1(t)$$

Thus, the solution of (3.33) can be obtained as

$$v_1(t) = e^{-\sigma t} \int^t \beta e^{\sigma m} z_1(m) dm + c_4 e^{-\sigma t}$$

$$= e^{-\sigma t} \int^t \left(\frac{p - p_1 e^{-\alpha m} + p_2 e^{-\mu_C m} + p_3 e^{(\tau + \sigma)m} - p_4 m e^{(\sigma - \alpha)m} + p_5 e^{(\sigma - 2\alpha)m}}{p_6 e^{(\tau + \sigma - \alpha)m} + p_7 e^{-(\sigma - \alpha - \mu_C)m} + p_8 e^{(\sigma + \tau - \mu_C)m} + p_9 e^{(\sigma - \alpha)m}} \right) dm + c_4 e^{-\sigma t}$$

$$= e^{-\sigma t} \left[\frac{pt + \frac{p_1}{\alpha} e^{-\alpha t} - \frac{p_2}{\mu_C} e^{-\mu_C t} + \frac{p_3}{(\tau + \sigma)} e^{(\tau + \sigma)t} - \frac{p_4}{(\alpha - \sigma)^2} (1 + (\alpha - \sigma)t) e^{-(\alpha - \sigma)t} + \frac{p_5}{\sigma - 2\alpha} e^{(\sigma - 2\alpha)t} + \frac{p_6}{(\tau + \sigma - \alpha)} e^{(\tau + \sigma - \alpha)t} + \frac{p_7}{(\sigma - \alpha - \mu_C)} e^{(\sigma - \alpha - \mu_C)t} + \frac{p_8}{(\sigma + \tau - \mu_C)} e^{(\sigma + \tau - \mu_C)t} + \frac{p_9}{\sigma - \alpha} e^{(\sigma - \alpha)t} \right] + c_4 e^{-\sigma t}$$

$$v_1(0) = \frac{p_1}{\alpha} - \frac{p_2}{\mu_C} + \frac{p_3}{(\tau + \sigma)} - \frac{p_4}{(\alpha - \sigma)^2} + \frac{p_5}{\sigma - 2\alpha} + \frac{p_6}{(\tau + \sigma - \alpha)} + \frac{p_7}{(\sigma - \alpha - \mu_C)}$$

$$+ \frac{p_8}{(\sigma + \tau - \mu_C)} + \frac{p_9}{\sigma - \alpha} + c_4 = 0$$

$$\Rightarrow c_4 = \left(\begin{array}{c} \frac{p_2}{\mu_C} - \frac{p_1}{\alpha} - \frac{p_3}{\tau + \sigma} - \frac{p_4}{(\alpha - \sigma)^2} - \frac{p_5}{\sigma - 2\alpha} - \frac{p_6}{(\tau + \sigma - \alpha)} - \frac{p_7}{(\sigma - \alpha - \mu_C)} \\ - \frac{p_8}{(\sigma + \tau - \mu_C)} - \frac{p_9}{\sigma - \alpha} \end{array} \right)$$

Therefore,

$$\begin{aligned} v_1(t) = & pte^{-\sigma t} + \frac{p_1}{\alpha} e^{-(\sigma+\alpha)t} - \frac{p_2}{\mu_C} e^{-(\sigma+\mu_C)t} + \frac{p_3}{(\tau+\sigma)} e^{\pi} + \frac{p_4}{(\alpha-\sigma)^2} (1 + (\alpha-\sigma)t) e^{-\alpha t} + \\ & \frac{p_5}{\sigma-2\alpha} e^{-2\alpha t} + \frac{p_6}{(\tau+\sigma-\alpha)} e^{(\tau+\sigma-\alpha)t} + \frac{p_7}{(\sigma-\alpha-\mu_C)} e^{(\sigma-\alpha)t} + \frac{p_8}{(\sigma+\tau-\mu_C)} e^{(\tau-\mu_C)t} + \\ & \frac{p_9}{\sigma-\alpha} e^{(\sigma-\alpha)t} + \left(\begin{array}{c} \frac{p_2}{\mu_C} - \frac{p_1}{\alpha} - \frac{p_3}{\tau+\sigma} - \frac{p_4}{(\alpha-\sigma)^2} - \frac{p_5}{\sigma-2\alpha} - \frac{p_6}{(\tau+\sigma-\alpha)} - \frac{p_7}{(\sigma-\alpha-\mu_C)} \\ - \frac{p_8}{(\sigma+\tau-\mu_C)} - \frac{p_9}{\sigma-\alpha} \end{array} \right) e^{-\sigma t} \end{aligned} \quad (3.34)$$

Where

$$\begin{aligned} p &= \frac{\beta a_1}{\alpha - \sigma}, \quad p_1 = \frac{\beta a_2}{\sigma}, \quad p_2 = \frac{\beta a_3}{(\alpha - \sigma - \mu_C)}, \quad p_3 = \frac{\beta a_4}{(\tau + \alpha)}, \quad p_4 = \beta a_5, \\ p_5 &= \frac{\beta a_6}{\alpha}, \quad p_6 = \frac{\beta a_7}{\tau}, \quad p_7 = \frac{\beta a_8}{\mu_C}, \quad p_8 = \frac{\beta a_9}{(\alpha + \tau - \mu_C)}, \\ p_9 &= \left(\frac{a_2}{\sigma} - \frac{a_1}{(\alpha - \sigma)} - \frac{a_3}{(\alpha - \sigma - \mu_C)} - \frac{a_4}{(\tau + \alpha)} - \frac{a_6}{\alpha} - \frac{a_7}{\tau} - \frac{a_8}{\mu_C} - \frac{a_9}{(\alpha + \tau - \mu_C)} \right) \end{aligned}$$

