

A short note on weak estimation of Sharp function

Abstract

In paper [2] Ahmad et al investigated the use of sharp function, known from functional analysis, in image processing. The sharp function gives a measure of variations of a function and can be used as an edge detector [3]. We extend the classical notion of sharp function for proving the classical Lebesgue differentiation theorem and Marcinkiewicz theorem for the sub-linear operator $T(x, y)$.

Key words: Maximal function, Bounded Mean Oscillations (BMO), Sharp function, Distribution, Sub-linear Operator.

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1 Introduction

The Sharp function is a well known functional analytic concept to measure the oscillatory behavior of functions. It goes back to the maximal function which was introduced by Hardy and Littlewood in 1930 to solve a problem in the theory of complex variable. Based on this idea John and Nirenberg introduced the concept of bounded mean oscillation (BMO) functions. In 1972 Fefferman and Stein introduced the sharp function (denoted by $f^\#$) and found that a function $f \in BMO$ is equivalent with $f^\# \in L^\infty$.

The idea of applying the sharp operator to measure the oscillation and classification of images was first proposed by Ahmad et al [2] where it was used to find a suitable compression technique.

In this paper, we use sharp function in proving the Marcinkiewicz Interpolation theorem for the weak sub-linear operator $T(x, y)$ and for generalization of the concept of distribution. This paper also includes the proof of Lebesgue differentiation theorem by the use of sharp function.

2 Maximal Function

In this section we give a short introduction to the maximal function and its background. There is a rich theory behind it, and we are going to point out the main results connected to it. The Hardy-Littlewood maximal function was developed to solve a problem in the theory of functions of complex variable.

Definition 2.1 Let $\mathbb{R}^n$ be the n -dimensional Euclidean space and $f(x)$ be a real valued measurable function on $\mathbb{R}^n$. For such a function f on $\mathbb{R}^n$ its Hardy-Littlewood maximal function is defined by the formula

$$Mf(x) = \sup_{Q: x \in Q} \left\{ \frac{1}{\lambda(Q)} \int_Q |f(y)| dy : Q \subset \mathbb{R}^n, x \in Q \right\}, \quad (1)$$

where the supremum ranges over all finite cubes Q in $\mathbb{R}^n$ and $\lambda(Q)$ is the Lebesgue measure of Q .

Definition 2.2 Let $\mathbb{R}^n$ be the n -dimensional Euclidean space and $f(x)$ be a real valued measurable function on $\mathbb{R}^n$. For such a function f on $\mathbb{R}^n$ its distribution is defined by the formula

$$d_f(t) = \lambda(\{x \in \mathbb{R}^n : |f(x)| > t\}), \text{ for } t \geq 0. \quad (2)$$

It is easy to find a function whose maximal function is unbounded.

Example 2.3 For $f(x) = |x|^t$ with $t > 0$, we get $Mf(x) = \infty$, for each $x \in \mathbb{R}$.

Theorem 2.4 [Hardy-Littlewood maximal theorem] For each function $f \in L^1(\mathbb{R}^n)$ we have

$$\lambda(\{x : Mf(x) > t\}) \leq 6^n t^{-1} \|f\|_1, \quad t > 0. \quad (3)$$

Proof: If $\|f\|_1 = \infty$, then it is trivial. Without loss of generality we can assume that $\|f\|_1 = 1$. For a fixed $t \geq 0$, let $E_t = \{x : Mf(x) > t\}$. Then for each $x \in E_t$ there is a cube Q_x such that $x \in Q_x \subset E_t$, and

$$\frac{1}{\lambda(Q_x)} \left\{ \int_{Q_x} |f(y)| dy \right\} > t. \quad (4)$$

Thus,

$$\frac{\|f\|_1}{t} \geq \lambda(Q_x) \quad \text{or} \quad \frac{1}{t} \geq \lambda(Q_x). \quad (5)$$

Let $\alpha_1 = \max\{\lambda(Q_x) : x \in E_t\}$. So $\alpha_1 \leq t^{-1}$ and let us fix a cube Q_x , call it Q_1 such that $\lambda(Q_1) > \frac{1}{2}\alpha_1$. Consider all cubes Q_x such that $Q_x \cap Q_1 = \emptyset$. If there are no such cubes we stop; if there are cubes we put $\alpha_2 = \max\{\lambda(Q_x) : x \in E_t \text{ and } Q_x \cap Q_1 = \emptyset\}$ and fix such a cube Q_x , call it Q_2 satisfying $\lambda(Q_2) > \frac{1}{2}\alpha_2$. Continuing in this way we get a sequence of cubes $Q_1, Q_2, \dots$, possibly finite such that

