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On Projectively Related Kropina Metric

Abstract : In the present paper, we study curvature properties of projectively flat Kropina metric and prove that two Finsler metrics $L = \alpha^2/\beta$ and $\bar{L} = \bar{\alpha}^2/\bar{\beta}$ are projectively related if and only if they have same Douglas tensors. Further we showed that corresponding Riemannian metrics are projectively related.

AMS Classification: 53C60, 53B40.

Key Words: Finsler metric, Randers metric, Kropina metric, Douglas tensor, Projectively flat, Constant sectional curvature.

1. Introduction

The projective changes between two Finsler spaces have been studied by many geometers like Z. Shen, H. Shimada, I. Y. Lee, H. S. Park etc. Two regular metrics on an n-space are said to be *pointwise projectively related* if they have the same geodesics as point sets i.e., two Finsler metrics on a manifold M^n are projectively equivalent if they have the same geodesics as point sets. Two regular metric spaces are said to be *projectively related* if there is a diffeomorphism between them such that the pull-back metric is pointwise projectively related to another one. The notion of an (α, β)-metric was introduced by M. Matsumoto and has been studied by many authors extensively such as [1] [2] [5] [11]. A Finsler metric $L(\alpha, \beta)$ is a differentiable manifold M^n is said to be (α, β)-metric, if L is a positively homogeneous function of degree one in two variables α and β, where α is Riemannian metric and β is a differential 1-form.

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We have specially interesting examples of an (α, β)-metric $L(\alpha, \beta)$ like Randers metric, Matsumoto metric, Kropina metric etc. In particular, a Finsler fundamental function $L(x, y)$ will be called the generalized Kropina metric if $L(x, y)$ is defined by $L(x, y) = \alpha^{m+1}\beta^{-m}(m \neq 0, -1)$. And the Kropina metric was first introduced by L. Berwald connection with a two dimensional Finsler space with rectilinear extremal and was investigated by V. K. Kropina [10].

Guojun Yang studied the m-Kropina metrics of scalar flag curvature [9]. The flag curvature is an analogue of sectional curvature in Riemann geometry. It measures the rate of changes of the Finsler metric modulo the changes of the geometry of tangent spaces. X. Chen, Z. Shen were studied on projectively flat Finsler metric with almost isotropic S-curvature [7]. I. Y. Lee, H. S. Park were worked on Finsler spaces with infinite series (α, β)-metric [11]. Then S. K. Narasimhamurthy, Latha Kumari G. N. and C. S. Bagewadi were studied on projectively flat Kropina metric with $K = 0$ [13]. Z. Shen worked on projectively flat Randers metric with constant flag curvature and also studied on projectively related Einstein metrics [17][18].

In the present paper we discussed the Kropina metric L and $\bar{L}$ are projectively related, they have the same Douglas tensor and also studied L and $\bar{L}$ are projectively flat with curvature properties.

2. Preliminaries

Definition 2.1. *A Finsler metric is a scalar field $L(x, y)$ which satisfies the following three conditions:*

(1) *It is defined and differential for any point of $TM\backslash\{0\}$,*
(2) *It is positively homogeneous of first degree in y^i, that is,*
$L(x, \lambda y) = \lambda L(x, y)$, *for any positive number λ,*
(3) *It is regular, that is,*
$g_{ij}(x, y) = \frac{1}{2}\dot{\partial}_i\dot{\partial}_j L^2$,
constitute the regular matrix g_{ij}, where $\dot{\partial}_i = \frac{\partial}{\partial y^i}$.

The manifold M^n equipped with a fundamental function $L(x, y)$ is called Finsler space $F^n = (M^n, L)$

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Example: Let M^n be a real n-dimensional differentiable manifold endowed with a Riemannian metric g and a differentiable 1-form ω. Let H be a closed subset of $\phi(U) \times R^n$ consisting of all points (x^i, y^i) such that $\omega_i(x) = 0$.
Let $L(x^i, y^i) = \frac{g_{ij}(x)y^iy^j}{\omega_i(x)y^i}$ be the real valued function defined on the open set $U^* = \phi(U) \times R^n - \{H\}$. Let B denote the union of all open sets $\phi^{-1}(U^*)$. It is clear that L satisfies the homogeneity property on B and satisfies $Rank(\dot{\partial}_i\dot{\partial}_j L^2/2) = n$ on an open sub manifold A of B. Then the pair $F^n = (TM^n, L)$ is a Finsler space called *Kropina space*.