Then, compare equation (3.17) with equation (3.20):

$$K_2 = \beta \quad \text{and} \quad \sigma = \mu_C$$

Therefore, the solution of (3.17) can be written as

$$\begin{aligned} x_1(t) = & ste^{-\sigma t} + \frac{S_1}{\alpha} e^{-(\mu_C+\alpha)t} - \frac{S_2}{\mu_C} e^{-2\mu_C t} + \frac{S_3}{(\tau+\mu_C)} e^{\pi} + \frac{S_4}{(\alpha-\mu_C)^2} (1 + (\alpha-\mu_C)t) e^{-\alpha t} + \\ & \frac{S_5}{\mu_C-2\alpha} e^{-2\alpha t} + \frac{S_6}{(\tau+\mu_C-\alpha)} e^{(\tau-\alpha)t} + \frac{S_7}{\alpha} e^{-\alpha t} + \frac{S_8}{\tau} e^{(\tau-\mu_C)t} + \frac{S_9}{\mu_C-\alpha} e^{(\mu_C-\alpha)t} + \\ & \left(\begin{array}{c} \frac{S_2}{\mu_C} - \frac{S_1}{\alpha} - \frac{S_3}{\tau+\mu_C} - \frac{S_4}{(\alpha-\mu_C)^2} - \frac{S_5}{\mu_C-2\alpha} - \frac{S_6}{(\tau+\mu_C-\alpha)} - \frac{S_7}{\alpha} \\ - \frac{S_8}{\tau} - \frac{S_9}{\mu_C-\alpha} \end{array} \right) e^{-\mu_C t} \end{aligned} \quad (3.34)$$

Where

$$s = \frac{k_2 a_1}{\alpha - \mu_C}, \quad s_1 = \frac{k_2 a_2}{\mu_C}, \quad p_2 = \frac{k_2 a_3}{(\alpha - 2\mu_C)}, \quad p_3 = \frac{k_2 a_4}{(\tau + \alpha)}, \quad p_4 = k_2 a_5,$$

$$p_5 = \frac{k_2 a_6}{\alpha}, \quad p_6 = \frac{k_2 a_7}{\tau}, \quad p_7 = \frac{k_2 a_8}{\mu_C}, \quad p_8 = \frac{k_2 a_9}{(\alpha + \tau - \mu_C)},$$

$$p_9 = \left(\frac{a_2}{\mu_C} - \frac{a_1}{(\alpha - \mu_C)} - \frac{a_3}{(\alpha - 2\mu_C)} - \frac{a_4}{(\tau + \alpha)} - \frac{a_6}{\alpha} - \frac{a_7}{\tau} - \frac{a_8}{\mu_C} - \frac{a_9}{(\alpha + \tau - \mu_C)} \right)$$

Then equation (3.18) becomes

$$\frac{dy_1}{dt} - \tau y_1 = \omega v_1 - x_0 y_0 \quad (3.35)$$

Compare (3.35) with (3.23), we have

$$a = -\tau, \quad g(t) = \omega v_1(t) - x_0(t) y_0(t)$$

Thus, the solution of (3.35) can be obtained as

$$y_1(t) = e^{\pi} \int^t e^{-m} (\omega v_1(m) - x_0(m) y_0(m)) dm + c_5 e^{\pi}$$

$$= e^{\pi} \int^t \left(r m e^{-(\tau+\sigma)m} + r_1 e^{-(\tau+\sigma+\alpha)m} - r_2 e^{-(\tau+\sigma+\mu_C)m} + r_3 + r_4 (1 + (\alpha - \sigma)m) e^{-(\alpha+\tau)m} + r_5 e^{-(\tau+2\alpha)m} + \right.$$

$$\left. r_6 e^{-\alpha m} + r_7 e^{(\sigma-\alpha-\mu_C-\tau)m} + r_8 e^{-\mu_C m} + r_9 e^{(\sigma-\alpha-\tau)m} + r_{10} e^{-(\tau+\sigma)m} - a_1 e^{-(\tau+\sigma)m} + \right.$$

$$\left. a_2 e^{-(\sigma+\tau+\alpha)m} + a_3 e^{-(\sigma+\tau+\mu_C)m} + a_4 e^0 - a_5 e^{-(\alpha+\tau)m} - a_6 e^{-(\tau+2\alpha)m} + \right.$$

$$\left. a_7 e^{-\alpha m} - a_8 e^{-(\alpha+\tau+\mu_C)m} + a_9 e^{-\mu_C m} \right) dm + c_5 e^{\pi}$$

$$e^{\pi} \left(\frac{-r(1 + (\sigma + \tau)t) e^{-(\tau+\sigma)t}}{(\tau + \sigma)^2} - \frac{r_1}{(\tau + \sigma + \alpha)} e^{-(\tau+\sigma+\alpha)t} + \frac{r_2 e^{-(\tau+\sigma+\mu_C)t}}{(\tau + \sigma + \mu_C)} + r_3 t - \right.$$

$$\left. \frac{r_4 ((\alpha^2 - \sigma\alpha + \alpha\tau - \sigma\tau)t + \sigma + \tau) e^{(\alpha+\tau)t}}{(\alpha + \tau)^2} - \frac{r_5 e^{-(\tau+2\alpha)t}}{(\tau + 2\alpha)^2} - \frac{r_6 e^{-\alpha t}}{\alpha} + \right.$$

$$\left. \frac{r_7 e^{-(\sigma-\alpha-\mu_C-\tau)t}}{(\sigma - \alpha - \mu_C - \tau)} - \frac{r_8}{\mu_C} e^{-\mu_C t} + \frac{r_9}{(\sigma - \alpha - \tau)} e^{(\sigma-\alpha-\tau)t} - \frac{r_{10}}{(\tau + \sigma)} e^{-(\tau+\sigma)t} + \right.$$

$$\left. \frac{a_1 e^{(\tau+\sigma)t}}{(\tau + \sigma)} - \frac{a_2 e^{-(\sigma+\tau+\alpha)t}}{(\sigma + \tau + \alpha)} + \frac{a_3 e^{-(\sigma+\tau+\mu_C)t}}{(\sigma + \tau + \mu_C)} + a_4 t + \frac{a_5}{(\alpha + \tau)} e^{-(\alpha+\tau)t} + \frac{a_6 e^{-(\tau+2\alpha)t}}{(\tau + 2\alpha)} \right.$$

$$\left. - \frac{a_7 e^{-\alpha t}}{\alpha} + \frac{a_8 e^{-(\alpha+\tau+\mu_C)t}}{(\alpha + \tau + \mu_C)} - \frac{a_9}{\mu_C} e^{-\mu_C t} \right) + c_5 e^{\pi}$$

