

Original Research Article

A New Generalized 'Useful' R-norm Information Measure and its Properties

ABSTRACT

In the present paper a new 'useful' R-norm information measure of type α and degree β is defined and characterized axiomatically. A new generalized 'useful' mean codeword length is introduced and its bounds in terms of 'useful' R-norm information measure of type α and degree β are studied. Properties of new generalized measure of information have also been studied.

Keywords: R-norm information; Mean codeword length; Kraft's inequality; Code Alphabets and Holder's inequality.

1. INTRODUCTION

We consider the set of positive real numbers, not equal to 1 and denote this by R^+ defined as $R^+ = \{R : R > 0 \neq 1\}$. Let Δ_n with $n \geq 2$ be the set of all probability distributions $P = \left\{ (p_1, p_2, \dots, p_n), p_i \geq 0 \text{ and } \sum_{i=1}^n p_i = 1 \right\}$.

Boekee and Lubbe [3] considered R-norm information of distribution P defined for $R \in R^+$ by

$$H_R(P) = \frac{R}{R-1} \left[1 - \left(\sum_{i=1}^n p_i^R \right)^{\frac{1}{R}} \right]. \quad (1.1)$$

The R-norm information measure (1.1) is a real function $\Delta_n \rightarrow R^+$ defined on Δ_n , where $n \geq 2$ and R^+ is the set of positive real numbers. This measure is different from entropies of Shannon [14], Renyi [13], Havrda and Charvat [6], and Daroczy [4]. The most interesting property is that when $R \rightarrow 1$, R-norm information measure approaches to Shannon's entropy and when $R \rightarrow \infty$, $H_R(P) \rightarrow 1 - \max p_i$, where $i = 1, 2, \dots, n$.

The measure (1.1) can be generalized in so many ways; Hooda and Ram [7] proposed and characterized the following parametric generalization of (1.1):

$$H_R^\beta(P) = \frac{R}{R + \beta - 2} \left[1 - \left(\sum_{i=1}^n p_i^{\frac{R}{2-\beta}} \right)^{\frac{2-\beta}{R}} \right], 0 < \beta \leq 1, R(>0) \neq 1. \quad (1.2)$$

The above measure (1.2) was called generalized R-norm information measure of degree β and it reduces to (1.1) when $\beta = 1$.

Hooda and Sharma [8] proposed and studied parametric generalization of (1.2) as given below:

$$H_R^{\alpha, \beta}(P) = \frac{R}{R + \beta - 2\alpha} \left[1 - \left(\sum_{i=1}^n p_i^{\frac{R}{2\alpha - \beta}} \right)^{\frac{2\alpha - \beta}{R}} \right], \alpha \geq 1, 0 < \beta \leq 1, R(>0) \neq 1, 0 < R + \beta \neq 2\alpha \quad (1.3)$$

They called (1.3) as the generalized R-norm information measure of type α and degree β . (1.3) reduces to (1.2) when $\alpha = 1$, which further reduces to (1.1) when $\beta = 1$.

Belis and Giasu [2] attached a utility distribution $U = \{(u_1, u_2, \dots, u_n), u_i > 0\}$ with the probability distribution P such that u_i is the utility of an event having probability p_i . They characterized a 'qualitative-quantitative' measure which was called 'useful' information by Longo [12] of the experiment E and is given as:

$$H(P; U) = H(p_1, p_2, \dots, p_n; u_1, u_2, \dots, u_n). \quad (1.4)$$

$$= - \sum_{i=1}^n u_i p_i \log p_i, u_i > 0, 0 < p_i \leq 1, \sum_{i=1}^n p_i = 1.$$

Hooda and Bhaker [10] characterized the following measure of 'useful' information:

$$H(P; U) = - \frac{\sum_{i=1}^n u_i p_i \log p_i}{\sum_{i=1}^n u_i p_i}. \quad (1.5)$$

Hooda et al. [9] characterized the following 'useful' R-norm information measure:

$$H_R(P; U) = \frac{R}{R-1} \left[1 - \frac{\left(\sum_{i=1}^n u_i p_i^R \right)^{\frac{1}{R}}}{\sum_{i=1}^n u_i p_i} \right], R(>0) \neq 1. \quad (1.6)$$

Analogous to (1.3) we consider a new measure of 'useful' R-norm information by attaching utility distribution as given below:

$$H_R^{\alpha, \beta}(P; U) = \frac{R}{R + \beta - 2\alpha} \left[1 - \frac{\left(\sum_{i=1}^n u_i p_i^{\frac{R}{2\alpha - \beta}} \right)^{\frac{2\alpha - \beta}{R}}}{\sum_{i=1}^n u_i p_i} \right], R(>0) \neq 1, 0 < \beta \leq 1, R + \beta \neq 2\alpha, \quad (1.7)$$

where U is the utility distribution corresponding to probability distribution P . This we may call (1.7) as the generalized 'useful' R-norm information measure of type α and degree β . In case $\alpha = 1$ and $\beta = 1$ measure (1.7) reduces to (1.6) and if $R \rightarrow 1$ it reduces to (1.5).

