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Conditions for existence or nonexistence of positive solutions to elliptic general model

Abstract

Two species of animals are competing or cooperating in the same environment. Under what conditions do they coexist peacefully? Or under what conditions does either one of the two species become extinct, that is, is either one of the two species excluded by the other? We investigate this phenomena in mathematical point of view.

In this paper we concentrate on coexistence solutions of the competition or cooperation model

$$\begin{cases} \Delta u + g(u,v) = 0 \\ \Delta v + h(u,v) = 0 \\ u|_{\partial\Omega} = v|_{\partial\Omega} = 0. \end{cases} \quad \text{in } \Omega,$$

This system is the general model for the steady state of a competitive or cooperative interacting system depending on growth conditions for g and h. The techniques used in this paper are super-sub solutions and some detailed properties of the solution of logistic equations.

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1 Introduction

A lot of research has been focused on reaction-diffusion equations modeling various systems in mathematical biology, especially the elliptic steady states of competitive and cooperative interacting processes with various boundary conditions. In earlier literature, investigations into mathematical biology models were concerned with studying those with homogeneous Neumann boundary conditions. Later on, the more important Dirichlet problems, which allow flux across the boundary, became the subject of study.

Suppose two species of animals, rabbits and squirrels for instance, are competing or cooperating in a bounded domain Ω. Let $u(x,t)$ and $v(x,t)$ be densities of the two habitats in the place x of Ω at time t. Then we have the dynamic competition model

$$\begin{cases} u_t(x,t) = \Delta u(x,t) + au(x,t) - bu^2(x,t) - cu(x,t)v(x,t) \\ v_t(x,t) = \Delta v(x,t) + dv(x,t) - fv^2(x,t) - eu(x,t)v(x,t) \\ u(x,t) = v(x,t) = 0 \ \text{ for } \ x \in \partial\Omega, \end{cases} \quad \text{in } \ \Omega \times [0,\infty),$$

or equivalently,

$$\begin{cases} u_t(x,t) = \Delta u(x,t) + u(x,t)(a - bu(x,t) - cv(x,t)) \\ v_t(x,t) = \Delta v(x,t) + v(x,t)(d - fv(x,t) - eu(x,t)) \\ u(x,t) = v(x,t) = 0 \ \text{ for } \ x \in \partial\Omega, \end{cases} \quad \text{in } \ \Omega \times [0,\infty),$$

or cooperation model

$$\begin{cases} u_t(x,t) = \Delta u(x,t) + au(x,t) - bu^2(x,t) + cu(x,t)v(x,t) \\ v_t(x,t) = \Delta v(x,t) + dv(x,t) - fv^2(x,t) + eu(x,t)v(x,t) \\ u(x,t) = v(x,t) = 0 \ \text{ for } \ x \in \partial\Omega, \end{cases} \quad \text{in } \ \Omega \times [0,\infty),$$

or equivalently,

$$\begin{cases} u_t(x,t) = \Delta u(x,t) + u(x,t)(a - bu(x,t) + cv(x,t)) \\ v_t(x,t) = \Delta v(x,t) + v(x,t)(d - fv(x,t) + eu(x,t)) \\ u(x,t) = v(x,t) = 0 \ \text{ for } \ x \in \partial\Omega. \end{cases} \quad \text{in } \ \Omega \times [0,\infty),$$

Here we are interested in the time independent, positive solutions, i.e. the positive solutions $u(x), v(x)$ of

$$\begin{cases} \Delta u(x) + u(x)(a - bu(x) - cv(x)) = 0 \\ \Delta v(x) + v(x)(d - fv(x) - eu(x)) = 0 \\ u|_{\partial\Omega} = v|_{\partial\Omega} = 0, \end{cases} \quad \text{in } \ \Omega, \tag{1}$$

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or

$$\begin{cases} \Delta u(x) + u(x)(a - bu(x) + cv(x)) = 0 \\ \Delta v(x) + v(x)(d - fv(x) + eu(x)) = 0 \\ u|_{\partial\Omega} = v|_{\partial\Omega} = 0, \end{cases} \text{ in } \Omega, \tag{2}$$

which are called the coexistence state or the steady state. The coexistence state is the positive density solution depending only on the spatial variable x, not on the time variable t, and so its existence means that the two species of animals can live peacefully and forever.