- (i) the cubes Q_j are disjoint,
- (ii) $\lambda(Q_j) > \frac{1}{2} \max\{\lambda(Q_x) : Q_x \cap Q_s = \emptyset ; \text{ for } s = 1, 2, 3, \dots, j-1\}$,
- (iii) if $Q_x \cap Q_s = \emptyset$ for $s = 1, 2, 3, \dots, j-1$, then $\lambda(Q_x) \leq 2\lambda(Q_j)$.

From (4) and (i), we get

$$\lambda\left(\bigcup_{i=1}^n Q_i\right) = \sum_{i=1}^n \lambda(Q_i) \leq \frac{1}{t} \sum_{i=1}^n \int_{Q_i} |f(y)| dy \leq \frac{1}{t} \int_{\bigcup Q_i} |f(y)| dy \leq \frac{\|f\|_1}{t} = \frac{1}{t}. \quad (6)$$

It is important to mention here that each Q_x intersects some Q_i . If there exists a Q_x disjoint from all Q_i 's then our process was infinite. So from (6), we see that $\lambda(Q_j) \rightarrow \infty$ which contradicts (iii). Now for a given Q_x , let Q_s be the first Q_i that intersects with Q_x . So by (iii), $\lambda(Q_x) \leq 2\lambda(Q_s)$ and hence $Q_x \subset 6 \diamond Q_s$ (by $c \diamond Q$ we mean a cube with the same center as Q but where sides are c times longer than sides of Q , $c > 0$). Thus from equation (6), we have

$$\begin{aligned} \lambda(\{x : Mf(x) > t\}) &= \lambda(\bigcup_x Q_x) \\ &\leq \lambda(\bigcup_i 6 \diamond Q_i) \\ &\leq \sum_i \lambda(6 \diamond Q_i) \\ &\leq 6^n \sum_i \lambda(Q_i) \leq 6^n \frac{1}{t} = 6^n \frac{\|f\|_1}{t}. \end{aligned} \quad (7)$$

An interesting application of the Maximal Theorem is a version of the Lebesgue differentiation theorem.

Theorem 2.5 *Let $f \in L^1(\mathbb{R}^n)$. For almost all $x \in \mathbb{R}^n$ and for every decreasing sequence of cubes $\{Q_j\}_{j=1}^\infty$ such that $\bigcap_{j=1}^\infty Q_j = \{x\}$, we have*

$$\lim_{j \rightarrow \infty} \frac{1}{\lambda(Q_j)} \int_{Q_j} f(y) dy = f(x). \quad (8)$$

Proof: If $f \in L^1(\mathbb{R}^n) \cap C(\mathbb{R}^n)$, then (8) holds for all $x \in \mathbb{R}^n$. Given $f \in L^1(\mathbb{R}^n)$ we take ϵ such that $0 < \epsilon < 1$ and write $f = g + h$ with $g \in L^1(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ and $\|h\|_1 < \epsilon$. Then by the above observation and by the definition of Hardy-Littlewood maximal function, we have,

$$\begin{aligned}
& \lim_{j \rightarrow \infty} \sup \left| \frac{1}{\lambda(Q_j)} \int_{Q_j} f(y) dy - f(x) \right| \\
&= \lim_{j \rightarrow \infty} \sup \left| \frac{1}{\lambda(Q_j)} \int_{Q_j} (g+h)(y) dy - (g+h)(x) \right| \\
&= \lim_{j \rightarrow \infty} \sup \left| \frac{1}{\lambda(Q_j)} \int_{Q_j} g(y) dy - g(x) + \frac{1}{\lambda(Q_j)} \int_{Q_j} h(y) dy - h(x) \right| \\
&= \lim_{j \rightarrow \infty} \sup \left| \frac{1}{\lambda(Q_j)} \int_{Q_j} h(y) dy - h(x) \right| \\
&\leq \lim_{j \rightarrow \infty} \sup \left\{ \frac{1}{\lambda(Q_j)} \int_{Q_j} |h(y)| dy \right\} + |h(x)| \\
&\leq |h(x)| + Mh(x).
\end{aligned}$$