In the present paper we characterize the 'useful' R-norm information measure type α and degree β axiomatically. In section 3 we give the properties of 'useful' R-norm information measure of type α and degree β . In section 4 we give a brief account of mean codeword length and their bounds. We derive the lower and upper bounds of generalized 'useful' mean codeword length of type (α, β) in terms of 'useful' R-norm information measure of type α and degree β in section 5.

2. AXIOMATIC CHARACTERIZATION

Let $S_n = \Delta_n \times \Delta_n^* \rightarrow R^+$, $n = 2, 3, \dots$ and G_n be a sequence of functions of p_i 's and u_i 's, $i = 1, 2, \dots, n$, defined over S_n satisfying the following axioms:

Axiom 2.1 $G_n(P:U) = a_1 + a_2 \sum_{i=1}^n h(p_i, u_i)$, where a_1 and a_2 are non zero constants and

$p, u \in J = \{(0,1) \times (0,\infty)\} \cup \{(0,y); 0 \leq y \leq 1\} \cup \{(\infty, y'); 0 \leq y' \leq \infty\}$.

This axiom is also called sum property.

Axiom 2.2 For $P \in \Delta_n, U \in \Delta_n^*, P' \in \Delta_m$, and $U' \in \Delta_m^*$, G_{mn} satisfies the following property

$$G_{mn}(PP':UU') = G_n(P:U) + G_m(P':U') - \frac{1}{a_1} G_n(P:U) G_m(P':U').$$

Axiom 2.3 $h(p, u)$ is a continuous function of its arguments p and u .

Axiom 2.4 Let all p_i 's and u_i 's are equiprobable and of equal utility of events respectively, then

$$G_n\left(\frac{1}{n}, \dots, \frac{1}{n}; u, \dots, u\right) = \frac{R}{R + \beta - 2\alpha} \left[1 - n^{\frac{-\beta + 2\alpha - R}{R}} \right],$$

where $n = 2, 3, \dots$, and $\alpha \geq 1$, $0 < \beta \leq 1$, $R(>0) \neq 1$, $0 < R + \beta \neq 2\alpha$.

First of all we prove the three lemmas to facilitate the proof of the main theorem.

Lemma 2.1 From axiom 2.1 and 2.2, it is very easy to arrive at the following functional equation:

$$\sum_{i=1}^n \sum_{j=1}^m h(p_i p'_j, u_i u'_j) = \left(\frac{-a_2}{a_1} \right) \sum_{i=1}^n h(p_i, u_i) \sum_{j=1}^m h(p'_j, u'_j), \quad (2.1)$$

where $(p_i, u_i), (p'_j, u'_j) \in J$ for $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, m$.

Lemma 2.2 The continuous solution that satisfies (2.1) is the same as that of the functional equation:

$$h(pp', uu') = \left(\frac{-a_2}{a_1} \right) h(p, u) h(p', u'), \quad (2.2)$$

Proof: Let a, b, c, d and a', b', c', d' be positive integers such that

$1 \leq a' \leq a, 1 \leq b' \leq b, 1 \leq c \leq c'$, and $1 \leq d \leq d'$.

Setting $n = a - a' + 1 = c' - c + 1$ and $m = b - b' + 1 = d' - d + 1$,

$$p_i = \frac{1}{a} (i = 1, 2, \dots, a - a'), \quad p_{a-a'+1} = \frac{a'}{a},$$

$$u_i = \frac{1}{c} (i = 1, 2, \dots, c' - c), \quad u_{c'-c+1} = \frac{c'}{c},$$

$$p'_j = \frac{1}{b} (j = 1, 2, \dots, b - b'), \quad p'_{b-b'+1} = \frac{b'}{b},$$

$$u'_j = \frac{1}{d} (j = 1, 2, \dots, d' - d), \quad u'_{d'-d+1} = \frac{d'}{d},$$

From equation (2.1) we have

$$\begin{aligned} & (a - a')(b - b') h\left(\frac{1}{ab}, \frac{1}{cd}\right) + (b - b') h\left(\frac{a'}{ab}, \frac{c'}{cd}\right) + (a - a') h\left(\frac{b'}{ab}, \frac{d'}{cd}\right) + h\left(\frac{a'b'}{ab}, \frac{c'd'}{cd}\right) \\ & = \left(\frac{-a_2}{a_1} \right) \left[(a - a') h\left(\frac{1}{a}, \frac{1}{c}\right) + h\left(\frac{a'}{a}, \frac{c'}{c}\right) \right] \left[(b - b') h\left(\frac{1}{b}, \frac{1}{d}\right) + h\left(\frac{b'}{b}, \frac{d'}{d}\right) \right] \end{aligned} \quad (2.3)$$