There is a class of Finsler metrics defined by a Riemannian metric and 1-form on a manifold, which is relatively simple with interesting curvature properties called (α, β)-metrics and these metrics are computable.

Definition 2.2. *The Finsler space $F^n = (M^n, L)$ is said to be have an (α, β)-metric if L is a positively homogeneous function of degree one in two variables $\alpha = \sqrt{a_{ij}y^iy^j}$ and $\beta = b_i(x)y^i$, where α is a Riemannian metric and β is differentiable 1-form.*

The space $R^n = (M^n, \alpha)$ is called the *associated Riemannian space* and the covariant vector field b_i is the *associated vector field*.
An (α, β)-metric is expressed in the following form,

$$L = \alpha\phi(s), \quad s = \frac{\beta}{\alpha}$$

where $\phi = \phi(s)$ is a C^∞ positive function on an open interval (b_0, b_0). The norm $\|\beta_x\|_\alpha$ of β with respect to α is defined by,

$$\|\beta_x\|_\alpha = \sup_{y\epsilon T_xM} \beta(x, y), \alpha(x, y) = a_{ij}(x)b_i(x)b_j(x).$$

In order to define L, β must satisfy the condition $\|\beta_x\|_\alpha < b_0$ for all $x\epsilon M^n$.
Example: Consider,

$$\alpha = \frac{(1-\varepsilon^2)\langle x, y\rangle^2 + \varepsilon|y|^2(1-\varepsilon|x|^2)}{(1-\varepsilon|x|^2)},$$

$$\beta = \frac{\sqrt{1-\varepsilon^2}(x, y)}{1-\varepsilon|x|^2},$$

where $x\epsilon R^2, y\epsilon R^2, \varepsilon\epsilon(0, 1)$, then

$$\|\beta\| = \sqrt{1-\varepsilon^2}\sqrt{\frac{|x|^2}{\varepsilon+|x|^2}} < 1.$$

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Hence $F = \alpha + \beta$ is a Randers metric.

Generalizing the above (α, β)-metric we obtain the different class of interesting (α, β)-metric. In this paper we consider one of them is known metric.

Definition 2.3. *A Finsler space having the fundamental function:*

$$L(x, y) = \frac{\alpha^2(x, y)}{\beta(x, y)}$$

is called a Kropina space.

The generalized (α, β)-metric:

$$L(x, y) = \frac{\alpha^{m+1}(x, y)}{\beta^m(x, y)}, \quad m \neq 0, -1,$$

is called a generalized m-Kropina metric and the Finsler space equipped with this metric is called a generalized m-Kropina space.

Definition 2.4. [19] *Consider*

$$D^i_{jkl} = \frac{\partial^3}{\partial y^j \partial y^k \partial y^l} \left(G^i - \frac{1}{n+1} \frac{\partial G^m}{\partial y^m} y^i \right), \tag{2.1}$$

where G^i are the spray coefficient of F. The tensor $D = D^i_{jkl} dx^j \otimes \partial_i \otimes dx^k \otimes dx^l$ is called the Douglas tensor [5].

A Douglas metric is called a Finsler metric if $D = 0$. In particular, in Riemannian geometry the Douglas tensor vanishes. The Douglas tensor is a non-Riemannian quantity. Berwald metrics are Douglas metrics. In Finsler geometry, the study of Douglas metrics shows the geometric meaning of non-Riemannian quantities.

Consider the Kropina metric $L = \alpha^2/\beta$ on a manifold M^n of dimension n, where $\alpha = \sqrt{a_{ij}(x) y^i y^j}$ is a Riemannian metric and $\beta(y) = b_i(x) y^i$ is a 1-form with $\| \beta \|_p = Sup_{y \epsilon T_p M} \frac{\beta(y)}{\alpha(y)} < 1$.

