

Original Research Article

**Computation of Eigenvalues of the Fourth Order Sturm-Liouville BVP
by Galerkin Weighted Residual Method**

Abstract

The aim of this paper is to find the eigenvalues of fourth order regular Sturm-Liouville problems (SLP) together with four boundary conditions which are specified at two boundary points. We use Bernstein polynomials as basis functions in the Galerkin weighted residual method to approximate the solutions. We derive rigorous matrix formulations for computing the eigenvalues of SLP. Special care is taken about how the polynomials satisfy the corresponding homogeneous form of *Dirichlet* boundary conditions. The approximate eigenvalues are compared with the exact result and also compared with the relevant studies by some authors. Excellent agreement is found in this study.

Keywords: Galerkin method, Bernstein polynomials, Sturm-Liouville problems, Eigenvalue.

1. Introduction

The concept of eigenvalue problem is rather important both in pure and applied mathematics, a physical system, such as pendulum, a vibrating or rotating shaft etc. The physical system such as pendulums and vibrating and rotating shafts are connected with eigenpairs of the system. The Sturm-Liouville systems arise from vibration problems in continuum mechanics. In the literature, we observe that many researchers studied second order Sturm Liouville eigenvalue problems. Some authors Abbasbandy and Shirazdi [13], Shi and Cao [14], Ycel and Boubaker [15], Gamel and Sameeh [16], Taher *et al* [17] paid their attention to develop various techniques for finding eigenvalues of fourth order Sturm Liouville BVP's. They applied different algorithms to minimize the convergence rates.

Chanane and Chanane [5, 6] introduced a novel series representation for the boundary/characteristic function associated with fourth-order Sturm-Liouville problems using the concepts of Fliess series and iterated integrals. Chawla [1] presented fourth-order finite-difference method for computing eigenvalues of fourth-order two-point boundary value problems. Usmani and Sakai [2] applied finite difference method of order two and four for computing eigenvalues of the fourth-order boundary value problems while Twizell and Matar [4] developed finite difference method for approximating the eigenvalues of fourth-order boundary value problems.

Jia *et al* [7] approximated the eigenvalues of fourth order BVP for a class of crosswise vibration equation of beam using Galerkin method and obtained the estimation of errors using the trigonometric polynomials that satisfies all the boundary conditions directly. The Adomian decomposition method (ADM) to solve fourth-order eigenvalue problems was used by Attili and Lesnic [8]. Syam and Siyyam [10] developed a variational iteration technique (VIM) for finding the eigenvalues of fourth-order non-singular Sturm-Liouville problems. Recently, Chanane [9] has enlarged the scope of the Extended Sampling method [11] which was devised initially for second-order Sturm-Liouville (SLE) problems to fourth-order ones. Abbasbandy and Shirzadi [13] applied the homotopy analysis method (HAM) to numerically approximate the eigenvalues of the second and fourth order Sturm-Liouville problems. Shi and Cao [14] presented a computational method for solving eigenvalue problems of high-order ordinary differential equations which based on the use of Haar wavelets. Ycel and Boubaker [15] applied differential quadrature method (DQM) and boubaker polynomial expansion scheme (BPES) for efficient computation of the eigenvalues of fourth-order Sturm-Liouville problems. Gamel and Sameeh [16] applied Chebychev method for finding eigenvalues of fourth order nonsingular Sturm-Liouville problems and compared the results to the other methods available in the literature.

Very recently Taher *et al* [17] applied an efficient technique using Chebychev spectral collocation method where Chebychev differentiation matrix is defined and computed the eigenvalues of SLP's. Since Bernstein polynomials have been used for the solution of differential equations by Doha *et al* [12] and also by Islam and Hossain [18], this partially motivates our interest to compute the eigenvalues of the SLP's using Bernstein polynomials. Another motivation is concerned with Galerkin weighted residual method which can provide solutions to many complicated problems. We organize this article as follows.

