

Original Research Article

Filtrations and Deductive Systems in BCL^+ Algebras

ABSTRACT

The aim of this paper is to introduce the two notions of filtrations and deductive systems in BCL^+ algebras, and investigate some of their related properties. Using this, we also prove that the filtration of BCL^+ algebras is a deductive system to the filtration of BCL^+ algebras with certain relations.

Keywords: BCL algebras; BCL^+ algebras; Filtrations; Deductive systems.

Mathematics Subject Classification: 03G25, 06F99

1. INTRODUCTION

The present author (2011 and 2012) [1, 2] first introduces of BCL/BCL^+ algebras. Later on, BCL algebras and related concepts were developed by Al-Kadi and Hosny (2013) [3] [6]. The present author (2014) [4, 5] [7] gave a characterization of a partial order and a topology in BCL^+ algebras, and discuss some distributions of BCL^+ algebras.

In this paper, we focus on how to generate a filtration by subset of BCL^+ algebras. We also introduce the notion of deductive systems in BCL^+ algebras, and discuss their interesting properties. Most importantly, using the notion of filtration we can to relate the deductive systems to the BCL^+ algebras.

2. PRELIMINARIES

Definition 2.1 (Definition 2.1, Liu [2]). A BCL^+ algebra is a triple $(Y; *, 1)$, where Y is a nonempty set, “ $*$ ” is a binary operation on Y , $1 \in Y$ is an element such that the following three axioms hold: for any $x, y, z \in Y$.

$$(BCL^+ 1) \quad x * x = 1.$$

$$(BCL^+ 2) \quad x * y = 1 \text{ and } y * x = 1 \text{ imply } x = y.$$

$$(BCL^+ 3) \quad ((x * y) * z) * ((x * z) * y) = (z * y) * x.$$

Definition 2.2 (Definition 2.2, Liu [2]). Suppose that $(Y; *, 1)$ is a BCL^+ algebra endowed with the ordered relation if $x \leq y$ if and only if $x * y = 1$, for all $x, y \in Y$. Then $(Y; \leq)$ is partially ordered set and $(Y; *, 1)$ is a BCL^+ algebra of partially ordered relation.

Note that the ordering 1 is the largest element of Y and that Y is a bounded BCL^+ algebra (unless otherwise specified) with a smallest element 0 relative to the natural ordering.

Definition 2.3 (Definition 2.3, Liu [4]). Let $(Y; *, 1)$ be a $BCL^+_{\leq}$ algebra, for all $x, y, z \in Y$, then

$$(BCL_{\leq}^+ 1) \ x \leq x.$$

$$(BCL_{\leq}^+ 2) \text{ If } x \leq y \text{ and } y \leq x \text{ then } x = y.$$

$$(BCL_{\leq}^+ 3) \ ((x * y) * z) * ((x * z) * y) \leq (z * y) * x.$$

Our key results are the following theorems.

Theorem 2.1 (Theorem 2.2, Liu [2]). Assume that $(Y; *, 1)$ is any a BCL^+ algebra. Then the following hold: for all $x, y, z \in Y$.

$$(i) \ (x * (x * y)) * y = 1.$$

$$(ii) \ x * 1 = x \text{ imply } x = 1.$$

$$(iii) \ ((x * y) * (x * z)) * (z * y) = 1.$$

$$(iv) \ (BCL^+ 2) \ x * y = 1 \text{ and } y * x = 1 \text{ imply } x = y.$$

Theorem 2.2 (Theorem 2.3, Liu [2]). An algebra $(Y; *, 1)$ is a BCL^+ algebra if and only if it satisfies the following conditions: for all $x, y, z \in Y$,

$$(i) \ (BCL^+ 1) \ x * x = 1.$$

$$(ii) \ (BCL^+ 2) \ x * y = 1 \text{ and } y * x = 1 \text{ imply } x = y.$$

$$(iii) \ (((x * y) * z) * ((x * z) * y)) * ((z * y) * x) = 1.$$

$$(iv) \ x * (1 * y) = x.$$

Theorem 2.3 (Theorem 2.1, Liu [4]). Let $(Y; *, 1)$ be a $BCL_{\leq}^+$ algebra. Then binary relation $\leq$ is a partial order on Y .

Theorem 2.4 (Theorem 2.2, Liu [4]). Suppose that $(Y; *, 1)$ is a BCL^+ algebra, we have that

$$((x * y) * (x * z)) * (z * y) = 1, \text{ for all } x, y, z \in Y.$$

Where $(Y; \leq)$ is partially ordered set. Then $BCL_{\leq}^+$ algebra is $(Y; *, 1)$.