Taking $a' = b' = c' = d' = 1$ in (2.3), we get

$$h\left(\frac{1}{ab}, \frac{1}{cd}\right) = \left(\frac{-a_2}{a_1}\right) h\left(\frac{1}{a}, \frac{1}{c}\right) h\left(\frac{1}{b}, \frac{1}{d}\right) \quad (2.4)$$

Taking $a' = c' = 1$ in (2.3) and using (2.4), we have

$$h\left(\frac{b'}{ab}, \frac{d'}{cd}\right) = \left(\frac{-a_2}{a_1}\right) h\left(\frac{1}{a}, \frac{1}{c}\right) h\left(\frac{b'}{b}, \frac{d'}{d}\right) \quad (2.5)$$

Again taking $b' = d' = 1$ in (2.3) and using (2.4), we get

$$h\left(\frac{a'}{ab}, \frac{c'}{cd}\right) = \left(\frac{-a_2}{a_1}\right) h\left(\frac{1}{b}, \frac{1}{d}\right) h\left(\frac{a'}{a}, \frac{c'}{c}\right) \quad (2.6)$$

Now (2.3) together with (2.4), (2.5) and (2.6) reduces to

$$h\left(\frac{a'b'}{ab}, \frac{c'd'}{cd}\right) = \left(\frac{-a_2}{a_1}\right) h\left(\frac{a'}{a}, \frac{c'}{c}\right) h\left(\frac{b'}{b}, \frac{d'}{d}\right) \quad (2.7)$$

Putting $\frac{a'}{a} = p, \frac{c'}{c} = u, \frac{b'}{b} = p', \frac{d'}{d} = u'$ in (2.7), we get the required results (2.2).

Next we obtain the general continuous solution of (2.2).

Lemma2.3 One of the general continuous solutions of equation (2.2) is given by

$$h(p, u) = \left(\frac{-a_1}{a_2}\right) \left(\frac{p^\mu u^\nu}{pu}\right)^{\frac{1}{\mu}}, \text{ where } \mu \neq 0, \nu \neq 0 \quad (2.8)$$

and

$$h(p, u) = 0 \quad (2.9)$$

Proof: Taking $g(p, u) = \left(\frac{-a_2}{a_1}\right) h(p, u)$ in (2.2), we have

$$g(pp', uu') = g(p, u)g(p', u') \quad (2.10)$$

The most general continuous solution of (2.10) (refer to Aczel [1]) is given by

$$g(p, u) = \left(\frac{p^\mu u^\nu}{pu}\right)^{\frac{1}{\mu}}, \mu \neq 0 \text{ and } \nu \neq 0 \quad (2.11)$$

and

$$g(p, u) = 0 \quad (2.12)$$

On substituting $g(p, u) = \left(\frac{-a_2}{a_1}\right) h(p, u)$ in (2.11) and (2.12) we get (2.8) and (2.9) respectively. This proves the lemma

2.3 for all rationals $p \in]0, 1[$ and $u > 0$, However, by continuity it holds for all reals $p \in]0, 1[$ and $u > 0$.

Theorem2.1 The measure (1.7) can be determined from the axioms 2.1 to 2.4.

Proof: Substituting the solution (2.8) in axiom 2.1, we have

$$G_n(P; U) = a_1 \left[1 - \sum_{i=1}^n \left(\frac{p_i^\mu u_i^\nu}{p_i u_i} \right)^{\frac{1}{\mu}} \right], \quad \mu \nu \neq 0 \quad (2.13)$$

Taking $p_i = \frac{1}{n}$ and $u_i = u$ for each i in (2.13), we get

$$G_n\left(\frac{1}{n}, \dots, \frac{1}{n}, u, \dots, u\right) = a_1 \left(1 - n^{\frac{1-\mu}{\mu}} u^{\frac{\nu-1}{\mu}} \right), \quad n = 2, 3, \dots, \quad (2.14)$$

Axiom 2.4 together with (2.14) gives

$$a_1 \left(1 - n^{\frac{1-\mu}{\mu}} u^{\frac{\nu-1}{\mu}} \right) = \frac{R}{R + \beta - 2\alpha} \left[1 - n^{\frac{-\beta+2\alpha-R}{R}} \right]$$

it implies

$$a_1 = \frac{R}{R + \beta - 2}, \quad \mu = \frac{R}{2\alpha - \beta}, \quad \nu = 1$$

Putting these values in (2.13) we have

$$G_n(P; U) = \frac{R}{R + \beta - 2\alpha} \left[1 - \left(\frac{\sum_{i=1}^n u_i p_i^{\frac{R}{2\alpha-\beta}}}{\sum_{i=1}^n u_i p_i} \right)^{\frac{2\alpha-\beta}{R}} \right]$$

$$= H_R^{\alpha, \beta}(P; U)$$

Hence this completes the proof of theorem 2.1.