All over the paper, the range of Latin indices is $1, 2, ..., n$ and the Einstein summation convection is adopted, $b_{i/j}$ denotes the covariant derivative of b_i with respect to α. From the differential 1-form $\beta(x, y) = b_i(x) y^i$ we define

$$2r_{ij} = b_{i/j} + b_{j/i}, \quad 2s_{ij} = b_{i/j} - b_{j/i},$$
$$s^i_j = a^{ih} s_{hj}, \quad s_j = b_i s^i_j, \quad s^i_0 = s^i_j y^j.$$

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3. Equations Of Geodesics Of Projectively Flat and Projectively Related Kropina Metric

In this section we find the geodesic equation of projectively flat Kropina metric, and then we show Kropina metric is projectively related.

It is well known that two Finsler metrics L and $\bar{L}$ are projectively equivalent if and only if

$$G^i = \bar{G}^i + Py^i, \tag{3.1}$$

where G^i and $\bar{G}^i$ are the spray coefficients of L and $\bar{L}$ respectively and $P = P(x, y)$ is positively y-homogeneous of degree one. We shall denote the quantities for $\bar{L}$ by the same letters with the bar and the corresponding indices. From (2.1) and (3.4) we have seen that the Douglas tensor of L is same as that of $\bar{L}$. Therefore the Douglas tensor is projective invariant. we have

Lemma 3.1. *Let $L = \alpha^2/\beta$ and $\bar{L} = \bar{\alpha}^2/\bar{\beta}$ be two Kropina metrics, D and $\bar{D}$ be the Douglas tensors of L and $\bar{L}$ respectively, if β and $\bar{\beta}$ closed. Then $\beta = \bar{\beta}$ and also $D = \bar{D}$ if and only if the following conditions holds:*

(1) $\alpha^2 \neq \lambda(x)\bar{\alpha}^2, s_{ij} = \bar{s}_{ij} = 0.$
(2) $\alpha^2 = \lambda(x)\bar{\alpha}^2, s_{ij} = \lambda(x)\bar{s}_{ij}.$

Proof: Let $G^i = G^i(x, y)$ and $\bar{G}^i = \bar{G}^i(x, y)$ be the spray coefficients of L and $\bar{L}$ in the coordinate system, $G^i_\alpha = G^i_\alpha(x, y)$ and $\bar{G}^i_{\bar{\alpha}} = \bar{G}^i_{\bar{\alpha}^i}(x, y)$ be the spray coefficients of α and $\bar{\alpha}$ respectively. Then we get,

$$G^i = G^i_\alpha - \frac{\alpha^2}{2\beta}s^i_0 + \left(\frac{\alpha^2}{\beta}s_0 + r_{00}\right)\frac{b^i}{2b^2} - \left(\frac{s_0}{b^2} + \frac{\beta r_{00}}{\alpha^2 b^2}\right)y^i. \tag{3.2}$$

$$\bar{G}^i = \bar{G}^i_{\bar{\alpha}} - \frac{\bar{\alpha}^2}{2\bar{\beta}}\bar{s}^i_0 + \left(\frac{\bar{\alpha}^2}{\bar{\beta}}\bar{s}_0 + \bar{r}_{00}\right)\frac{\bar{b}^i}{2\bar{b}^2} - \left(\frac{\bar{s}_0}{\bar{b}^2} + \frac{\bar{\beta}\bar{r}_{00}}{\bar{\alpha}^2\bar{b}^2}\right)y^i. \tag{3.3}$$

where $r_{00} = r_{ij}y^iy^j$.

Set $\widetilde{G}^i = G^i_\alpha - \frac{\alpha^2}{2\beta}s^i_0 + \left(\frac{\alpha^2}{\beta}s_0 + r_{00}\right)\frac{b^i}{2b^2}$. Clearly from (3.1), $\widetilde{G}^i$ and G^i are projectively equivalent. Therefore, they have the same Douglas tensors i.e., $D = \widetilde{D}$. We can calculate D^i_{jkl} by means of $\widetilde{G}^i$.