3. MAIN RESULTS

Consider a bounded BCL^+ algebra $(Y; *, 1)$, we define a unary operation " $\mathfrak{R}$ " on Y by $\mathfrak{R} := x * 0$ for all $x \in Y$, and we define

$$x \vee y := (y * x) * x.$$

Note that $x \vee y$ is an upper bound of x and y .

Definition 3.1. Let Y be a BCL^+ algebra. For all $x, y, z \in Y$, we have the following identities:

$$(Y1) \ x * x = 1, 1 * x = x \text{ imply } x = 1.$$

$$(Y2) \ x * (y * z) \leq (x * y) * (x * z).$$

$$(Y3) \ (x * y) * z \leq (x * z) * (y * z).$$

$$(Y4) \ x \leq y \text{ implies } z * x \leq z * y \text{ and } y * z \leq x * z.$$

$$(Y5) \ \mathfrak{R}(0) = 1 \text{ and } \mathfrak{R}(1) = 0.$$

$$(Y6) \ x \leq y \text{ implies } \mathfrak{R}(y) \leq \mathfrak{R}(x).$$

$$(Y7) \ x \leq \mathfrak{R}(\mathfrak{R}(x)).$$

$$(Y8) \ x * y \leq \mathfrak{R}(y) * \mathfrak{R}(x).$$

$$(Y9) \ x * \mathfrak{R}(y) = y * \mathfrak{R}(x).$$

$$(Y10) \ ((x * y) * z) * ((x * z) * y) \leq (z * y) * x.$$

Definition 3.2. Let H be a nonempty subset of a BCL^+ algebra $(Y; *, 1)$. Then we say that H is a filtration if the following three axioms are satisfied:

$$(H1) \ 0 \in H.$$

$$(H2) \text{ For all } x, y \in Y, \mathfrak{R}(\mathfrak{R}(y) * \mathfrak{R}(x)) \in H \text{ and } y \in H \text{ implies either } x \in H.$$

84 (H3) For all $x, y, z \in Y$, $(z * (y * x)) * x \in H$.

85 **Definition 3.3.** If D be a nonempty subset of a BCL^+ algebra $(Y; *, 1)$. Then we say that D be a deductive system of Y
 86 if the following two axioms are satisfied:

87 (D1) $1 \in D$.

88 (D2) $x \in D$ and $x * y \in D$ implies either $y \in D$.

89 **Theorem 3.1.** Let H be a filtration of a BCL^+ algebra $(Y; *, 1)$, and suppose $y \in H$. If $\mathfrak{K}(y) \leq \mathfrak{K}(x)$ and $x \in Y$, then
 90 $x \in H$.

91 **Proof.** If $\mathfrak{K}(y) \leq \mathfrak{K}(x)$, then

$$\mathfrak{K}(y) * \mathfrak{K}(x) = 1$$

92 and hence

$$\mathfrak{K}(\mathfrak{K}(y) * \mathfrak{K}(x)) = \mathfrak{K}(1) = 0 \in H.$$

93 By Definition 3.2 (H2), we conclude that $x \in H$. $\square$

94 **Corollary 3.1.** In the situation of Theorem 3.1, if $x \leq y$ and $x \in Y$, then $x \in H$.

95 **Theorem 3.2.** Suppose that H is a nonempty subset of a BCL^+ algebra $(Y; *, 1)$, and let H be a filtration if and only if
 96 $a_1 * (a_2 * (\dots * (a_i * (\dots * (a_n * x) \dots)))) \in H$. Then $x \in H$.

97 **Proof.** Let H be a filtration, and $a_1, a_2, \dots, a_i, \dots, a_n \in H$, since

$$a_1 * (a_2 * (\dots * (a_i * (\dots * (a_n * x) \dots)))) \in H,$$

100 we have

$$a_1 \in H \text{ implies } a_2 * (\dots * (a_i * (\dots * (a_n * x) \dots))) \in H,$$

101 here

$$a_i \in H \text{ implies } a_{i+1} * (\dots * (a_n * x) \dots) \in H.$$

102 We continue like this, in the "extreme" case implies $x \in H$.

103 Conversely, let $a \in H$ and $a * x \in H$, then $x \in H$. $\square$

104 **Corollary 3.2.** Let H be a filtration of a BCL^+ algebra $(Y; *, 1)$, and suppose $a \in H$ for all $x \in Y$. If $a \leq x$, then there
 105 exists $x \in H$ such that

$$(a * x) * x \in H.$$

106 **Proof.** Let $a \leq x$, by Theorem 3.2 $a * x \in H$. Since $x \in H$, then $(a * x) * x \in H$. $\square$

107 **Theorem 3.3.** Suppose that B be subalgebra of Y , then B is a filtration and $x \in B$ if and only if $1 * x \in B$.