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3. PROPERTIES OF ‘USEFUL’ R-NORM INFORMATION MEASURE OF TYPE α AND DEGREE β

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The generalized ‘useful’ R-norm information measure $H_R^{\alpha, \beta}(P; U)$ satisfies the following properties:

Property 3.1 $H_R^{\alpha, \beta}(P; U)$ is a symmetric function of their arguments provided that permutation of p_i^s and u_i^s are taken together.

$$H_R^{\alpha, \beta}(p_1, p_2, \dots, p_{n-1}, p_n; u_1, u_2, \dots, u_{n-1}, u_n) = H_R^{\alpha, \beta}(p_n, p_1, p_2, \dots, p_{n-1}; u_n, u_1, u_2, \dots, u_{n-1}).$$

Property 3.2 $H_R^{\alpha, \beta}\left(\frac{1}{8}, \frac{1}{8}; 1, 1\right) = 1$, when $\alpha = \beta = 1$ and $R = 2$.

$$\textbf{Proof: } H_R^{\alpha, \beta}(P; U) = \frac{R}{R + \beta - 2\alpha} \left[1 - \left(\frac{\sum_{i=1}^n u_i p_i^{\frac{R}{2\alpha-\beta}}}{\sum_{i=1}^n u_i p_i} \right)^{\frac{2\alpha-\beta}{R}} \right]$$

for $i = 1, 2$

$$H_R^{\alpha, \beta}(P; U) = \frac{R}{R + \beta - 2\alpha} \left[1 - \left(\frac{u_1 p_1^{\frac{R}{2\alpha-\beta}}}{u_1 p_1} + \frac{u_2 p_2^{\frac{R}{2\alpha-\beta}}}{u_2 p_2} \right)^{\frac{2\alpha-\beta}{R}} \right]$$

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Setting

$$p_1 = \frac{1}{8}, \quad p_2 = \frac{1}{8}, \quad u_1 = 1, \quad u_2 = 1, \quad \alpha = 1, \quad \beta = 1 \quad \text{and} \quad R = 2$$

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$$\begin{aligned}
 H_2^{1,1}\left(\frac{1}{8}, \frac{1}{8}; 1, 1\right) &= 2 \left[1 - \left\{ \frac{\left(\frac{1}{8}\right)^2}{\frac{1}{8}} + \frac{\left(\frac{1}{8}\right)^2}{\frac{1}{8}} \right\}^{\frac{1}{2}} \right] \\
 &= 2 \left[1 - \left(\frac{1}{4}\right)^{\frac{1}{2}} \right] = 1
 \end{aligned}$$

Property 3.3 Addition of two events whose probability of occurrence is zero or utility is zero has no effect on 'useful' information, i.e.

$$\begin{aligned}
 H_R^{\alpha, \beta}(p_1, p_2, \dots, p_n, 0; u_1, u_2, \dots, u_{n+1}) &= H_R^{\alpha, \beta}(p_1, p_2, \dots, p_n; u_1, u_2, \dots, u_n) \\
 &= H_R^{\alpha, \beta}(p_1, p_2, \dots, p_{n+1}; u_1, u_2, \dots, u_n, 0)
 \end{aligned}$$

Proof: Let us consider

$$\begin{aligned}
 &H_R^{\alpha, \beta}(p_1, p_2, \dots, p_n, 0; u_1, u_2, \dots, u_{n+1}) \\
 &= \frac{R}{R + \beta - 2\alpha} \left[1 - \left\{ \frac{u_1 p_1^{\frac{R}{2\alpha - \beta}}}{u_1 p_1} + \frac{u_2 p_2^{\frac{R}{2\alpha - \beta}}}{u_2 p_2} + \dots + \frac{u_n p_n^{\frac{R}{2\alpha - \beta}}}{u_n p_n} + \dots + 0 \right\}^{\frac{2\alpha - \beta}{R}} \right] \\
 &= H_R^{\alpha, \beta}(P; U)
 \end{aligned}$$

Property 3.4 $H_R^{\alpha, \beta}(P; U)$ satisfies the non additivity of the following form:

$$H_R^{\alpha, \beta}(P * Q; U * V) = H_R^{\alpha, \beta}(P; U) + H_R^{\alpha, \beta}(Q; V) - \frac{R + \beta - 2\alpha}{R} H_R^{\alpha, \beta}(P; U) H_R^{\alpha, \beta}(Q; V), \text{ where}$$

$$P * Q = (p_1 q_1, \dots, p_1 q_m; p_2 q_1, \dots, p_2 q_m, p_n q_1, \dots, p_n q_m)$$

$$U * V = (u_1 v_1, \dots, u_1 v_m, u_2 v_1, \dots, u_2 v_m, u_n v_1, \dots, u_n v_m)$$