$$\begin{aligned}
 y_0(t) = & -\frac{r}{(\tau+\sigma)^2} - \frac{r_1}{(\tau+\sigma+\alpha)} + \frac{r_2}{(\tau+\sigma+\mu_c)} - \\
 & \frac{r_4(\sigma+\tau)}{(\alpha+\tau)^2} - \frac{-r_5}{(\tau+2\alpha)^2} - \frac{r_6}{\alpha} + \frac{r_7}{(\sigma-\alpha-\mu_c-\tau)} - \frac{r_8}{\mu_c} + \frac{r_9}{(\sigma-\alpha-\tau)} - \frac{r_{10}}{(\tau+\sigma)} + \\
 & \frac{a_1}{(\tau+\sigma)} + \frac{a_2}{(\sigma+\tau+\alpha)} - \frac{a_3}{(\sigma+\tau+\mu_c)} + \frac{a_5}{(\alpha+\tau)} + \frac{a_6}{(\tau+2\alpha)} \\
 & - \frac{a_7}{\alpha} + \frac{a_8}{(\alpha+\tau+\mu_c)} - \frac{a_9}{\mu_c} + c_5 = 0
 \end{aligned}$$

$$\Rightarrow c_5 = \left(\begin{aligned} & \frac{r}{(\tau+\sigma)^2} + \frac{(r_1-a_2)}{(\tau+\sigma+\alpha)} - \frac{(r_2-a_3)}{(\tau+\sigma+\mu_c)} - \frac{r_4(\sigma+\tau)}{(\alpha+\tau)^2} + \frac{-r_5}{(\tau+2\alpha)^2} + \frac{r_6}{\alpha} - \\ & \frac{r_7}{(\sigma-\alpha-\mu_c-\tau)} + \frac{r_8}{\mu_c} - \frac{r_9}{(\sigma-\alpha-\tau)} + \frac{r_{10}}{(\tau+\sigma)} - \frac{a_1}{(\tau+\sigma)} - \frac{a_5}{(\alpha+\tau)} - \\ & \frac{a_6}{(\tau+2\alpha)} - \frac{a_7}{\alpha} - \frac{a_8}{(\alpha+\tau+\mu_c)} + \frac{a_9}{\mu_c} \end{aligned} \right)$$

Therefore,

$$\begin{aligned}
 y_1(t) = & -\frac{r(1+(\sigma+\tau)t)e^{-\sigma t}}{(\tau+\sigma)^2} - \frac{r_1}{(\tau+\sigma+\alpha)}e^{-(\sigma+\alpha)t} + \frac{r_2e^{-(\sigma+\mu_c)t}}{(\tau+\sigma+\mu_c)} + r_3te^\pi - \\
 & \frac{r_4((\alpha^2-\sigma\alpha+\alpha\tau-\sigma\tau)t+\sigma+\tau)e^{(\alpha+2\tau)t}}{(\alpha+\tau)^2} - \frac{-r_5e^{-2\alpha t}}{(\tau+2\alpha)^2} - \frac{r_6}{\alpha}e^{-(\alpha-\tau)t} + \\
 & \frac{r_7e^{-(\sigma-\alpha-\mu_c)t}}{(\sigma-\alpha-\mu_c-\tau)} - \frac{r_8}{\mu_c}e^{-(\mu_c-\tau)t} + \frac{r_9}{(\sigma-\alpha-\tau)}e^{(\sigma-\alpha)t} - \frac{r_{10}}{(\tau+\sigma)}e^{-\sigma t} + \\
 & \frac{a_1e^{(2\tau+\sigma)t}}{(\tau+\sigma)} - \frac{a_2e^{-(\sigma+\alpha)t}}{(\sigma+\tau+\alpha)} + \frac{a_3e^{-(\sigma+\mu_c)t}}{(\sigma+\tau+\mu_c)} + a_4te^\pi + \frac{a_5}{(\alpha+\tau)}e^{-\alpha t} + \frac{a_6e^{-2\alpha t}}{(\tau+2\alpha)} \\
 & - \frac{a_7e^{-(\alpha-\tau)t}}{\alpha} + \frac{a_8e^{-(\alpha+\mu_c)t}}{(\alpha+\tau+\mu_c)} - \frac{a_9}{\mu_c}e^{-(\mu_c-\tau)t} + c_5e^\pi,
 \end{aligned} \tag{3.36}$$

Where

$$\begin{aligned}
 r = p\omega, \quad r_1 = \frac{p_1\omega}{\alpha}, \quad r_2 = \frac{p_2\omega}{\mu_c}, \quad r_3 = \frac{p_3\omega}{(\tau+\sigma)}, \quad r_4 = \frac{p_4\omega}{(\alpha-\sigma)}, \\
 r_5 = \frac{p_5\omega}{\sigma-2\alpha}, \quad r_6 = \frac{p_6\omega}{(\tau+\sigma-\alpha)}, \quad r_7 = \frac{p_7\omega}{(\sigma-\alpha-\mu_c)} \\
 r_8 = \frac{p_8\omega}{(\sigma+\tau-\mu_c)}, \quad r_9 = \frac{p_9\omega}{\sigma-\alpha}
 \end{aligned}$$

$$r_{10} = \left(\begin{array}{c} \frac{p_2}{\mu_C} - \frac{p_1}{\alpha} - \frac{p_3}{\tau + \sigma} - \frac{p_4}{(\alpha - \sigma)^2} - \frac{p_5}{\sigma - 2\alpha} - \frac{p_6}{(\tau + \sigma - \alpha)} - \frac{p_7}{(\sigma - \alpha - \mu_C)} - \\ \frac{p_8}{(\sigma + \tau - \mu_C)} - \frac{p_9}{\sigma - \alpha} \end{array} \right)$$

4. Results and Discussion

In this work, we have extended Wilson (2004) model to incorporate relevant features parameters such as recovery class and drug administration. We investigated the existence of unique solution of the extended model under certain conditions. We have obtained the non-zero equilibrium states of a mathematical model of Chlamydia Trachomatis dynamics and conducted a linear stability analysis.

The model equation is solved analytically using parameter expanding method also known as asymptotic method. The computations were done using Computer Symbolic Algebraic Package Maple.

The graphs of concentration free extracellular Chlamydia particles $x(t)$, number of uninfected mucosal epithelial cells $y(t)$, number of Chlamydia infected epithelial cells $z(t)$ and number of epithelial cell which recovered from Chlamydia infection $v(t)$ are presented in Figures 4.1 to 4.8. In Figure 4.1 displays the graph of $x(t)$ against t for different values of γ . It is seen that the concentration of free extracellular Chlamydia particles decreases as the rate of clearance (recovery) of infected cells due to cell-mediated immunity increases.