108 **Proof.** If B is a filtration, then $1 * x \in B$ implies $x \in B$.

109 Conversely, let $x \in B$ and $x * y \in B$. Since B is subalgebra, we have

$$\begin{aligned} 1 * y &= (x * x) * y \\ &= (x * y) * x \in B. \end{aligned}$$

110 We see that $y \in B$, and B is a filtration. $\square$

111 **Theorem 3.4.** Suppose that $\wp$ be a nonempty family filtrations of BCL^+ algebras $(Y; *, 1)$. Then $\bigcap \wp$ is also a filtration
 112 of BCL^+ algebras $(Y; *, 1)$.

113 **Proof.** Let $H = \bigcap \wp$, we consider

$$\wp = \{\wp_j \mid j \in J\},$$

114 such that $0 \in \wp_j$, then $0 \in \wp$. Let $y * x \in \wp$ and $x \in \wp$, then $j \in J$, we have $y * x \in \wp_j$ and $x \in \wp_j$. Since $\wp_j$ is
 115 filtrations of Y , then $y \in \wp_j$ and, in fact, j be arbitrary, and so $y \in \wp$ also, thus $\wp$ be filtrations of Y . $\square$

116 **Lemma 3.1.** Let D be a deductive system of a BCL^+ algebra $(Y; *, 1)$, and suppose $a \leq x$ whenever $a \in D$. Then
 117 $x \in D$.

118 **Proof.** If $a \leq x$, then

$$a * x = 1 \in D.$$

126 By Definition 3.3 (D2), we conclude that $x \in D$. $\square$

127 If Y be commutative, then $(y * x) * x = (x * y) * y$, for all $x, y \in Y$. The key to this is the following lemma.

128 **Lemma 3.2.** *Let Y be commutative and let for all $x, y, z \in Y$. Then the following equalities are satisfied:*

129 (Y11) $x * (y * z) = y * (x * z)$.

130 (Y12) $(x * y) * z = (x * z) * y$.

131 **Proof.** For part (Y11), suppose $x * y = y$ and $y * x = x$. By Definition 3.1 (Y2), we see that

132
$$x * (y * z) \leq (x * y) * (x * z) = y * (x * z),$$

$$y * (x * z) \leq (y * x) * (y * z) = x * (y * z).$$

133 We obtain

134
$$x * (y * z) = y * (x * z).$$

135 For part (Y12), suppose $y * z = y$ and $z * y = z$. By Definition 3.1 (Y3), we also use the same method for

136 $(x * y) * z = (x * z) * y$. $\square$

137 **Lemma 3.3.** *Let Y be commutative and let for all $x, y \in Y$. Then the following equalities are satisfied:*

138 (Y13) $\mathfrak{R}(\mathfrak{R}(x)) = x$.

139 (Y14) $\mathfrak{R}(\mathfrak{R}(x) * \mathfrak{R}(y)) = \mathfrak{R}(y * x)$.

140 **Proof.** For part (Y13), we have

141
$$\begin{aligned} x \vee 0 &= (0 * x) * x \\ &= (x * 0) * 0 \\ &= \mathfrak{R}(x * 0) \\ &= \mathfrak{R}(\mathfrak{R}(x)). \\ 0 \vee x &= (x * 0) * 0 \\ &= (0 * x) * x \\ &= x. \end{aligned}$$

143 It follows that if $x \leq (0 * x) * x$ and $x \geq (0 * x) * x$, then

144
$$\begin{aligned} x * ((0 * x) * x) &= ((0 * x) * x) * x \\ &= x * x \\ &= 1. \end{aligned}$$

145 By the commutativity, we get $\mathfrak{R}(\mathfrak{R}(x)) = x$.

146 Now, for part (Y14). By Definition 3.1 (Y6) and (Y8), we write

147
$$\mathfrak{R}(\mathfrak{R}(x) * \mathfrak{R}(y)) \leq \mathfrak{R}(y * x),$$

148 and use Lemma 3.2 (Y11) and (Y12), Definition 3.1 (Y5) and (Y1) to write

149
$$\begin{aligned} \mathfrak{R}(y * x) * \mathfrak{R}(\mathfrak{R}(x) * \mathfrak{R}(y)) &= ((y * x) * 0) * (((x * 0) * (y * 0)) * 0) \\ &= ((x * 0) * (y * 0)) * (((y * x) * 0) * 0) \\ &= (y * ((x * 0) * 0)) * (((y * x) * 0) * 0) \\ &= (y * ((0 * x) * x)) * ((0 * (y * x)) * (y * x)) \\ &= (y * x) * (y * x) \\ &= 1. \end{aligned}$$