We give a brief introduction of Bernstein polynomials in section 2 along with their properties. The formulation of the general linear fourth order Sturm-Liouville problems by utilizing the technique of Galerkin weighted residual method incorporated with the boundary conditions have been discussed in Section 3. In Section 4, we consider numerical examples to verify the efficiency of the proposed method.

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68 **2. Bernstein polynomials**

69 The general form of the Bernstein polynomials of n -th degree over the interval $[a, b]$ defined
70 by Islam and Hossain [18].

$$B_{i,n}(x) = \binom{n}{i} \frac{(x-a)^i (b-x)^{n-i}}{(b-a)^n}, \quad a \leq x \leq b, \quad i = 0, 1, 2, \dots, n. \quad (1)$$

Note that each these $n+1$ polynomials satisfies the following properties

i) $B_{i,n}(x) = 0$, if $i < 0$ or $i > n$,

$$\text{ii) } B_{i,n}(a) = 0 = B_{i,n}(b), \quad 1 \leq i \leq n-1 \quad \text{iii) } \sum_{i=0}^n B_{i,n}(x) = 1$$

For simplicity we denote $B_{i,n}(x)$ as B_i throughout the paper. Since the piecewise polynomials are differentiable and integrable, Bernstein polynomials defined in equation (1) form a complete basis over the finite interval.

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79 3. Matrix Formulation

80 Consider the following general fourth order nonsingular Sturm-Liouville problem (SLP)

$$\frac{d^2}{dx^2} \left[p(x) \frac{d^2 u}{dx^2} \right] - \frac{d}{dx} \left[q(x) \frac{du}{dx} \right] + r(x)u = \lambda \mu(x)u, \quad 0 < x < L \quad (2)$$

82 here L is finite number; $p(x), q(x), r(x)$ and $\mu(x)$ are all piecewise continuous functions
83 and $p(x), \mu(x) \geq 0$ subject to some specified conditions and at these conditions mean that
84 equation (2) is regular, i.e., nonsingular.

85 We can rewrite the equation (2) in the following form as a general fourth order Sturm-
86 Liouville problems (SLP)

$$\frac{d^4 u}{dx^4} + a_3 \frac{d^3 u}{dx^3} + a_2 \frac{d^2 u}{dx^2} + a_1 \frac{du}{dx} + a_0 u = \lambda \phi u, \quad a \leq x \leq b \quad (3)$$

$$\text{88 where } a_3(x) = -\frac{2p'(x)}{p(x)}, \quad a_2(x) = \frac{p''(x) - q(x)}{p(x)}, \quad a_1(x) = \frac{q'(x)}{p(x)}, \quad a_0(x) = \frac{r(x)}{p(x)}, \quad \phi(x) = \frac{\mu(x)}{p(x)}$$

89 in which $\alpha_0, \alpha_1, \beta_0, \beta_1$ are finite real constants and a_0, a_1, a_2, a_3, ϕ are all continuous
90 functions of x defined on the interval $[a, b]$.

91 Let us consider the fourth order SLP (3) subject to the boundary conditions

$$92 \quad u(a) = \alpha_0, \quad u(b) = \beta_0, \quad u'(a) = \alpha_1, \quad u'(b) = \beta_1. \quad (4)$$

93 To approximate the solution of SLP (3), we express in terms of Bernstein

$$94 \quad \text{polynomials basis as } \tilde{u}(x) = \theta_0(x) + \sum_{i=1}^{n-1} c_i B_i(x) \quad (5)$$

95 where $\theta_0(x)$ is specified by the Dirichlet boundary conditions and $B_i(a) = 0$ and $B_i(b) = 0$

96 for each $i = 1, 2, 3, \dots, n-1$.