$$\text{Proof: R.H.S.} = H_R^{\alpha, \beta}(P; U) + H_R^{\alpha, \beta}(Q; V) - \frac{R + \beta - 2\alpha}{R} H_R^{\alpha, \beta}(P; U) H_R^{\alpha, \beta}(Q; V)$$

$$\begin{aligned}
 &= \frac{R}{R + \beta - 2\alpha} \left[1 - \left\{ \frac{\sum_{i=1}^n u_i v_j (p_i q_j)^{\frac{R}{2\alpha - \beta}}}{\sum_{i=1}^n u_i v_j p_i q_j} \right\}^{\frac{2\alpha - \beta}{R}} \right] \\
 &= H_R^{\alpha, \beta}(P; Q; U; V) \\
 &= \text{L.H.S.}
 \end{aligned}$$

Property 3.5 Let A_i, A_j be two events having probabilities p_i, p_j and utilities u_i, u_j respectively, then define the utility u of compound event $A_i \cap A_j$ as

$$u(A_i \cap A_j) = \frac{u_i p_i + u_j p_j}{p_i + p_j} \quad (3.1)$$

Theorem 3.1 under the composition law (3.1) following holds:

$$\begin{aligned} & {}_{n+1}H_R^\beta(p_1, p_2, \dots, p_{n+1}, p', p''; u_1, u_2, \dots, u_{n+1}, u', u'') \\ &= {}_nH_R^{\alpha, \beta}(p_1, p_2, \dots, p_{n+1}; u_1, u_2, \dots, u_{n+1}) + (p' + p'')H_R^{\alpha, \beta}\left(\frac{p'}{p' + p''}, \frac{p''}{p' + p''}; u', u''\right) \end{aligned}$$

$$\textbf{Proof: } {}_{n+1}H_R^{\alpha, \beta}(p_1, p_2, \dots, p_{n+1}, p', p''; u_1, u_2, \dots, u_{n+1}, u', u'')$$

$$\begin{aligned} &= {}_nH_R^{\alpha, \beta}(p_1, p_2, \dots, p_{n+1}; u_1, u_2, \dots, u_{n+1}) + \frac{R}{R + \beta - 2\alpha} \left[1 - \left\{ \frac{u' p'^{\frac{R}{2\alpha - \beta}}}{u' p'} + \frac{u'' p''^{\frac{R}{2\alpha - \beta}}}{u'' p''} \right\}^{\frac{2\alpha - \beta}{R}} \right] \\ &= {}_nH_R^{\alpha, \beta} + \frac{R}{R + \beta - 2\alpha} \left[(p' + p'') - \left\{ \frac{u' \left(\frac{p'}{p' + p''} \right)^{\frac{R}{2\alpha - \beta}}}{u' \frac{p'}{p' + p''}} + \frac{u'' \left(\frac{p''}{p' + p''} \right)^{\frac{R}{2\alpha - \beta}}}{u'' \frac{p''}{p' + p''}} \right\} (p' + p'') \right] \\ &= {}_nH_R^{\alpha, \beta} + (p' + p'')H_R^{\alpha, \beta}\left[\frac{p'}{p' + p''}, \frac{p''}{p' + p''}; u', u''\right] \end{aligned}$$

This completes the proof of theorem 3.1.

4. MEAN CODEWORD LENGTH AND ITS BOUNDS

Let a finite set of n input symbols $X = \{x_1, x_2, \dots, x_n\}$ be encoded using D size alphabets with Probability distribution $P = \left\{ (p_1, p_2, \dots, p_n), p_i \geq 0, \sum_{i=1}^n p_i = 1 \right\}$. It was shown by Kraft [11] that there is a unique decipherable code with code words lengths l_i ($i = 1, 2, \dots, n$) satisfying the following inequality:

$$\sum_{i=1}^n D^{-l_i} \leq 1, \quad (4.1)$$

which is Known as Kraft's inequality

$$\text{Let } L = \sum_{i=1}^n p_i l_i \log D \quad (4.2)$$

be the mean codeword length associated with input symbols $\{x_1, x_2, \dots, x_n\}$ then under the Kraft's inequality Shannon [14] proved the following result for a noiseless channel:

$$H(P) \leq L < H(P) + \log D, D \geq 2, \quad (4.3)$$

with equality if and only if $l_i = -\log_D p_i$.