Now by equations (2) and (3), we get

$$\lambda(\{x \in \mathbb{R}^n : |h(x)| > \sqrt{\epsilon}\}) \leq \frac{\|h\|_1}{\sqrt{\epsilon}} \leq \sqrt{\epsilon}$$

and

$$\lambda(\{x : Mh(x) > \sqrt{\epsilon}\}) \leq 6^n \frac{\|h\|_1}{\sqrt{\epsilon}} \leq 6^n \sqrt{\epsilon}$$

respectively. This shows that outside the set $\{x \in \mathbb{R}^n : Mh(x) > \sqrt{\epsilon}\} \cup \{x \in \mathbb{R}^n : |h(x)| > \sqrt{\epsilon}\}$ which has measure at most $C\sqrt{\epsilon}$, we have

$$\lim_{j \rightarrow \infty} \sup \left| \frac{1}{\lambda(Q_j)} \int_{Q_j} f(y) dy - f(x) \right| \leq 2\sqrt{\epsilon}.$$

Since ϵ was arbitrary small, we obtain

$$\lim_{j \rightarrow \infty} \frac{1}{\lambda(Q_j)} \int_{Q_j} f(y) dy = f(x).$$

The Hardy-Littlewood maximal operator M is a significant non-linear operator used in real and harmonic analysis. It takes a locally integrable function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ and returns another function Mf that at each $x \in \mathbb{R}^n$ gives the maximal average value that f can have on cubes containing the point $x \in Q_x$. Hardy-Littlewood maximal inequality states that M is bounded as a sub-linear operator from $L^p(\mathbb{R}^n)$ to itself for $p > 1$.

Definition 2.6 A measurable function f on $\mathbb{R}^n$ has bounded p -mean oscillation, $1 \leq p < \infty$, if

$$\|f\|_{BMO_p} = \sup_Q \left\{ \frac{1}{\lambda(Q)} \int_Q |f(x) - f_Q|^p dx \right\}^{1/p} < \infty$$

where the sup ranges over all finite cubes Q in $\mathbb{R}^n$ and $f_Q = \frac{1}{\lambda(Q)} \int_Q f(x) dx$ is the mean value of the function f on the cube Q .

The set of all functions of bounded p -mean oscillation is denoted by $BMO_p(\mathbb{R}^n)$. $\|f\|_{BMO_p}$ is “almost” a norm since it has the following properties

- (i) $\|f + g\|_{BMO_p} \leq \|f\|_{BMO_p} + \|g\|_{BMO_p}$
- (ii) $\|\alpha f\|_{BMO_p} \leq |\alpha| \|f\|_{BMO_p}$
- (iii) $\|f\|_{BMO_p} = 0$ iff $f = \text{constant a.e.}$,

where f and g are the measurable functions on $\mathbb{R}^n$ and α is some scalar quantity.

Example 2.7 *The function*

$$f(x) = \begin{cases} 1 & \text{if } x \in [0, 1] \\ 0 & \text{if } x \notin [0, 1] \end{cases}$$

is in $BMO_1(\mathbb{R})$.

Let $Q = [0, 1]$, then

$$\frac{1}{b-a} \int_a^b |f(x) - f_Q| dx = \int_0^1 |f(x) - 1| dx = 0 < \infty.$$

Every bounded function has a bounded p -mean oscillations.

That is, $L^\infty(\mathbb{R}^n) \subset BMO_1(\mathbb{R}^n)$, $1 \leq p < \infty$.