97 Using (5) into equation (3), the Galerkin weighted residual equations are:

$$\int_a^b \left[\frac{d^4 \tilde{u}}{dx^4} + a_3 \frac{d^3 \tilde{u}}{dx^3} + a_2 \frac{d^2 \tilde{u}}{dx^2} + a_1 \frac{d \tilde{u}}{dx} + a_0 \tilde{u} - \lambda \phi \tilde{u} \right] B_j dx = 0, \quad j = 1, 2, 3, \dots, n. \quad (6)$$

Now integrating each term of (6) by parts, we have

$$\begin{aligned} \int_a^b \frac{d^4 \tilde{u}}{dx^4} B_j(x) dx &= \left[B_j(x) \frac{d^3 \tilde{u}}{dx^3} \right]_a^b - \int_a^b \frac{d}{dx} [B_j(x)] \frac{d^3 \tilde{u}}{dx^3} dx \\ &= - \left[\frac{d}{dx} [B_j(x)] \frac{d^2 \tilde{u}}{dx^2} \right]_a^b + \int_a^b \frac{d^2}{dx^2} [B_j(x)] \frac{d^2 \tilde{u}}{dx^2} dx \\ &= - \left[\frac{d}{dx} [B_j(x)] \frac{d^2 \tilde{u}}{dx^2} \right]_a^b + \left[\frac{d^2}{dx^2} [B_j(x)] \frac{d \tilde{u}}{dx} \right]_a^b - \int_a^b \frac{d^3}{dx^3} [B_j(x)] \frac{d \tilde{u}}{dx} dx \\ &= \beta_1 \frac{d^2 B_j(b)}{dx^2} - \alpha_1 \frac{d^2 B_j(a)}{dx^2} - \frac{dB_j(b)}{dx} \frac{d^2 \tilde{u}(b)}{dx^2} + \frac{dB_j(a)}{dx} \frac{d^2 \tilde{u}(a)}{dx^2} \\ &\quad - \int_a^b \frac{d^3}{dx^3} [B_j(x)] \frac{d \tilde{u}}{dx} dx \end{aligned} \quad (8)$$

$$\text{Since } \left[B_j(x) \frac{d^3 \tilde{u}}{dx^3} \right]_a^b = 0 \text{ by the Dirichlet boundary conditions.}$$

Similarly,

$$\begin{aligned} \int_a^b a_3(x) \frac{d^3 \tilde{u}}{dx^3} B_j(x) dx &= \left[a_3(x) B_j(x) \frac{d^2 \tilde{u}}{dx^2} \right]_a^b - \int_a^b \frac{d}{dx} [a_3(x) B_j(x)] \frac{d^2 \tilde{u}}{dx^2} dx \\ &= - \left[\frac{d}{dx} [a_3(x) B_j(x)] \frac{d \tilde{u}}{dx} \right]_a^b + \int_a^b \frac{d^2}{dx^2} [a_3(x) B_j(x)] \frac{d \tilde{u}}{dx} dx \\ &= - \left[\frac{d}{dx} [a_3(x) B_j(x)] \right]_{x=b} \times \beta_1 + \left[\frac{d}{dx} [a_3(x) B_j(x)] \right]_{x=a} \times \alpha_1 \\ &\quad + \int_a^b \frac{d^2}{dx^2} [a_3(x) B_j(x)] \frac{d \tilde{u}}{dx} dx \end{aligned} \quad (9)$$

Equations (8) and (9) are obtained by imposing boundary conditions in equation (4).

Also,

$$\int_a^b a_2(x) \frac{d^2 \tilde{u}}{dx^2} B_j(x) dx = \left[a_2(x) B_j(x) \frac{d \tilde{u}}{dx} \right]_a^b - \int_a^b \frac{d}{dx} [a_2(x) B_j(x)] \frac{d \tilde{u}}{dx} dx$$

$$= - \int_a^b \frac{d}{dx} [a_2(x) B_j(x)] \frac{d\tilde{u}}{dx} dx \quad (10)$$

$$\int_a^b a_1(x) \frac{d\tilde{u}}{dx} B_j(x) dx = [a_1(x) B_j(x) \tilde{u}(x)]_a^b - \int_a^b \frac{d}{dx} [a_1(x) B_j(x)] \tilde{u}(x) dx. \quad (11)$$

Inserting $B_j(a) = B_j(b) = 0$ in the above integrals, we finally obtain the equations (8), (9), (10) and (11)