Guiasu and Picard [5] considered the problem of encoding by means of a single letter prefix code whose code words $w_1, w_2, \dots, w_n$ are of lengths $l_1, l_2, \dots, l_n$ respectively and satisfy the Kraft's inequality (4.1)

They introduced the following 'useful' mean codeword length:

$$L = \frac{\sum_{i=1}^n u_i p_i l_i}{\sum_{i=1}^n u_i p_i}. \quad (4.4)$$

Later on Longo [12] interpreted as the average transmission cost of letters x_i^s and obtained the following bounds for the cost measure (4.4):

$$H(P; U) \leq L < H(P; U) + 1. \quad (4.5)$$

Hooda et al. [9] considered following generalization of (4.4)

$$L_R(P; U) = \frac{R}{R-1} \left[1 - \frac{\sum_{i=1}^n u_i p_i D^{-l_i \left(\frac{R-1}{R} \right)}}{\sum_{i=1}^n u_i p_i} \right] \quad (4.6)$$

and proved

$$H_R(P; U) \leq L_R(P; U) < H_R(P; U) + 1 \quad (4.7)$$

under the condition

$$\sum_{i=1}^n u_i D^{-l_i} \leq \sum_{i=1}^n u_i p_i. \quad (4.8)$$

Here we introduce a new generalized 'useful' mean code word length as given below:

$$L_R^{\alpha, \beta}(P; U) = \frac{R}{R + \beta - 2\alpha} \left[1 - \frac{\sum_{i=1}^n u_i p_i D^{-l_i \left(\frac{R + \beta - 2\alpha}{R} \right)}}{\sum_{i=1}^n u_i p_i} \right], \quad (4.9)$$

In case $\alpha = 1$ and $\beta = 1$, then (4.9) reduces to (4.6) and again if utilities are ignored for all $i = 1, 2, \dots, n$ it reduces to

$$L_R(P) = \frac{R}{R-1} \left[1 - \sum_{i=1}^n p_i D^{-l_i \left(\frac{R-1}{R} \right)} \right], \quad (4.10)$$

which is average codeword length due to Boekke and Lubbe [3]. Thus (4.9) is a valid non-additive 'useful' mean codeword length.

5. A CODING THEOREM

In this section we study the lower and upper bounds of $L_R^{\alpha,\beta}(P;U)$ given by (4.9) in terms of 'useful' R-norm information measure of type α and degree β given by (1.7).

Theorem 5.1 If $l_i, i=1,2,\dots,n$ is length of codeword x_i satisfying (4.1) then

$$H_R^{\alpha,\beta}(P;U) \leq L_R^{\alpha,\beta}(P;U), R(\neq 1) > 0, \alpha \geq 1, 0 < \beta \leq 1. \quad (5.1)$$

under the condition

$$\sum_{i=1}^n u_i D^{-l_i} \leq \sum_{i=1}^n u_i p_i. \quad (5.2)$$

Proof: By Holder's inequality we have

$$\left(\sum_{i=1}^n x_i^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^n y_i^q \right)^{\frac{1}{q}} \leq \sum_{i=1}^n x_i y_i, \quad (5.3)$$

where $x_i, y_i \geq 0$ for each i and $\frac{1}{p} + \frac{1}{q} = 1$

$$\text{setting } x_i = \left(\frac{u_i p_i}{\sum_{i=1}^n u_i p_i} \right)^{\frac{R}{R+\beta-2\alpha}} D^{-l_i}, \quad y_i = \left(\frac{u_i p_i^{\frac{R}{2\alpha-\beta}}}{\sum_{i=1}^n u_i p_i} \right)^{\frac{2\alpha-\beta}{2\alpha-\beta-R}}$$

$$p = \frac{R+\beta-2\alpha}{R}, \quad q = \frac{2\alpha-\beta-R}{2\alpha-\beta}$$

Put all these values in (5.3) we have

$$\left(\frac{\sum_{i=1}^n u_i p_i D^{-l_i \left(\frac{R+\beta-2\alpha}{R} \right)}}{\sum_{i=1}^n u_i p_i} \right)^{\frac{R}{R+\beta-2\alpha}} \left(\frac{\sum_{i=1}^n u_i p_i^{\frac{R}{2\alpha-\beta}}}{\sum_{i=1}^n u_i p_i} \right)^{\frac{2\alpha-\beta}{2\alpha-\beta-R}} \leq \frac{\sum_{i=1}^n u_i D^{-l_i}}{\sum_{i=1}^n u_i p_i} \leq 1$$

It implies

$$\left(\frac{\sum_{i=1}^n u_i p_i^{\frac{R}{2\alpha-\beta}}}{\sum_{i=1}^n u_i p_i} \right)^{\frac{2\alpha-\beta}{2\alpha-\beta-R}} \leq \left(\frac{\sum_{i=1}^n u_i p_i D^{-l_i \left(\frac{R+\beta-2\alpha}{R} \right)}}{\sum_{i=1}^n u_i p_i} \right)^{\frac{R}{2\alpha-\beta-R}} \quad (5.4)$$

226

227

Case 1 Let $R + \beta < 2\alpha$ Raising Power $\frac{2\alpha - \beta - R}{R} > 0$ both sides of (5.4),