Example 2.8 *For $f(x) = |x|^\alpha$ with $\alpha \in (-1, 0)$, we find maximal function $Mf(x)$ of $f(x)$ as*

$$Mf(x) = \sup_{x \in I} \left\{ \frac{1}{|I|} \int_I |f(t)| dt \right\}.$$

We know that $Mf(x) = \max\{Mf_+(x), Mf_-(x)\}$. Assume that $x > 0$, then clearly,

$$Mf(x) = Mf_-(x) = \sup_{y < x} \frac{1}{x-y} \int_y^x |f(t)| dt = \sup_{y < x} \frac{1}{x-y} \int_y^x |t|^\alpha dt.$$

Further, to obtain sup we consider $y < 0$, then we have

$$Mf(x) = \sup_{y < 0} \frac{1}{x-y} \left\{ \frac{|t|^{\alpha+1}}{\alpha+1} \right\}_y^x.$$

Since $y < 0$, so that choose $y = -s$, where $s > 0$ then

$$\begin{aligned} Mf(x) &= \sup_{s > 0} \frac{1}{x+s} \left\{ \frac{x^{\alpha+1} + s^{\alpha+1}}{\alpha+1} \right\} \\ &= \sup_{s > 0} \frac{1}{x(1+(s/x))} \left\{ \frac{x^{\alpha+1}(1+(s/x)^{\alpha+1})}{\alpha+1} \right\} \\ &= \sup_{s > 0} x^\alpha \left\{ \frac{(1+(s/x)^{\alpha+1})}{(1+(s/x))(1+\alpha)} \right\}. \end{aligned}$$

Putting $\frac{s}{x} = t > 0$, we get

$$Mf(x) = x^\alpha \sup_{t>0} \left\{ \frac{1 + t^{\alpha+1}}{(1+t)(1+\alpha)} \right\} = x^\alpha C_\alpha.$$

C_α tends to 0 as t goes to infinity and the supremum is obtained.

3 Sharp Function

Sharp function is a very powerful tool that mediates between L^p spaces and the space of bounded mean oscillations, i.e., BMO_p .

Definition 3.1 For locally integrable function f on $\mathbb{R}^n$ the sharp function is given by

$$f^\#(x) = \sup_{Q:x \in Q} \left\{ \frac{1}{\lambda(Q)} \int_{\mathbb{R}^n} |f(y) - f_Q|^p dy \right\}^{1/p} < \infty.$$

Thus, $f \in BMO$ is identical with the statement $f^\# \in L^\infty$. The interest in $f^\#$ lies in the fact that $f^\# \in L^p, p < \infty$ implies that $f \in L^p$.

Proposition 3.2 If $f^\#$ is a function in $L^p(\mathbb{R}^n)$, and $1 \leq p < \infty$, then

$$d_{f^\#}(t) \leq t^{-p} \|f^\#(x)\|_p^p \quad (9)$$

and

$$\|f^\#(x)\|_p^p = p \int_0^\infty t^{p-1} d_{f^\#}(t) dt. \quad (10)$$

Proof. By the definition of the distribution

$$\begin{aligned} d_{f^\#}(t) &= \lambda(\{x \in \mathbb{R}^n : |f^\#(x)| > t\}) \\ &= \int_{\{x \in \mathbb{R}^n : |f^\#(x)| > t\}} dx \end{aligned} \quad (11)$$

or,

$$\begin{aligned} t^p d_{f^\#}(t) &= \int_{\{x \in \mathbb{R}^n : |f^\#(x)| > t\}} t^p dx \\ &\leq \int_{\mathbb{R}^n} |f^\#(x)|^p dx. \end{aligned} \quad (12)$$

Since $f^\#$ is integrable on $\mathbb{R}^n$, then we have

$$t^p d_{f^\#}(t) \leq \|f^\#(x)\|_p^p$$

$$d_{f^\#}(t) \leq t^{-p} \|f^\#(x)\|_p^p.$$

For the remaining part we define the set $A \subset \mathbb{R}^n \times [0, \infty)$ as $A = \{(x, s) : s < |f^\#(x)|^p\}$. Using Fubini's theorem, we get

$$\begin{aligned} \int_{\mathbb{R}^n} |f^\#(x)|^p dx &= \int_{\mathbb{R}^n} \int_0^{|f^\#(x)|^p} \mathbf{1} ds dx \\ &= \int_{\mathbb{R}^n \times [0, \infty)} \mathbf{1}_A(x, s) ds dx \\ &= \int_0^\infty \lambda(\{x : |f^\#(x)|^p > s\}) ds. \end{aligned}$$

Putting $s = t^p \Rightarrow ds = pt^{p-1}dt$ in the above integral, we obtain

$$\begin{aligned} \int_{\mathbb{R}^n} |f^\#(x)|^p dx &= \int_0^\infty pt^{p-1} \lambda(\{x : |f^\#(x)|^p > t^p\}) dt \\ &= p \int_0^\infty t^{p-1} \lambda(\{x : |f^\#(x)| > t\}) dt. \end{aligned} \tag{13}$$

Therefore,

$$\|f^\#(x)\|_p^p = p \int_0^\infty t^{p-1} d_{f^\#}(t) dt.$$

This completes the proof.