Substituting (8), (9), (10) and (11) into (6) and after rearranging the terms we have

$$\begin{aligned} & \int_a^b \left[\frac{d^3}{dx^3} [B_j(x)] \frac{d\tilde{u}}{dx} + \frac{d^2}{dx^2} [a_3(x) B_j(x)] \frac{d\tilde{u}}{dx} - \frac{d}{dx} [a_2(x) B_j(x)] \frac{d\tilde{u}}{dx} \right. \\ & \left. - \frac{d}{dx} [a_1(x) B_j(x)] \tilde{u} + a_0(x) B_j \tilde{u} - \lambda \phi(x) B_j \tilde{u} \right] dx \\ & + \beta_1 \frac{d^2 B_j(b)}{dx^2} - \alpha_1 \frac{d^2 B_j(a)}{dx^2} - \frac{dB_j(b)}{dx} \frac{d^2 \tilde{u}(b)}{dx^2} + \frac{dB_j(a)}{dx} \frac{d^2 u(a)}{dx^2} \\ & - \left[\frac{d}{dx} [a_3(x) B_j(x)] \right]_{x=b} \times \beta_1 + \left[\frac{d}{dx} [a_3(x) B_j(x)] \right]_{x=a} \times \alpha_1 = 0 \end{aligned} \quad (12)$$

Also from (5), we have

$$\tilde{u}(a) = \sum_{i=1}^{n-1} c_i B_i(a), \quad \tilde{u}(b) = \sum_{i=1}^{n-1} c_i B_i(b) \quad (13a)$$

$$\frac{d^2 \tilde{u}}{dx^2} = \frac{d^2 \theta_0}{dx^2} + \sum_{i=1}^{n-1} c_i \frac{d^2 B_i}{dx^2} \quad (13b)$$

Using equations (13a) and (13b) into equation (12) we obtain

$$\begin{aligned} & - \sum_{i=1}^{n-1} \left[\int_a^b \left[\frac{d^3 B_j}{dx^3} \frac{dB_i}{dx} + \frac{d^2}{dx^2} [a_3(x) B_j(x)] \frac{dB_i}{dx} - \frac{d}{dx} [a_2(x) B_j(x)] \frac{dB_i}{dx} \right. \right. \\ & \left. \left. - \frac{d}{dx} [a_1(x) B_j(x)] B_i(x) + a_0(x) B_i(x) B_j(x) \right] dx \right] c_i \\ & - \left[\frac{d}{dx} [a_3(x) B_j(x)] \right]_{x=b} \times \beta_1 + \left[\frac{d}{dx} [a_3(x) B_j(x)] \right]_{x=a} \times \alpha_1 \\ & + \int_a^b \left[\frac{d^3}{dx^3} [B_j(x)] \frac{d\theta_0}{dx} + \frac{d^2}{dx^2} [a_3(x) B_j(x)] \frac{d\theta_0}{dx} - \frac{d}{dx} [a_2(x) B_j(x)] \frac{d\theta_0}{dx} \right. \end{aligned}$$

$$\begin{aligned}
 & -\frac{d}{dx} \left[a_1(x) B_j(x) \right] \theta_0(x) + a_0(x) \theta_0 B_j(x) - \lambda \phi(x) \theta_0(x) B_j \Big] dx \\
 & + \sum_{i=1}^{n-1} \left[-\frac{dB_j(b)}{dx} \frac{d^2 B_i(b)}{dx^2} + \frac{dB_j(a)}{dx} \frac{d^2 B_i(a)}{dx^2} \right] c_i + \left[-\frac{dB_j(b)}{dx} \frac{d^2 \theta_0(b)}{dx^2} + \frac{dB_j(a)}{dx} \frac{d^2 \theta_0(a)}{dx^2} \right] \\
 & = \lambda \sum_{i=1}^{n-1} \int_a^b \left[\phi(x) B_i B_j dx \right] c_i + \alpha_1 \frac{d^2 B_j(a)}{dx^2} - \beta_1 \frac{d^2 B_j(b)}{dx^2} \tag{14}
 \end{aligned}$$