Figure 4.2 displays the graph of $y(t)$ against t for different values of γ . It is seen that the number of uninfected mucosal epithelial cells decreases as the rate of clearance (recovery) of infected cells due to cell-mediated immunity increases. Figure 4.3 displays the graph of $z(t)$ against t for different values of γ . It is seen that the number of Chlamydia infected epithelial cells decreases as the rate of clearance (recovery) of infected cells due to cell-mediated immunity increases.

Figure 4.4 displays the graph of $v(t)$ against t for different values of γ . It is seen that the number of epithelial cell which recovered from Chlamydia infection increases as the rate of clearance (recovery) of infected cells due to cell-mediated immunity decreases.

Figure 4.5 displays the graph of $x(t)$ against t for different values of δ . It is seen that the concentration of free extracellular Chlamydia particles decreases as the rate of the recovery

due to drug administration increases. Figure 4.6 displays the graph of $y(t)$ against t for different values of δ . It is seen that the number of uninfected mucosal epithelial cells decreases as the recovery rate due to drug administration increases. Also, figure 4.7 displays the graph of $z(t)$ against t for different values of δ . It is seen that the number of Chlamydia infected epithelial cells decreases as recovery rate due to drug administration increases. While in figure 4.8, it displays the graph of $v(t)$ against t for different values of δ . which shows that the number of epithelial cell which recovered from Chlamydia infection increases as the recovery rate due to drug administration decreases.

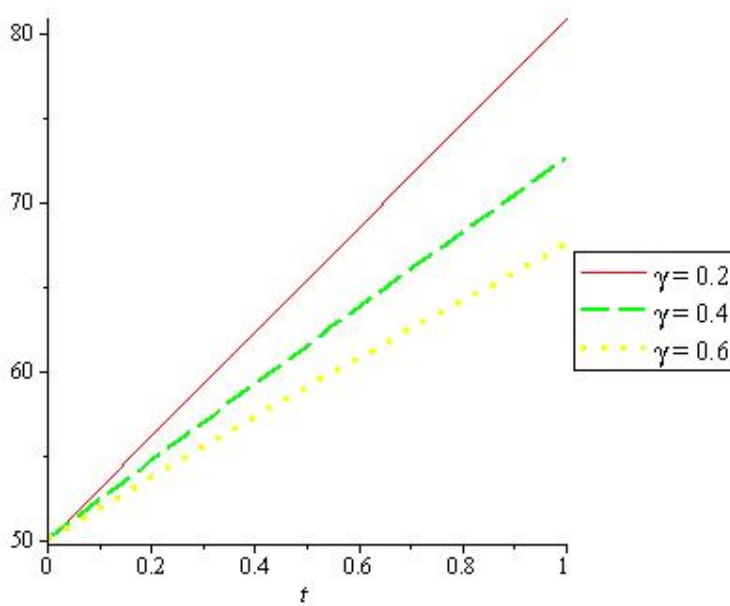


Figure 3.13: Plots of $x(t)$ against t for different values of γ and $P_C = 0.1, P_S = 0.1, K_1 = 1, K_2 = 0.1, \mu_C = 0.02, \mu_S = 0.01, \delta = 0.1, \omega = 0.04, v_0 = 0, x_0 = 50, y_0 = 30, z_0 = 10$

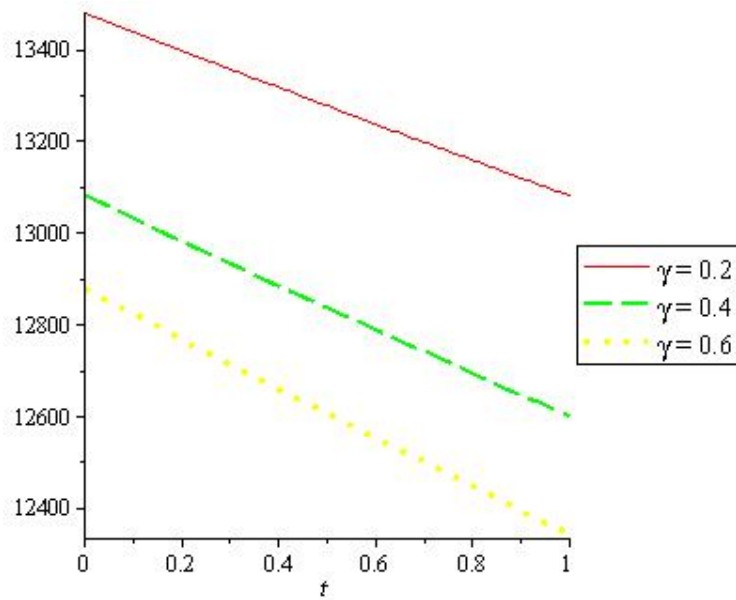


Figure 3.14: Plots of $y(t)$ against t for different values of γ and $P_C = 0.1, P_S = 0.1, K_1 = 1, K_2 = 0.1, \mu_C = 0.02, \mu_S = 0.01, \delta = 0.1, \omega = 0.04, v_0 = 0, x_0 = 50, y_0 = 30, z_0 = 10$

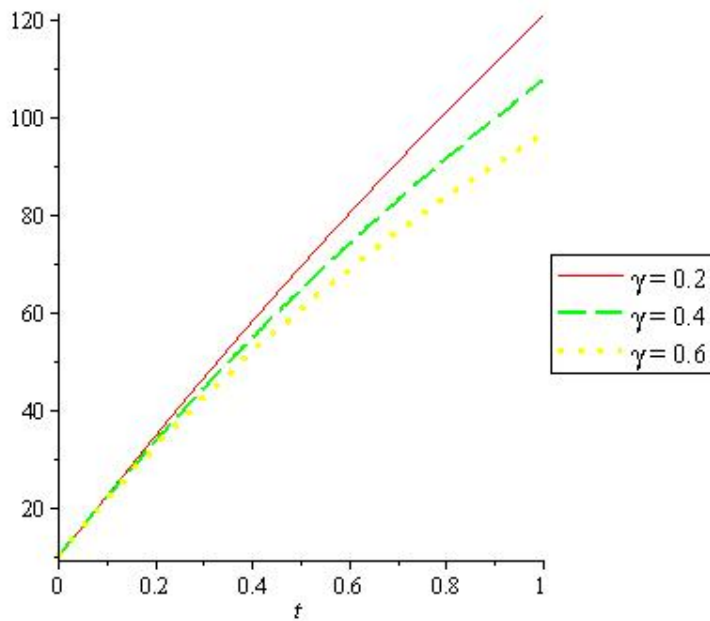


Figure 3.15: Plots of $z(t)$ against t for different values of γ and $P_C = 0.1, P_S = 0.1, K_1 = 1, K_2 = 0.1, \mu_C = 0.02, \mu_S = 0.01, \delta = 0.1, \omega = 0.04, v_0 = 0, x_0 = 50, y_0 = 30, z_0 = 10$