Definition 3.3 *An operator T defined on some class of measurable functions and mapping it into measurable functions is called sublinear if*

$$|T(f+g)(x)| \leq |T(f)(x)| + |T(g)(x)| \quad a.e.$$

and

$$|T(\lambda f)(x)| \leq |\lambda| |T(f)(x)|, \quad a.e.,$$

for all admissible functions f, g and all scalars λ .

Definition 3.4 *A sublinear operator T defined on $L^1(\mathbb{R}^n)$ is of weak type $(1,1)$, if there exists a constant C such that for each $f \in L^1(\mathbb{R}^n)$ and each $t > 0$ we have*

$$\lambda(\{x : |T(f)(x)| > t\}) \leq \frac{C}{t} \|f\|_1.$$

It is clear that each linear operator is sublinear. It immediately follows from (9) that if a sublinear operator T satisfies

$$\|T(f)\|_1 \leq C \|f\|_1,$$

then T is of weak type $(1,1)$.

Theorem 3.5 [Marcinkiewicz] Suppose T is a sub-linear operator defined on $L^1(\mathbb{R}^n) + L^\infty(\mathbb{R}^n)$ which is of weak type $(1,1)$ and for some C satisfies

$$\|Tf^\#\|_\infty \leq C\|f^\#\|_\infty, \quad (14)$$

then for each $p, 1 < p < \infty$, there exists a constant $C(p)$ such that

$$\|Tf^\#\|_p \leq C(p)\|f^\#\|_p, \text{ where } C(p) = \frac{2^p p}{(p-1)}.$$

Proof. We are given that T is a sub-linear operator defined on $L^1(\mathbb{R}^n) + L^\infty(\mathbb{R}^n)$ and is of weak type $(1,1)$ then by the definition of T we can say without loss of generality that for $C = 1$ and $t > 0$,

$$\lambda(\{x : |Tf^\#(x)| > t\}) \leq \frac{\|f^\#\|_1}{t}.$$

And also T satisfies (14), i.e.,

$$\|Tf^\#\|_\infty \leq C\|f^\#\|_\infty$$

or,

$$\|Tf^\#\|_\infty \leq \|f^\#\|_\infty.$$

Now for $f^\# \in L^p(\mathbb{R}^n)$ and given $t > 0$, we write $f^\#(x) = f_t(x) + f^t(x)$, where $f_t(x) \in L^1(\mathbb{R}^n)$, $f^t(x) \in L^\infty(\mathbb{R}^n)$ and

$$f^t(x) = \begin{cases} f^\#(x) & x \geq t \\ 0 & x < t, \end{cases}$$

$$f_t(x) = \begin{cases} f^\#(x) & x \leq t \\ 0 & x > t. \end{cases}$$

Thus $f_t(x) = \mathbf{1}_{\{s : |f^\#(s)| \leq t\}} f^\#(x)$. Since $f^\#$ is a measurable function and T is a sub-linear operator, therefore $Tf^\#$ is also a measurable function. Then by countable sub-additivity, we have

$$\{x : |Tf^\#(x)| > t\} \subset \left\{x : |Tf_{t/2}(x)| > \frac{t}{2}\right\} \cup \left\{x : |Tf^{t/2}(x)| > \frac{t}{2}\right\}.$$

So from (10), we obtain

$$\begin{aligned} \|Tf^\#\|_p^p &= \int_{\mathbb{R}^n} |Tf^\#(x)|^p dx \\ &= p \int_0^\infty t^{p-1} \lambda(\{x : |Tf^\#(x)| > t\}) dt \end{aligned} \quad (15)$$

or,

$$\|Tf^\#\|_p^p \leq p \int_0^\infty t^{p-1} \lambda\left(\left\{x : |Tf_{\frac{t}{2}}(x)| > \frac{t}{2}\right\}\right) dt + p \int_0^\infty t^{p-1} \lambda\left(\left\{x : |Tf^{t/2}(x)| > \frac{t}{2}\right\}\right) dt.$$