Finally, the eigenvalues are obtained in matrix form as below

$$\sum_{j=1}^n \left[D_{i,j} - E_j - \lambda (F_{i,j} - G_j) \right] c_i = 0 \tag{15a}$$

where,

$$\begin{aligned}
 D_{i,j} = & \int_a^b \left\{ -\frac{d^3 B_j}{dx^3} \frac{dB_i}{dx} + \frac{d^2}{dx^2} \left[a_3(x) B_j(x) \right] \frac{dB_i}{dx} - \frac{d}{dx} \left[a_2(x) B_j(x) \right] \frac{dB_i}{dx} \right\} dx \\
 & - \int_a^b \left\{ -\frac{d}{dx} \left[a_1(x) B_j(x) \right] B_i(x) + a_0(x) B_i(x) B_j(x) \right\} dx + \\
 & \left[-\frac{dB_j(b)}{dx} \frac{d^2 B_i(b)}{dx^2} + \frac{dB_j(a)}{dx} \frac{d^2 B_i(a)}{dx^2} \right] \tag{15b}
 \end{aligned}$$

$$\begin{aligned}
 E_j = & \alpha_1 \frac{d^2 B_j(a)}{dx^2} - \beta_1 \frac{d^2 B_j(b)}{dx^2} - \left[a_1(x) B_j(x) \right]_{x=b} \times \beta_0 + \left[a_1(x) B_j(x) \right]_{x=a} \times \alpha_0 \\
 & - \left[\frac{d}{dx} \left[a_3(x) B_j(x) \right] \right]_{x=b} \times \beta_1 + \left[\frac{d}{dx} \left[a_3(x) B_j(x) \right] \right]_{x=a} \times \alpha_1 \\
 & - \int_a^b \left[\frac{d^3}{dx^3} \left[B_j(x) \right] \frac{d\theta_0}{dx} + \frac{d^2}{dx^2} \left[a_3(x) B_j(x) \right] \frac{d\theta_0}{dx} - \frac{d}{dx} \left[a_2(x) B_j(x) \right] \frac{d\theta_0}{dx} \right. \\
 & \left. - \frac{d}{dx} \left[a_1(x) B_j(x) \right] \theta_0(x) + a_0(x) \theta_0(x) \right] dx + \alpha_1 \frac{d^2 B_j(a)}{dx^2} - \beta_1 \frac{d^2 B_j(b)}{dx^2} \\
 & - \left[\frac{dB_j(b)}{dx} \frac{d^2 \theta_0(b)}{dx^2} - \frac{dB_j(a)}{dx} \frac{d^2 \theta_0(a)}{dx^2} \right] \tag{15c}
 \end{aligned}$$

$$F_{i,j} = \int_a^b \left[\phi(x) B_i B_j dx \right] \tag{15d}$$

$$G_j = \int_a^b \phi(x) \theta_0(x) B_j dx. \tag{15e}$$

Hence, the eigenvalues can be obtained by solving the determinant of the coefficient matrix in equation (15a) such that

$$\det(D_{i,j} - E_j I - \lambda(F_{i,j} - GI)) = 0. \quad (16)$$

4. Test examples

In this section we present five numerical examples of fourth order SLP problems, using the method outlined in the previous section. The convergence of the Galerkin method is measured by the relative error

$$\mathcal{E}_k = \left| \frac{\lambda^{Exact} - \lambda^{(Gal)}}{\lambda^{exact}} \right| < 10^{-10} \quad (17)$$

Example 1(a): We first consider the Sturm-Liouville BVP examined by Yucel and Boubaker [15], Gamel and Sameeh [16] and Chebychev spectral collocation method Taher *et al* [17].

$$\frac{d^4 u}{dx^4} - \lambda u(x) = 0, \quad 0 < x < 1 \quad (18a)$$

$$\left. \begin{aligned} u(0) &= u'(0) = 0 \\ u(1) &= u''(1) = 0 \end{aligned} \right\}, \quad (18b)$$

which corresponds to the case $a_0(x) = a_1(x) = a_2(x) = a_3(x) = 0$, $a = 0$ and $b = 1$ in equation (3).