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Putting $Q^i = -\frac{\alpha^2}{2\beta}s_0^i + \left(\frac{\alpha^2}{\beta}s_0 + r_{00}\right)\frac{b^i}{2b^2}$, we get

$$\frac{\partial Q^m}{\partial y^m} = -\frac{\alpha^2}{2\beta}s_m^m - \left(\frac{2\beta y_m - \alpha^2 b_m}{2\beta^2}\right)s_0^m + \left[s_0\left(\frac{2\beta y_m - \alpha^2 b_m}{\beta^2}\right) + r_{00}\right]\frac{b^m}{2b^2} = 0$$

Using the definition of the Douglas tensor, we obtain

$$D^i_{jkl} = \widetilde{D}^i_{jkl} = \widehat{D}^i_{jkl} + \frac{\partial^3 Q^i}{\partial y^j \partial y^k \partial y^l},$$

where D^i_{jkl} and $\widehat{D}^i_{jkl}$ of L and α respectively. Nothing that $\widehat{D}^i_{jkl} = 0$, we obtain

$$D^i_{jkl} = \frac{\partial^3 Q^i}{\partial y^j \partial y^k \partial y^l}.$$

Similarly, for $\bar{L}$, we get

$$\bar{D}^i_{jkl} = \frac{\partial^3 \bar{Q}^i}{\partial y^j \partial y^k \partial y^l}.$$

where $\bar{Q}^i = -\frac{\bar{\alpha}^2}{2\bar{\beta}}\bar{s}_0^i + \left(\frac{\bar{\alpha}^2}{\bar{\beta}}\bar{s}_0 + \bar{r}_{00}\right)\frac{\bar{b}^i}{2\bar{b}^2}$.

Let us assume that $Q^i = \bar{Q}^i + P^i$, where P^i is y-homogeneous of degree two, then we get

$$\frac{\partial^3 Q^i}{\partial y^j \partial y^k \partial y^l} = \frac{\partial^3 \bar{Q}^i}{\partial y^j \partial y^k \partial y^l} + \frac{\partial^3 P^i}{\partial y^j \partial y^k \partial y^l}.$$

Since $D = \bar{D}$, we get

$$\frac{\partial^3 P^i}{\partial y^j \partial y^k \partial y^l} = 0.$$

This equation satisfies if and only if $P^i = H^i_{jk}(x)y^j y^k$ for some set of local functions $H^i_{jk}(x)$.

Since P^i is positively y-homogeneous of degree two, so we have

$$-\frac{\alpha^2}{2\beta}s_0^i + \left(\frac{\alpha^2}{\beta}s_0 + r_{00}\right)\frac{b^i}{2b^2} = -\frac{\bar{\alpha}^2}{2\bar{\beta}}\bar{s}_0^i + \left(\frac{\bar{\alpha}^2}{\bar{\beta}}\bar{s}_0 + \bar{r}_{00}\right)\frac{\bar{b}^i}{2\bar{b}^2} + P^i. \tag{3.4}$$

Replacing y by $-y$, we have

$$\frac{\alpha^2}{2\beta}s_0^i + \left(-\frac{\alpha^2}{\beta}s_0 + r_{00}\right)\frac{b^i}{2b^2} = \frac{\bar{\alpha}^2}{2\bar{\beta}}\bar{s}_0^i + \left(-\frac{\bar{\alpha}^2}{\bar{\beta}}\bar{s}_0 + \bar{r}_{00}\right)\frac{\bar{b}^i}{2\bar{b}^2} + P^i. \tag{3.5}$$

By (3.4) and (3.5), we obtain

$$\alpha^2\bar{\beta}\left(s_0^i + s_0\frac{b^i}{b^2}\right) = \bar{\alpha}^2\beta\left(\bar{s}_0^i + \bar{s}_0\frac{\bar{b}^i}{\bar{b}^2}\right) \tag{3.6}$$

If $\alpha^2 \neq \lambda(x)\bar{\alpha}^2$, we get $s_0^i = \bar{s}_0^i$, i.e., $s_{ij} = \bar{s}_{ij} = 0$.

If $\alpha^2 = \lambda(x)\bar{\alpha}^2$, we obtain $s_0^i = \frac{\bar{s}_0^i}{\lambda(x)}$ and therefore $s_{ij} = \lambda(x)\bar{s}_{ij}$.