But we suppose that T satisfies (14) with $C = 1$, and this implies that $\|Tf_{t/2}\|_\infty \leq \frac{t}{2}$ and thus, the second integral is zero. Then by the definition of sub-linear operator and (10) we get,

$$\begin{aligned} \|Tf^\#\|_p^p &\leq p \int_0^\infty t^{p-1} \frac{2}{t} \|f_{t/2}\|_1 dt \\ &= 2p \int_0^\infty t^{p-2} \int_{\{x: |f^\#(x)| > \frac{t}{2}\}} |f^\#(x)| dx dt \\ &= 2p \int_{\mathbb{R}^n} |f^\#(x)| \int_0^{2|f^\#(x)|} t^{p-2} dt dx \\ &= 2p \int_{\mathbb{R}^n} |f^\#(x)| \frac{1}{p-1} (2|f^\#(x)|)^{p-1} dx \\ &\leq \frac{2^p p}{(p-1)} \int_{\mathbb{R}^n} |f^\#(x)|^p dx \\ &\leq C(p) \|f^\#\|_p^p \quad 1 \leq p < \infty. \end{aligned}$$

This completes the proof. The relationship between Weak (p, q) inequalities and the almost convergence is given by the following result.

Theorem 3.6 *Let $\{T_t\}$ be a family of linear operators on $L^p(\mathbb{R}^n)$; $1 \leq p < \infty$ and define*

$$T^* f^\#(x) = \sup_t |T_t f^\#(x)|.$$

If T^* is weak (p, q) then the set

$$\{f^\# \in L^p(\mathbb{R}^n) : \lim_{t \rightarrow t_0} T_t f^\#(x) = f^\#(x) \text{ a.e.}\}$$

is closed in $L^p(\mathbb{R}^n)$.

Proof. Let $\{f_n\}$ be a sequence of functions which converges to $f^\#$ in $L^p(\mathbb{R}^n)$ such that $Tf^\#(x)$ converges to $f^\#(x)$ a.e.

Now,

$$\begin{aligned} &\mu(\{f^\# \in L^p(\mathbb{R}^n) : \limsup_{t \rightarrow t_0} |T_t f^\#(x) - f^\#(x)| > \lambda\}) \\ &= \mu(\{f^\# \in L^p(\mathbb{R}^n) : \limsup_{t \rightarrow t_0} |T_t f^\#(x) - f_n(x) + f_n(x) - f^\#(x)| > \lambda\}) \\ &\leq \mu(\{f^\# \in L^p(\mathbb{R}^n) : \limsup_{t \rightarrow t_0} |T_t(f^\# - f_n)(x)| > \frac{\lambda}{2}\}) \\ &\quad + \mu(\{f^\# \in L^p(\mathbb{R}^n) : \limsup_{t \rightarrow t_0} |(f^\# - f_n)(x)| > \frac{\lambda}{2}\}) \\ &\leq \mu(\{f^\# \in L^p(\mathbb{R}^n) : |T_t^*(f^\# - f_n)(x)| > \frac{\lambda}{2}\}) \\ &\quad + \mu(\{f^\# \in L^p(\mathbb{R}^n) : |f^\#(x) - f_n(x)| > \frac{\lambda}{2}\}) \\ &\leq \left(\frac{2C(p)}{\lambda} \|f^\#(x) - f_n(x)\|_p \right)^q + \left(\frac{2}{\lambda} \|f^\#(x) - f_n(x)\|_p \right)^p. \end{aligned}$$

Since T^* is weak (p, q) and $f_n \rightarrow f^\#$ as $n \rightarrow \infty$, therefore

$$\begin{aligned} & \mu(\{f^\# \in L^p(\mathbb{R}^n) : \limsup_{t \rightarrow t_0} |T_t f^\#(x) - f^\#(x)| > 0\}) \\ & \leq \sum_{k=1}^{\infty} \mu(\{f^\# \in L^p(\mathbb{R}^n) : \limsup_{t \rightarrow t_0} |T_t f^\#(x) - f^\#(x)| > \frac{1}{k}\}) \\ & = 0. \end{aligned}$$

This completes the proof.

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