The exact solution of (18a) is obtained by solving

$$\tanh(\sqrt{\lambda}) - \tan(\sqrt{\lambda}) = 0. \quad (19)$$

Using the method illustrated in section 3, we approximate $u(x)$ as

$$\tilde{u}(x) = \theta_0(x) + \sum_{i=1}^{n-1} c_i B_i(x). \quad (20)$$

Here $\theta_0(x) = 0$ as specified by the Dirichlet boundary conditions of equation (18b).

The weighted residual, equation (18a) becomes

$$\sum_{j=1}^n [D_{i,j} - \lambda F_{i,j}] c_j = 0, \quad j = 1, 2, 3, \dots, n-1 \quad (21a)$$

where ,

$$D_{i,j} = \int_0^1 \frac{d^3 B_j}{dx^3} \frac{dB_i}{dx} dx + \left[\frac{d^2 B_i(0)}{dx^2} \frac{dB_j(0)}{dx} + \frac{d^2 B_j(0)}{dx^2} \frac{dB_i(1)}{dx} \right] \quad (21b)$$

$$F_{i,j} = \lambda \int_0^1 B_i B_j dx \quad (21c)$$

Table 1: Observed relative errors of the eigenvalues for example 1(a)

Exact eigenvalues	Relative error present (Bernstein) $n=20$	Relative error present (Bernstein) $n=29$	Relative error Taher <i>et al</i> [17]. (Cheby. spect. coll.)	Relative errors Cheby Gamel and Sameeh [16]	Relative errors Yucel and Boubaker [15] PDQ $N=20$	Relative Errors Yucel and Boubaker [15]) PDQ $N=30$
237.72106753	4.697E-012	4.206E-012	2.03E-009	4.697 E-012	7.59E-009	7.59 E -09
2496.48743786	1.269E-012	1.269E-012	7.93E-010	3.046 E-12	4.44E-008	4.45E-008
10867.58221698	1.022E- 013	1.022E-013	2.33E-010	5.104E-12	1.94E-009	1.71E-008
31780.09645408	3.880E- 014	3.388E-014	8.61E-009	8.605E-09	4.50E-008	2.36E-008
74000.84934915	1.994E-012	6.851E-015	7.51E-011		3.97E-005	2.99E-008
148634.47728577	3.053E-009	2.153E-015	2.24E-010		1.43E-004	4.77E-008
269123.43482664	2.262E-013	6.272E-015			4.08E-003	9.611E-10
451247.99471928	1.640E-005	2. 966E-015			1.11E-002	1.74E-008
713126.24789600	1.362E-004	1.257E-012			9.02E-002	3.16E-006
1075214.10347396	4.104E-003	2.230E-010			2.06E-002	9.31E-006

Solving the determinant of the system in (21a), we get the approximate eigenvalues for different values of n .

Exact eigenvalues and relative errors are tabulated in Table 1 using different degrees of polynomials with the relative error for the Chebychev method [16], differential quadrature method [15] and Chebychev spectral collocation method [17].

The result obtained using $n =20$, for the first six eigenvalues of the problem using Bernstein polynomials.

Table 2: Comparison of eigenvalues for example1 (a)

Result of Gamel and Sameeh [16] $\lambda_k^{(Cheby)}$	Results of Attili and Lesnic [8]	Results of Abbasbandy and Shirazdi [13]	Results of Syam and Siyyam [10]	Eigenvalue (Bernstein) (present) $\lambda_k^{(Gal.)}$
237.72106753	237.72106753	237.72106753	237.72106754	237.72106753
2496.48743786	2496.48743785	2496.48743785	2496.48743843	2496.48743786
10867.58221704	10867.59367146	10867.58221697	10867.58221699	10867.58221698
31780.09645409	31475.48355038	31780.09645277	31780.09650785	31780.09645408
.....		74000.84934655	74000.85036550	74000. 84934930
.....		148634.47747229	148634.47728684	148634. 47773948

Example 1(b) : Consider the Sturm-Liouville BVP worked out by Syam & Siyyam [10], Gamel and Sameeh [16].

$$\frac{d^4 u}{dx^4} - \lambda u(x) = 0, \quad (22a)$$

$$u''(0) = u''(1) = u'''(0) = u'''(1) = 0. \quad (22b)$$

Table 3 shows the comparison of our result obtained using $n=22$ for Bernstein polynomial, with the first six eigenvalues of the problem with Gamel and Sameeh [16], Syam and Siyyam [10].