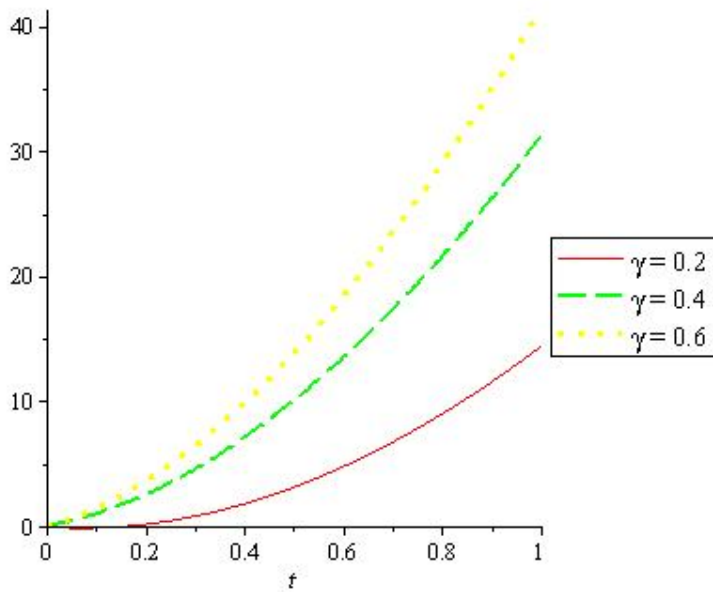


Figure 3.16: Plots of $v(t)$ against t for different values of γ and $P_C = 0.1, P_S = 0.1, K_1 = 1, K_2 = 0.1, \mu_C = 0.02, \mu_S = 0.01, \delta = 0.1, \omega = 0.04, v_0 = 0, x_0 = 50, y_0 = 30, z_0 = 10$

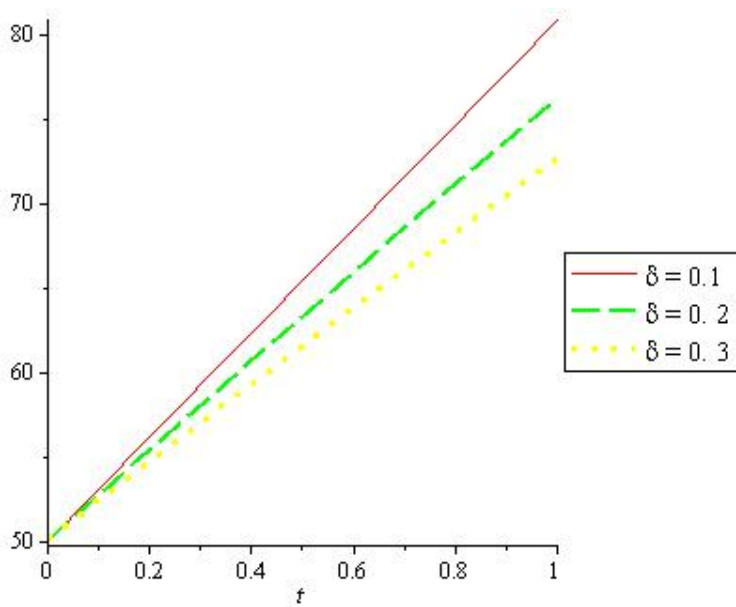


Figure 3.17: Plots of $x(t)$ against t for different values of δ and $P_C = 0.1, P_S = 0.1, K_1 = 1, K_2 = 0.1, \mu_C = 0.02, \mu_S = 0.01, \gamma = 0.2, \omega = 0.04, v_0 = 0, x_0 = 50, y_0 = 30, z_0 = 10$

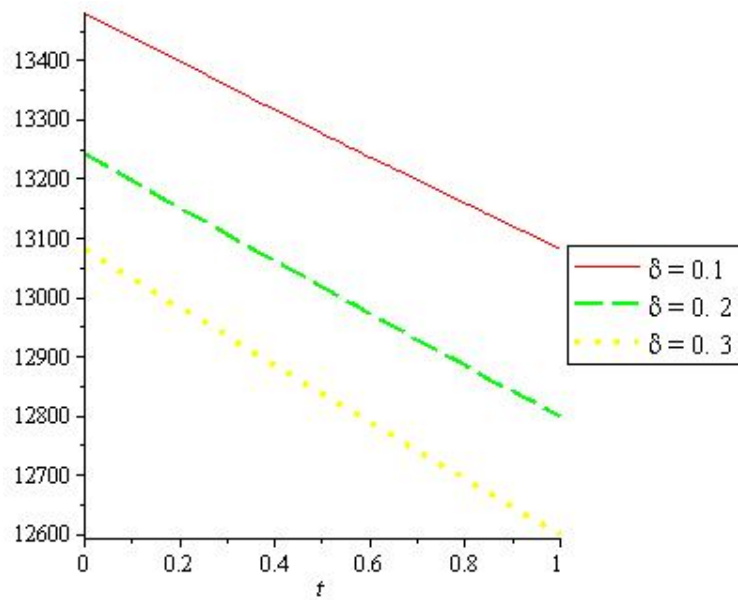


Figure 3.18: Plots of $y(t)$ against t for different values of δ and $P_C = 0.1, P_B = 0.1, K_1 = 1, K_2 = 0.1, \mu_C = 0.02, \mu_B = 0.01, \gamma = 0.2, \omega = 0.04, v_0 = 0, x_0 = 50, y_0 = 30, z_0 = 10$