We use the following lemma: proved by A. Rapcsak [16]:

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Lemma 3.2. *Let (M, L) be a Finsler manifold. A Finsler metric $\bar{L}$ is pointwise projectively related to L if and only if*

$$\frac{\partial \bar{L}_{;k}}{\partial y^l} y^k - \bar{L}_{;l} = 0,$$

where $\bar{L}_{;k}$ represents the covariant derivative of $\bar{L}$ with respect to L, i.e.,

$$\bar{L}_{;k} = \frac{\partial \bar{L}}{\partial x^k} - \frac{\partial G^l}{\partial y^k} \frac{\partial \bar{L}}{\partial y^l}.$$

Now, we state and prove the main theorem.

Theorem 3.1. *A Kropina metric $L = \alpha^2/\beta$ is pointwise projectively related to another Kropina metric $\bar{L} = \bar{\alpha}^2/\bar{\beta}$ and $\beta = \bar{\beta}$ if and only if the following cases holds:*

(1) *$\alpha^2 \neq \lambda(x)\bar{\alpha}^2$, α is pointwise projectively related to $\bar{\alpha}$.*
(2) *$\alpha^2 = \lambda(x)\bar{\alpha}^2$ for some scalar λ and $\bar{\beta} - \lambda\beta = 0$ is closed provided $b^i = \bar{b}^i$ and $r_{00} = \bar{r}_{00}$.*

Proof: First we show necessary condition.
Since L is projectively related to $\bar{L}$, then $D = \bar{D}$. From lemma 3.1, we take two case study respectively.
Case (1): $\alpha^2 \neq \lambda(x)\bar{\alpha}^2$, $s_{ij} = \bar{s}_{ij} = 0$. Then from (3.1) we have,

$$G^i = G^i_\alpha + \left(\frac{r_{00} b^i}{2b^2 y^i} - \frac{\beta r_{00}}{\alpha^2 b^2} \right) y^i,$$

by which we have

$$\begin{aligned} \bar{L}_{;k} &= \frac{\partial \bar{L}}{\partial x^k} - \frac{G^l_\alpha}{\partial y^k} \frac{\partial \bar{L}}{\partial y^l} - \frac{\partial P}{\partial y^k} y^l + P \delta^l_k \frac{\partial \bar{L}}{\partial y^l}, \\ &= \bar{L}_{|k} - \frac{\partial P}{\partial y^k} y^l + P \delta^l_k \frac{\partial \bar{L}}{\partial y^l}, \end{aligned}$$

where $\bar{L}_{|k} = \frac{\partial \bar{L}}{\partial x^k} - \frac{G^l_\alpha}{\partial y^k} \frac{\partial \bar{L}}{\partial y^l}$ defines the covariant derivative of $\bar{L}$ with respect to α and $P = \left(\frac{r_{00} b^i}{2b^2 y^i} - \frac{\beta r_{00}}{\alpha^2 b^2} \right)$.
It is easy to see that

$$\begin{aligned} \frac{\partial \bar{L}_{;k}}{\partial y^l} y^k - \bar{L}_{;l} &= \frac{\partial \bar{L}_{;k}}{\partial y^l} y^k - \bar{L}_{;l} - \frac{\partial P}{\partial y^l} y^m + P \delta^m_l \frac{\partial \bar{L}}{\partial y^m} + \frac{\partial P}{\partial y^l} + P \delta^m_l \frac{\partial \bar{L}}{\partial y^m}, \\ &= \frac{\partial \bar{L}_{|k}}{\partial y^l} y^k - \bar{L}_{|l}. \end{aligned} \tag{3.7}$$

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From lemma 3.2, the left side of (3.7) is equal to zero. Therefore, the right side of (3.7) which implies that $\bar{L}$ is pointwise projectively related to α. Thus , from (3.1), $\bar{\alpha}$ is projectively related to α.

Case (2) : $\alpha^2 = \lambda(x)\bar{\alpha}^2, s_{ij} = \lambda(x)\bar{s}_{ij}$.

We obtain $\alpha^2\bar{\beta}\left(s_0^i + s_0\frac{b^i}{b^2}\right) = \bar{\alpha}^2\beta\left(\bar{s}_0^i + \bar{s}_0\frac{\bar{b}^i}{\bar{b}^2}\right)$.