Table 3: Comparison of eigenvalues for example 1(b)

$\lambda_k^{(Galerkin)}$ Bernstein	Results of Gamel and Sameeh Cheby.coll.[16]	Results of Syam and Siyyam [10]
500.563901740	500.563901740	500.563901756
3803.53708049	3803.53708058	3803.53708049
14617.6301311	14617.6301777	14617.6301311
39943.7990057		39943.7990057
89135.4076573		89135.4076571

Example 1(c): Consider the Sturm-Liouville BVP which is taken from Attili and Lesnic [8]

$$\frac{d^4 u}{dx^4} - \lambda u(x) = 0, \quad (23a)$$

$$u''(0) = u'''(0) = 0, \quad u(1) = u'(1) = 0. \quad (23b)$$

Table 4 shows the comparison of our result obtained using $n = 22$ for Bernstein polynomial, with the first nine eigenvalues of the problem with the results of Attili and Lesnic [8].

Table 4: Comparison of eigenvalues for example 1(c)

k	Computed eigenvalue by present method $\lambda_k^{(Gal.)}$	Results of Attili and Lesnic [8]
1	12.3623633683259	12.3623633683262
2	485.518818513372	485.518818513372
3	3806.54626639151	3806.54626639145
4	14617.2733051187	14617.2733051100
5	39943.8317785095	39943.8317790386
6	89135.4050714239	89135.4050444342
7	173881.315656105	173881.315656105
8	308208.452093651	308208.438655408
9	508481.543266068	508481.270992137

Example 2(a): Consider the Sturm-Liouville BVP taken from the articles of Attili and Lesnic [8] and Taher *et al* [17], respectively.

$$\frac{d^4 u}{dx^4} = 0.02 x^2 \frac{d^2 u}{dx^2} + 0.04 x \frac{du}{dx} - (0.0001 x^4 - 0.02)u(x) + \lambda u(x), \quad (24a)$$

$$\left. \begin{aligned} u(0) &= u''(0) = 0 \\ u(5) &= u''(5) = 0 \end{aligned} \right\}. \quad (24b)$$

The above problem can be written as self-adjoint form as

$$\frac{d^4 u}{dx^4} - 0.02 \frac{d}{dx} \left(x^2 \frac{du}{dx} \right) + (0.0001 x^4 - 0.02)u(x) = \lambda u(x), \quad (25)$$

Table 5 shows the comparison of our result obtained using $n = 22$ for Bernstein polynomial, of the first six eigenvalues of the problem with the results of Attili and Lesnic [8], Syam and Siyyam [10], Chanane [11], Abbasbandy and Shirazdi [13], Yucel and Boubaker [15], Gamel and Sameeh [16] and Taher *et al* [17].

Table 5 : Comparison of eigenvalues for example 2(a)

Our method $n=22$ $\lambda_k^{(galerkin)}$	Results of Taher <i>et al</i> [17]	Results of Gamel and Sameeh [16]	Result of Attili and Lesnic [8]	Results of Yucel and Boubaker [15]	Results of Syam and Siyyam [10]
0.21505086437	0.21505086432	0.21505086437	0.2150508643697	0.21505086437	0.21505086437
2.75480993468	2.75480993362	2.7548099346829	2.7548099346829	2.75480993468	2.75480993468
13.21535154056	13.21535154059	13.215351540416	13.215351540558	13.2153515406	13.2153515406
40.95081975916	40.95081975814	40.950820029821	40.950819759137	40.9508197591	40.9508197591
99.05347806349	99.05347803835		99.053478138138	99.0534780633	99.0534781381
204.35573226893	204.35573547934		204.35449348957	204.355732256	204.3544934895
Our method $n=22$ $\lambda_k^{(galerkin)}$	Results of Abbasbandy and Shirazdi [13]	Results of Chanane [11]	Chanane [11]	Attili and Lesnic [8]	
0.21505086437	0.21505086	0.21505086437	0.86690250239947	0.86690250239 97106	
2.75480993468	2.75480888	2.75480993468	6.35768644817446	6.35768644814 5815	
13.21535154056	13.21539518	13.2153515406	23.9927469506674 7	23.9927468502 81375	
40.95081975916	40.94821490	40.9508193487	64.9786359159700 7	64.9786675957 1622	
99.05347806349	----	----		144.280628038 44648	
204.35573226893	----	-----		280.586020481 95377	