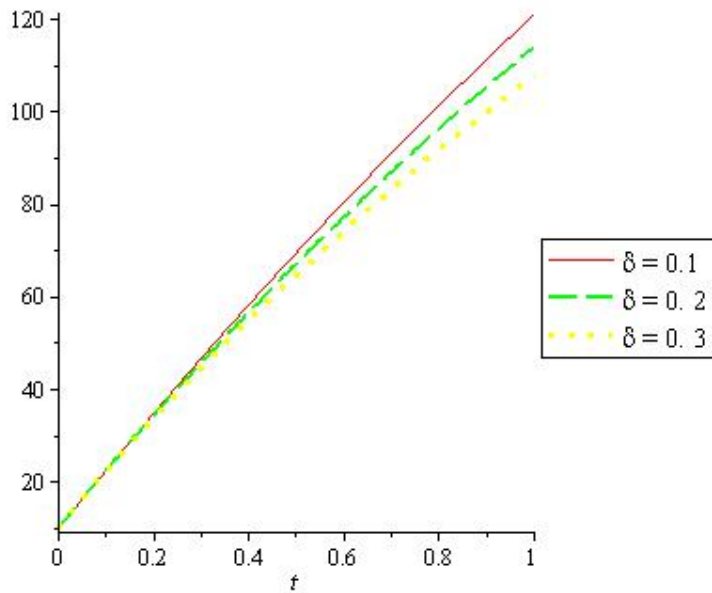


Figure 3.19: Plots of $z(t)$ against t for different values of δ and $P_C = 0.1, P_B = 0.1, K_1 = 1, K_2 = 0.1, \mu_C = 0.02, \mu_B = 0.01, \gamma = 0.2, \omega = 0.04, v_0 = 0, x_0 = 50, y_0 = 30, z_0 = 10$

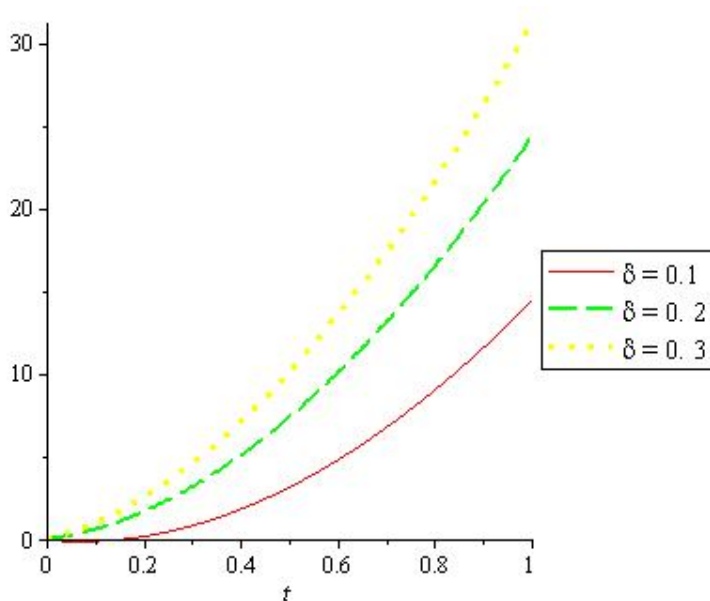


Figure 3.20: Plots of $v(t)$ against t for different values of δ and $P_C = 0.1, P_E = 0.1, K_1 = 1, K_2 = 0.1, \mu_C = 0.02, \mu_E = 0.01, \gamma = 0.2, \omega = 0.04, v_0 = 0, x_0 = 50, y_0 = 30, z_0 = 10$

5. Conclusion

We established the non-zero equilibrium states of a mathematical model of the dynamics of a sexually transmitted infection (Chlamydia Trachomatis). We conducted stability analysis of non-zero steady states. The results show that it is asymptotically stable under some conditions. A computer symbolic algebraic package Maple was used to numerically and analytically calculate the concentration of free extracellular Chlamydia particles and number of infected, uninfected and recovered epithelial cells with different parameter values parameter expanding method. The results obtained shows that the parameters involved play a crucial role in the increase and decrease of number of infected, uninfected and recovered epithelial cells.

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Figure 4.1: Plots of $x(t)$ against t for different values of γ and $P_C = 0.1, P_E = 0.02, K_1 = 1, K_2 = 0.1, \mu_C = 0.02, \mu_E = 0.01, \delta = 0.1, \omega = 0.04, \nu_0 = 0, x_0 = 50, y_0 = 30, z_0 = 10$

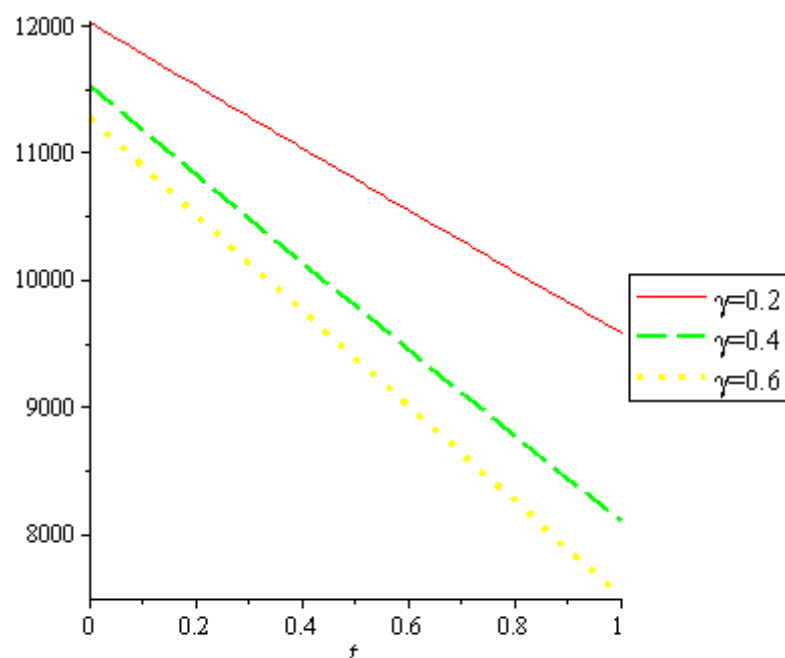


Figure 4.2: Plots of $y(t)$ against t for different values of γ and $P_C = 0.1, P_E = 0.02, K_1 = 1, K_2 = 0.1, \mu_C = 0.02, \mu_E = 0.01, \delta = 0.1, \omega = 0.04, \nu_0 = 0, x_0 = 50, y_0 = 30, z_0 = 10$