228

we have

229

$$\left(\frac{\sum_{i=1}^n u_i p_i^{\frac{R}{2\alpha-\beta}}}{\sum_{i=1}^n u_i p_i} \right)^{\frac{2\alpha-\beta}{R}} \leq \frac{\sum_{i=1}^n u_i p_i D^{-l_i \left(\frac{R+\beta-2\alpha}{R} \right)}}{\sum_{i=1}^n u_i p_i}$$

230

Subtracting both sides from 1, we get

231

$$1 - \left(\frac{\sum_{i=1}^n u_i p_i^{\frac{R}{2\alpha-\beta}}}{\sum_{i=1}^n u_i p_i} \right)^{\frac{2\alpha-\beta}{R}} \geq 1 - \frac{\sum_{i=1}^n u_i p_i D^{-l_i \left(\frac{R+\beta-2\alpha}{R} \right)}}{\sum_{i=1}^n u_i p_i} \quad (5.5)$$

232

Multiplying (5.5) by $\frac{R}{R + \beta - 2\alpha} < 0$ throughout we get

233

$$\frac{R}{R + \beta - 2\alpha} \left[1 - \left(\frac{\sum_{i=1}^n u_i p_i^{\frac{R}{2\alpha-\beta}}}{\sum_{i=1}^n u_i p_i} \right)^{\frac{2\alpha-\beta}{R}} \right] \leq \frac{R}{R + \beta - 2\alpha} \left[1 - \frac{\sum_{i=1}^n u_i p_i D^{-l_i \left(\frac{R+\beta-2\alpha}{R} \right)}}{\sum_{i=1}^n u_i p_i} \right]$$

234

$$H_R^{\alpha, \beta}(P; U) \leq L_R^{\alpha, \beta}(P; U) \quad (5.6)$$

235

Case 2 Let $R + \beta > 2\alpha$. Raising power $\frac{2\alpha - \beta - R}{R} < 0$ both sides of (5.4).

236

$$\left(\frac{\sum_{i=1}^n u_i p_i^{\frac{R}{2\alpha-\beta}}}{\sum_{i=1}^n u_i p_i} \right)^{\frac{2\alpha-\beta}{R}} \geq \frac{\sum_{i=1}^n u_i p_i D^{-l_i \left(\frac{R+\beta-2\alpha}{R} \right)}}{\sum_{i=1}^n u_i p_i} \quad (5.7)$$

237

Subtracting both sides from 1, we get

238

$$1 - \left(\frac{\sum_{i=1}^n u_i p_i^{\frac{R}{2\alpha-\beta}}}{\sum_{i=1}^n u_i p_i} \right)^{\frac{2\alpha-\beta}{R}} \leq 1 - \frac{\sum_{i=1}^n u_i p_i D^{-l_i \left(\frac{R+\beta-2\alpha}{R} \right)}}{\sum_{i=1}^n u_i p_i}$$

239

Multiplying (5.7) by $\frac{R}{R + \beta - 2\alpha} > 0$ throughout we get:

$$\frac{R}{R+\beta-2\alpha} \left[1 - \frac{\left(\sum_{i=1}^n u_i p_i^{\frac{R}{2\alpha-\beta}} \right)^{\frac{2\alpha-\beta}{R}}}{\sum_{i=1}^n u_i p_i} \right] \leq \frac{R}{R+\beta-2\alpha} \left[1 - \frac{\sum_{i=1}^n u_i p_i D^{-l_i \left(\frac{R+\beta-2\alpha}{R} \right)}}{\sum_{i=1}^n u_i p_i} \right]$$

$$H_R^{\alpha,\beta}(P;U) \leq L_R^{\alpha,\beta}(P;U) \quad (5.8)$$

Hence theorem 5.1 is proved in both cases.

In (5.1) equality holds if and only if

$$D^{-l_i} = \frac{p_i^{\frac{R}{2\alpha-\beta}}}{\frac{\sum_{i=1}^n u_i p_i^{\frac{R}{2\alpha-\beta}}}{\sum_{i=1}^n u_i p_i}}, \quad R > 0 (\neq 1).$$

or

$$l_i = -\log_D p_i^{\frac{R}{2\alpha-\beta}} + \log_D \left[\frac{\sum_{i=1}^n u_i p_i^{\frac{R}{2\alpha-\beta}}}{\sum_{i=1}^n u_i p_i} \right]$$

$$\log_D p_i^{-\frac{R}{2\alpha-\beta}} \left[\frac{\sum_{i=1}^n u_i p_i^{\frac{R}{2\alpha-\beta}}}{\sum_{i=1}^n u_i p_i} \right] \leq l_i < \log_D p_i^{-\frac{R}{2\alpha-\beta}} \left[\frac{\sum_{i=1}^n u_i p_i^{\frac{R}{2\alpha-\beta}}}{\sum_{i=1}^n u_i p_i} \right] + 1 \quad (5.9)$$

It implies

$$p_i^{-\frac{R}{2\alpha-\beta}} \left[\frac{\sum_{i=1}^n u_i p_i^{\frac{R}{2\alpha-\beta}}}{\sum_{i=1}^n u_i p_i} \right] \leq D^{l_i} < D p_i^{-\frac{R}{2\alpha-\beta}} \left[\frac{\sum_{i=1}^n u_i p_i^{\frac{R}{2\alpha-\beta}}}{\sum_{i=1}^n u_i p_i} \right] \quad (5.10)$$

In the next theorem, we obtain an upper bound on $L_R^{\alpha,\beta}(P;U)$ in terms of $H_R^{\alpha,\beta}(P;U)$.