Example 2(b): Consider the Sturm-Liouville BVP worked out by Attili and Lesnic [8], Chanane [11], Yucel and Boubaker [15], Taher *et al* [17].

$$\frac{d^4 u}{dx^4} = 0.02 x^2 \frac{d^2 u}{dx^2} + 0.04 x \frac{du}{dx} - (0.0001 x^4 - 0.02) u(x) + \lambda u(x), \quad (26a)$$

$$u(0) = u'(0) = 0,$$

$$u(5) = u'(5) = 0 \quad (26b)$$

Table 6 shows the comparison of our result obtained using the degree of polynomial $n = 22$, for Bernstein basis, for the first six eigenvalues of the problem with the results of Attili and Lesnic, Chanane, Yucel and Boubaker, Taher *et al* [8, 11, 15, 17] respectively.

Table 6 : Comparison of eigenvalues for example 2(b)

Our method $\lambda_k^{(galerkin)}$	Taher <i>et al</i> [17]	Yucel and Boubaker [15]	Chanane [11]	Attili and Lesnic [8]
0.86690250239970	0.86690250239196	0.86690250224260	0.86690250239947	0.8669025023997106
6.35768644814590	6.35768644814386	6.35768644843984	6.35768644817446	6.357686448145815
23.99274685030238	23.99274685032633	23.9927468509660	23.99274695066747	23.992746850281375
64.97866759050172	64.97866759484157	64.97866761311830	64.97863591597007	64.97866759571622
144.2806269274497	144.28062688384347	144.2806269273480		144.28062803844648
280.6009633049182	280.60096699712966	280.60096374439620		280.58602048195377

Conclusions

We have discussed in details the formulations of Sturm-Liouville problem by the Galerkin weighted residual method using Bernstein polynomials as a basis function. To verify accuracy of our method we have considered four examples. In Table 1, we have computed first 10 eigenvalues and compare our results with other published works available in the literature. In Table I, the first seven eigenvalues using Bernstein polynomials are very close to the exact results and the computed values for the lower eigenvalues have a better accuracy than those for the higher eigenvalues. At the same time it is also observed in Table 1 that all 10 eigenvalues obtained using Bernstein polynomials converge more rapidly than those obtained by the other methods. In fact relative error decreases as the degree of polynomials increase from $n = 20$ to $n = 29$ in the case of Bernstein basis. But on the other hand estimated eigenvalues show less convergent especially at present method for $n = 20$. It is obviously observed that eigenvalues obtained by Galerkin-Bernstein method are most accurate than the other results have been achieved by various methods. Excellent agreement is being observed in Table 1 between results of present work and the results of previously published works by Yucel and Boubaker [15], Gamel and Sameeh [16] and Taher *et al* [17]

In table 2, 3 and 5 we computed 6 eigenvalues and compared our results with Attili and Lesnic [8], Syam and Siyyam [10], Chanane [11], Abbasbandi and Shirazdi [13], Gamel and Sameeh [16], Taher *et al* [17]. Also using 29 Bernstein polynomials, we obtain the first 9 eigenvalues and compared our results with Attili and Lesnic [8] summarized in Table 4. Therefore, we conclude that Bernstein polynomial scheme produces much accurate results than all other previously published works available in the literature.

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