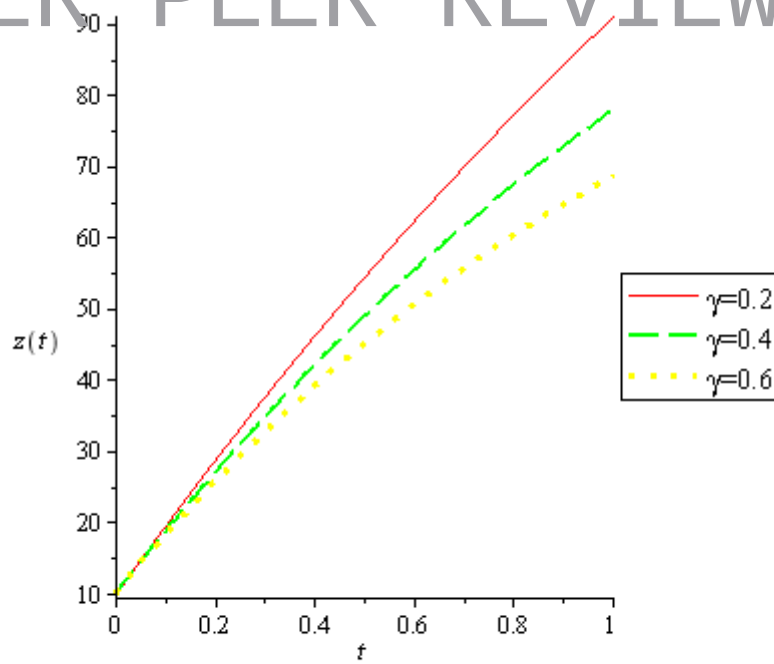


Figure 4.3: Plots of $z(t)$ against t for different values of γ and $P_C = 0.1, P_E = 0.02, K_1 = 1, K_2 = 0.1, \mu_C = 0.02, \mu_E = 0.01, \delta = 0.1, \omega = 0.04, v_0 = 0, x_0 = 50, y_0 = 30, z_0 = 10$

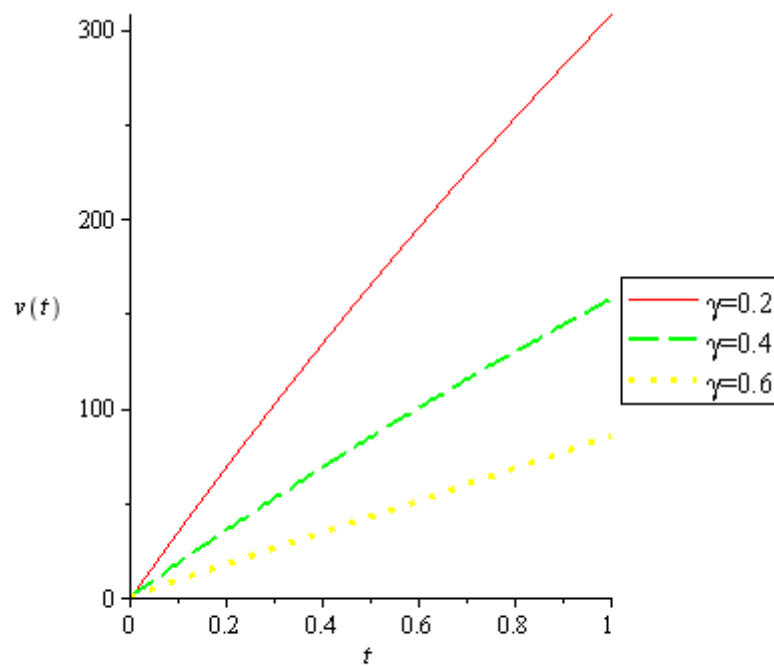


Figure 4.4: Plots of $v(t)$ against t for different values of γ and $P_C = 0.1, P_E = 0.02, K_1 = 1, K_2 = 0.1, \mu_C = 0.02, \mu_E = 0.01, \delta = 0.1, \omega = 0.04, v_0 = 0, x_0 = 50, y_0 = 30, z_0 = 10$

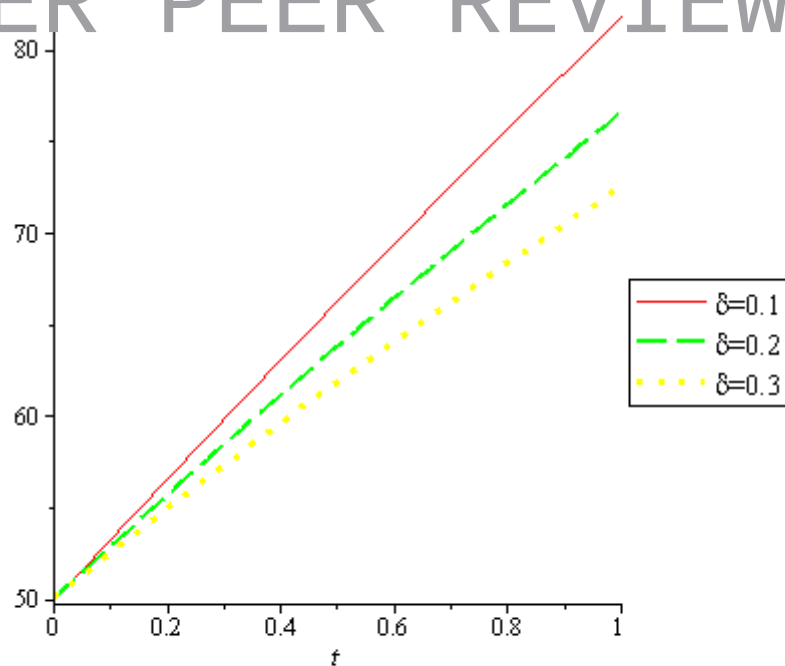


Figure 4.5: Plots of $x(t)$ against t for different values of δ and $P_C = 0.1, P_E = 0.02, K_1 = 1, K_2 = 0.1, \mu_C = 0.02, \mu_E = 0.01, \gamma = 0.2, \omega = 0.04, \nu_0 = 0, x_0 = 50, y_0 = 30, z_0 = 10$