Theorem 5.2 Let $l_1, l_2, \dots, l_n$ be the codeword lengths satisfying (510), then following inequality holds:

$$L_R^{\alpha,\beta}(P;U) \leq D^{\frac{2\alpha-\beta-R}{R}} H_R^{\alpha,\beta}(P;U) + \frac{R}{R+\beta-2\alpha} \left(1 - D^{\frac{2\alpha-\beta-R}{R}} \right) \quad (5.11)$$

Proof: From the right hand inequality of (5.10).

we have

$$D^{l_i} < D p_i^{-\frac{R}{2\alpha-\beta}} \left[\frac{\sum_{i=1}^n u_i p_i^{\frac{R}{2\alpha-\beta}}}{\sum_{i=1}^n u_i p_i} \right] \quad (5.12)$$

Here we two cases arise:

Case 1 Let $0 < R + \beta < 2\alpha$. Raising both sides of (5.12) to the power $\frac{2\alpha - \beta - R}{R} > 0$,

we get

$$D^{-l_i \left(\frac{R+\beta-2\alpha}{R} \right)} < D^{\frac{2\alpha-\beta-R}{R}} p_i^{\frac{R+\beta-2\alpha}{2\alpha-\beta}} \left[\frac{\sum_{i=1}^n u_i p_i^{\frac{R}{2\alpha-\beta}}}{\sum_{i=1}^n u_i p_i} \right]^{\frac{2\alpha-\beta-R}{R}} \quad (5.13)$$

Multiplying both sides of (5.13) by $\frac{u_i p_i}{\sum_{i=1}^n u_i p_i}$ and summing over i , we have

$$\frac{\sum_{i=1}^n u_i p_i D^{-l_i \left(\frac{R+\beta-2\alpha}{R} \right)}}{\sum_{i=1}^n u_i p_i} < D^{\frac{2\alpha-\beta-R}{R}} \left[\frac{\sum_{i=1}^n u_i p_i^{\frac{R}{2\alpha-\beta}}}{\sum_{i=1}^n u_i p_i} \right]^{\frac{2\alpha-\beta-R}{R}} \left[\frac{\sum_{i=1}^n u_i p_i^{\frac{R}{2\alpha-\beta}}}{\sum_{i=1}^n u_i p_i} \right]$$

or

$$\frac{\sum_{i=1}^n u_i p_i D^{-l_i \left(\frac{R+\beta-2\alpha}{R} \right)}}{\sum_{i=1}^n u_i p_i} < D^{\frac{2\alpha-\beta-R}{R}} \left[\frac{\sum_{i=1}^n u_i p_i^{\frac{R}{2\alpha-\beta}}}{\sum_{i=1}^n u_i p_i} \right]^{\frac{2\alpha-\beta}{R}}$$

Subtracting both sides from 1 and multiplying by $\frac{R}{R + \beta - 2\alpha} < 0$, we have

$$L_R^{\alpha, \beta}(P; U) < D^{\frac{2\alpha-\beta-R}{R}} H_R^{\alpha, \beta}(P; U) + \frac{R}{R + \beta - 2\alpha} \left[1 - D^{\frac{2\alpha-\beta-R}{R}} \right] \quad (5.14)$$

Similarly we can prove that (5.14) holds when $R + \beta > 2\alpha$. Hence theorem 5.2 is proved

Thus (5.1) and (5.11) together give

$$H_R^{\alpha,\beta}(P:U) \leq L_R^{\alpha,\beta}(P;U) < D^{\frac{2\alpha-\beta-R}{R}} H_R^{\alpha,\beta}(P;U) + \frac{R}{R+\beta-2\alpha} \left(1 - D^{\frac{2\alpha-\beta-R}{R}} \right),$$

which is analogous to Shannon's coding theorem for noiseless channel.

CONCLUSION

In this paper we have defined a new generalized 'useful' R-norm information measure analogous to Hooda and Sharma's (2008) R-norm information measure and characterized axiomatically. Some important properties have also been studied.

The new generalized measure has been applied in obtaining lower and upper bounds of the generalized 'useful' mean code word lengths. This can be generalized parametrically and applied in source coding.

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