On the other hand, by (3.1), we get $G^i = \bar{G}^i + Py^i$, where $P = P(x, y)$ is positively y-homogeneous of degree one. From virtue of (3.2) and (3.3), we get

$$G^i_\alpha = \bar{G}^i_{\bar{\alpha}} + (P - P_1 + P_2)y^i, \tag{3.8}$$

where, $P_1 = -\left(\frac{s_0}{b^2} + \frac{\beta r_{00}}{\alpha^2 b^2}\right)$, $P_2 = -\left(\frac{\bar{s}_0}{\bar{b}^2} + \frac{\bar{\beta}\bar{r}_{00}}{\bar{\alpha}^2\bar{b}^2}\right)$.

Because $\alpha^2 = \lambda(x)\bar{\alpha}^2$, we have

$$G^i_\alpha = \bar{G}^i_{\bar{\alpha}} + \left[\left(\frac{r_{00}b^i}{2b^2y^i} - \frac{\beta r_{00}}{\alpha^2 b^2}\right) + \left(\frac{s_0}{b^2} + \frac{\beta r_{00}}{\alpha^2 b^2}\right) - \left(\frac{\bar{s}_0}{\bar{b}^2} + \frac{\bar{\beta}\bar{r}_{00}}{\bar{\alpha}^2\bar{b}^2}\right)\right] y^i.$$

After long computation and $s_{ij} = \lambda\bar{s_{ij}}$ we get,

$$G^i_\alpha \;=\; \bar{G}^i_{\bar{\alpha}} + \left[\frac{s_0}{b^2} - \frac{\bar{s_0}}{\bar{b}^2} - \frac{\beta r_{00}}{\alpha^2 b^2} + \frac{\beta r_{00}}{\alpha^2 b^2} - \frac{\bar{\beta}\bar{r}_{00}\lambda}{\alpha^2\bar{b}^2}\right] y^i + \frac{r_{00}b^i}{2b^2} \tag{3.9}$$

Since, $s_{ij} = \lambda\bar{s}_{ij}$

$$\beta r_{00}\bar{b}^2 y^i - \beta\bar{r_{00}}\lambda b^2 y^i = 0 \tag{3.10}$$

After calculation we get $r_{00}\bar{\beta}\bar{b^i} - \bar{r_{00}}\lambda\beta b^i$. Therefore λ is positive constant and $G^i_\alpha = \bar{G}^i_{\bar{\alpha}}$. Hence, we conclude that $s_{ij} = \lambda\bar{s}_{ij}$ which implies $\bar{\beta} - \lambda\beta = 0$ is closed provided $b^i = \bar{b}^i$ and $r_{00} = \bar{r_{00}}$. Thus we proved necessary condition.

Now we prove sufficient condition. Consider condition (1), from lemma 3.1, we have seen that $D = \bar{D}$.

Since β and $\bar{\beta}$ are closed, then $s_0^i = \bar{s}_0^i$. Hence, (3.2) and (3.3) implies L and α are projectively related. Similarly $\bar{L}$ and $\bar{\alpha}$. Since $\bar{\alpha}$ is pointwise projectively related to α, then L is projectively related to $\bar{L}$.

Now consider condition (2), since $\alpha^2 = \lambda\bar{\alpha}^2$ for some scalar λ, then $\bar{G}^i_{\bar{\alpha}} = G^i_\alpha$ implies that $\bar{\alpha}$ is pointwise projectively related to α. Since $\bar{\beta} - \lambda\beta$ is closed, then $s_{ij} = \lambda\bar{s}_{ij}$, which implies (3.6) holds.

Therefore, from (3.2) and (3.3), we have seen that (3.1) holds, that is L and $\bar{L}$ are pointwise

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projectively related. Hence, Theorem 3.1 is proved.

By using the above theorem we conclude that:

Theorem 3.2. *The Kropina metrics $L = \alpha^2/\beta$ and $\bar{L} = \bar{\alpha}^2/\bar{\beta}$ are pointwise projectively related and $\beta = \bar{\beta}$ if and only if $D = \bar{D}$ and the corresponding Riemannian metrics α and $\bar{\alpha}$ are projectively related.*

4. Kropina Metric Of Constant Flag Curvature

Here we assume that the Kropina metric $L = \alpha^2/\beta$ is of constant flag curvature [9]. $\bar{L} = \bar{\alpha}^2/\bar{\beta}$ is pointwise projectively related to L.