150 In this case, we get

151
$$\mathfrak{R}(y * x) \leq \mathfrak{R}(\mathfrak{R}(x) * \mathfrak{R}(y)).$$

152 Thus $\mathfrak{R}(\mathfrak{R}(x) * \mathfrak{R}(y)) = \mathfrak{R}(y * x)$. $\square$

153 **Lemma 3.4.** *Let $(Y; *, 1)$ be a BCL^+ algebra and let for all $x, y \in Y$. Then*

154 (Y15) $x \leq (x * y) * y$

Proof. The proof is an easy consequence of [Lemma 3.2](#) and [Definition 3.1 \(Y1\)](#). $\square$

Theorem 3.5. Let H be a filtration of a BCL^+ algebra $(Y; *, 1)$, and assume that H is a nonempty subset of Y and Y is commutative. Then

$$\Re(x * y) \in H \text{ and } y \in H \text{ implies } x \in H \text{ for all } x, y \in Y.$$

Proof. The proof is immediate by [Lemma 3.3 \(Y14\)](#). $\square$

Theorem 3.6. Let H be a nonempty subset of a BCL^+ algebra $(Y; *, 1)$. Then H is a filtration if and only if it is a deductive system.

Proof. Let H be a filtration and suppose $a \leq x$ with $a \in H$. Since $1 \in H$, then H satisfy [Definition 3.3 \(D1\)](#).

Next, we prove [Definition 3.3 \(D2\)](#). Let $a \in H$ and use [Theorem 3.2](#) to write $a * x = a_1 \in H$, let $x \in Y$, by [Corollary 3.2](#) we have

$$a_2 = (a * x) * x \in H.$$

This yields

$$\begin{aligned} 1 * x &= (((a * x) * x) * ((a * x) * x)) * x \\ &= (a_2 * (a_1 * x)) * x \in H, \end{aligned}$$

and $1 \in H$ implies $x \in H$. Hence H is a deductive system as desired.

Conversely, if H is a deductive system, then $1 \in H$, and we see that

$$\begin{aligned} a * (x * a) &= x * (a * a) \\ &= a * (x * x) \\ &= a * 1 \in H. \end{aligned}$$

This implies $a \in H$. Since also $x * a \in H$ for each element $x \in Y$, this conditions [Definition 3.2 \(H1\)](#) and [Definition 3.2 \(H2\)](#) are satisfied.

Now, we prove [Definition 3.2 \(H3\)](#). By [Lemma 3.2](#), [Definition 3.1 \(Y2\)](#) and [Lemma 3.4 \(Y15\)](#), then we have

$$\begin{aligned} z * (((z * (y * x)) * x)) &= (z * (y * x)) * (z * x) \\ &\geq (y * x) * x \\ &\geq y. \end{aligned}$$

Thus

$$y \leq z * (((z * (y * x)) * x)),$$

and $y, z \in Y$ implies $(z * (y * x)) * x \in H$. $\square$

REFERENCES

- [1] Liu Y. H. A new branch of the pure algebra: BCL-algebras. *Advances in Pure Mathematics*. 2011;**1**:297-299.
<http://dx.doi.org/10.4236/apm.2011.15054>
- [2] Liu Y. H. On BCL+-algebras. *Advances in Pure Mathematics*. 2012;**2**:59-61.
<http://dx.doi.org/10.4236/apm.2012.21012>
- [3] Al-Kadi D. Hosny R. On BCL-algebra. *Journal of Advances in Mathematics*. 2013;**3**:184-190
- [4] Liu Y. H. Partial Orders in BCL+-algebra. *Journal of Advances in Mathematics*. 2013;**5**:630-634
- [5] Liu Y. H. Topological BCL+-algebras. *Pure and Applied Mathematics Journal*. 2014;**3**:11-13.
<http://doi:10.11648/j.pamj.20140301.13>
- [6] Al-Kadi D. Soft BCL-algebra, *International Journal of Algebra*, 2014;**8**:57-65.
<http://dx.doi.org/10.12988/ija.2014.311122>
- [7] Liu Y. H. Some distributions of BCL+-algebras. *International Journal of Algebra*. 2014;**8**:495-503.
<http://dx.doi.org/10.12988/ija.2014.4556>