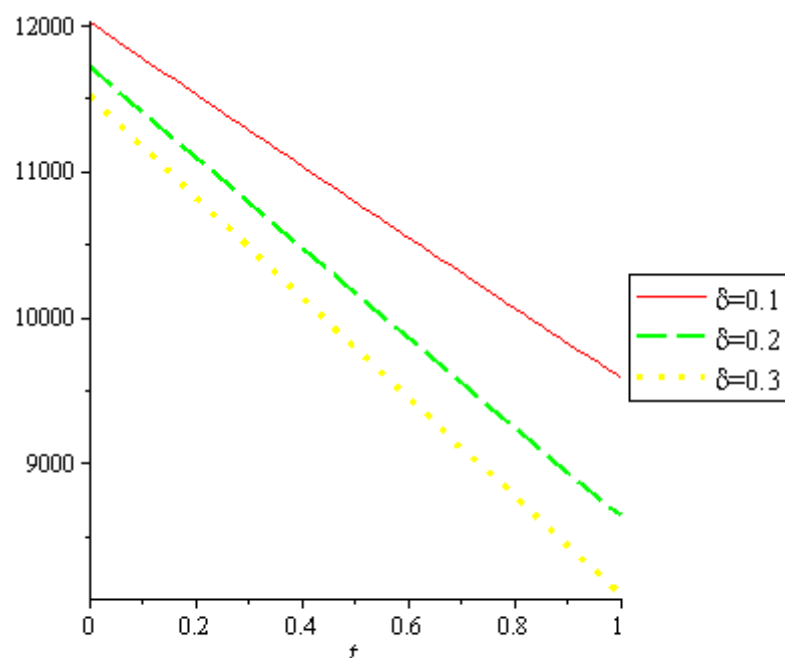


Figure 4.6: Plots of $y(t)$ against t for different values of δ and $P_C = 0.1, P_E = 0.02, K_1 = 1, K_2 = 0.1, \mu_C = 0.02, \mu_E = 0.01, \gamma = 0.2, \omega = 0.04, \nu_0 = 0, x_0 = 50, y_0 = 30, z_0 = 10$

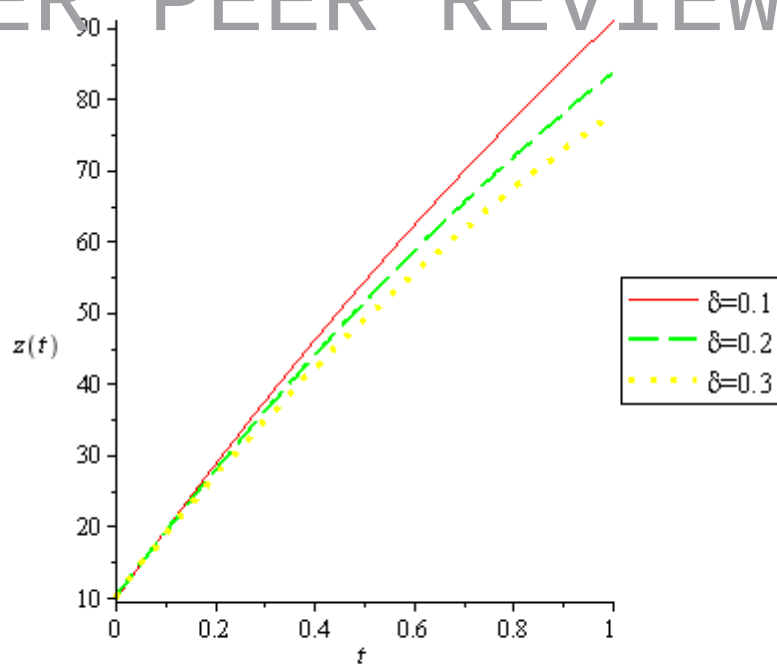


Figure 4.7: Plots of $z(t)$ against t for different values of δ and $P_C = 0.1, P_E = 0.02, K_1 = 1, K_2 = 0.1, \mu_C = 0.02, \mu_E = 0.01, \gamma = 0.2, \omega = 0.04, v_0 = 0, x_0 = 50, y_0 = 30, z_0 = 10$

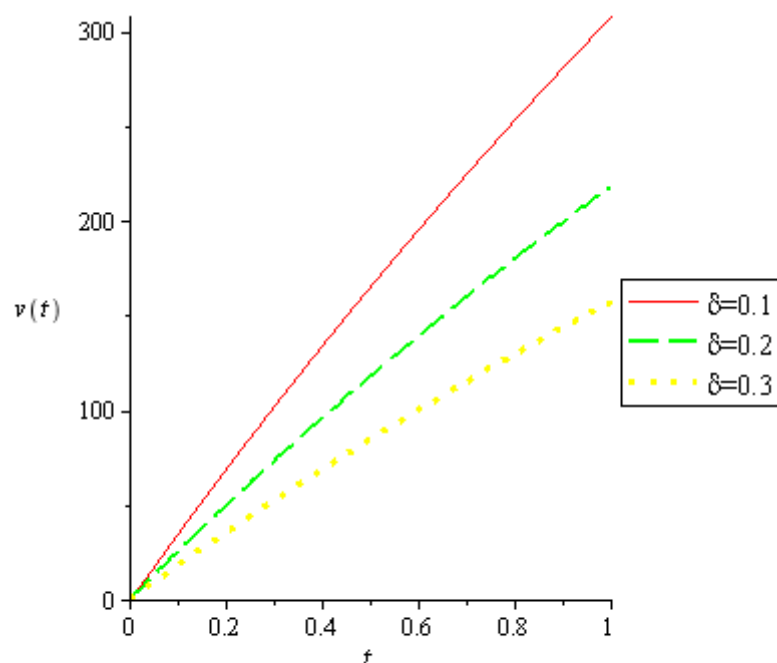


Figure 4.8: Plots of $v(t)$ against t for different values of δ and $P_C = 0.1, P_E = 0.02, K_1 = 1, K_2 = 0.1, \mu_C = 0.02, \mu_E = 0.01, \gamma = 0.2, \omega = 0.04, v_0 = 0, x_0 = 50, y_0 = 30, z_0 = 10$