In [9], it is already shown that $L = \alpha^2/\beta$ has constant flag curvature K if and only if there exist a constant σ such that the following equations holds:

$$\begin{aligned} r_{00}\beta(\alpha^2 b^i - 4\beta) &= \sigma\alpha^2, && (4.1)\\ s_{ij|k} &= t_j - \frac{t^l_l - (n-3)s^l s_l}{(n-1)} b_j a_{ik} + r_{ik}s_j + q_{ki}b_j + s_{j|k}b_i - (i/j), && (4.2)\\ \bar{R}^i_k &= \frac{(n-3)s^l s_l - t^l_l}{n-1}(\alpha^2\delta^i_k - y^i y_k) - B_{00}\delta^i_k - B^i_k\alpha^2 + B_{0k}y^i + B^i_0 y_k && (4.3)\\ &+ r^i_0 r_{0k} - r_{00}r^i_k, \end{aligned}$$

where, the symbol (i/j) above denotes the terms obtained from the proceeding terms by the interchange of the indices i and j, $\bar{R}^i_k$ is the Riemannian curvature tensor of α and

$$\begin{aligned} \sigma_i &= 2[(n-3)s^l s_l - (n-1)\sigma - t^l_l]b_i + 2(n-1)b^p b^l(r_{lp|i} - r_{li|p}),\\ B_{ik} &= \frac{1}{2}(r_{il}r^l_k + b^l r_{lk|i}) + \frac{b_k\sigma_i}{4(n-1)} + s_{i|k} + (i/k), \end{aligned}$$

and $\sigma = \sigma(x)$ is a scalar function.

$r_{ij} = 0, s_i = 0$ and from (4.3) we have,

$$\bar{R}^i_k = \sigma(\alpha^2\delta^i_k - y^i y_k), \tag{4.4}$$

by [9] we get, $K = \frac{\sigma}{4}$ and $\sigma = 1$. Thus α has constant sectional curvature σ. Because $\bar{\alpha}$ is pointwise projectively related to α, $\bar{\alpha}$ has constant sectional curvature. By theorem 3.1 β and $\bar{\beta}$ are closed. Therefore, L and $\bar{L}$ are projectively flat. Thus we have the following result;

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Theorem 4.3. *Let $L = \alpha^2/\beta$ be an Kropina metric with $\| \beta \|_\alpha = 1$ of constant flag curvature and $\alpha^2 \neq \lambda(x)\bar{\alpha}^2$. Then $\bar{L} = \bar{\alpha}^2/\bar{\beta}$ is pointwise projectively related to $L = \alpha^2/\beta$ if and only if L and $\bar{L}$ are projectively flat and α has non-negative constant sectional curvature.*

5. Conclusion

A Riemann metric has two features, length and angle, while a Finsler metric has only one feature i.e., length. In projective geometry, the second feature of Riemann metrics is generally not used [18]. There are some quantities in the projective Finsler geometry which are projective invariant. It is the Hilbert fourth problem in the regular case to study and characterize Finsler metric on an open domain $\Omega \subset R^n$ such that geodesics are straight lines. Finsler metrics on Ω with this property are said to be projectively flat [20].

Beltrami theorem states that [4] "A Riemannian metric is locally projectively flat if and only if it is of constant sectional curvature". It is well-known that the local structure of Riemann metrics of constant sectional curvature has been solved. In Finsler geometry the flag curvature is a natural extension of the sectional curvature in Riemann geometry, and a Riemann metric of scalar flag curvature is nothing but of constant sectional curvature in dimension $n \geq 3$.

In this paper we have studied in the third section, Kropina metric L is pointwise projectively related to another Kropina metric $\bar{L}$ on two conditions and in the fourth section we proved that if two Kropina metric, L and $\bar{L}$ are projectively flat then both α and $\bar{\alpha}$ has non-negative sectional curvature.

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