A lot of work about the existence and uniqueness of the coexistence state of the above steady state models has already been done during the last decades.(See [2], [3], [4], [7], [6], [12], [13].)

In [4], Cosner and Lazer established a sufficient and necessary conditions for the existence of positive solution to the competing system.
The following is their result:

Theorem 1.1 *In order that there exist positive smooth functions u and v in Ω satisfying (1) with $a = d$, it is necessary and sufficient that one of the following three sets of conditions hold, where λ_1 is as described in the Lemma 2.2.*

$$(1) a > \lambda_1, b > e, c < f$$
$$(2) a > \lambda_1, b = e, c = f$$
$$(3) a > \lambda_1, b < e, c > f$$

Biologically, the Theorem 1.1 implies that they can coexist peacefully if their reproduction rates are large enough and their self-limitation and competition rates are balanced each other.

In [13], Korman and Leung established a sufficient and necessary conditions for the existence of positive solution to the cooperation system. The following is their result:

Theorem 1.2 *For existence of a positive solution to (2) with $b = f = 1$, it is necessary and sufficient that $ce < 1$.*

Biologically, the Theorem 1.2 indicates that their strong cooperation may decrease reproduction capacities and increase self-limitation abilities that cause their extinction.

However, in reality, the rates of change of population densities may vary

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in a more complicated and irregular manner than can be described by the simple competition or cooperation models. In this paper we study rather general types of the system. We are concerned with the existence of positive coexistence when the relative growth rates are nonlinear, more precisely, the existence and uniqueness of a positive steady state of

$$\begin{cases} \Delta u + g(u,v) = 0 \\ \Delta v + h(u,v) = 0 \\ u|_{\partial\Omega} = v|_{\partial\Omega} = 0, \end{cases} \quad \text{in } \Omega,$$

where g and h are C^1 functions, Ω is a bounded domain in R^n and u, v are densities of the two competitive or cooperative species.

The functions g and h describe how species 1 (u) and 2 (v) interact among themselves and with each other.

The followings are questions raised in the general model with nonlinear growth rates.

Problem 1 : Under what conditions do the species coexist?

Problem 2 : When does either one of the species become extinct?

In Section 3, some sufficient and necessary conditions for the existence of positive solution in the competition system are obtained that generalizes the Theorem 1.1, and we can also see some nonexistence result. In Section 4. we establish sufficient and necessary conditions for the existence of positive solution in the cooperating system that generalizes the Theorem 1.2.

2 Preliminaries

Before entering into our primary arguments and results, we must first present a few preliminary items that we later employ throughout the proofs detailed in this paper. The following definition and lemmas are established and accepted throughout the literature on our topic.

Definition **2.1** *(Super and sub solutions)*
Consider

$$\begin{cases} \Delta u + f(x,u) = 0 \ \ in \ \ \Omega, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{3}$$

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where $f \in C^{\alpha}(\bar{\Omega} \times R)$ and Ω is a bounded domain in R^n.
(A) A function $\bar{u} \in C^{2,\alpha}(\bar{\Omega})$ satisfying

$$\begin{cases} \Delta \bar{u} + f(x, \bar{u}) \leq 0 \ \ in \ \ \Omega, \\ \bar{u}|_{\partial\Omega} \geq 0 \end{cases}$$

is called a super solution to (3).
(B) A function $\underline{u} \in C^{2,\alpha}(\bar{\Omega})$ satisfying

$$\begin{cases} \Delta \underline{u} + f(x, \underline{u}) \geq 0 \ \ in \ \ \Omega, \\ \underline{u}|_{\partial\Omega} \leq 0 \end{cases}$$

is called a sub solution to (3).

Lemma 2.1 *Let $f(x, \xi) \in C^{\alpha}(\bar{\Omega} \times R)$ and let $\bar{u}, \underline{u} \in C^{2,\alpha}(\bar{\Omega})$ be, respectively, super and sub solutions to (3) which satisfy $\underline{u}(x) \leq \bar{u}(x), x \in \bar{\Omega}$. Then (3) has a solution $u \in C^{2,\alpha}(\bar{\Omega})$ with $\underline{u}(x) \leq u(x) \leq \bar{u}(x), x \in \bar{\Omega}$.*

In our proof, we also employ accepted conclusions concerning the solutions of the following logistic equations.

Lemma 2.2 *(Established in [15])*
Consider

$$\begin{cases} \Delta u + uf(u) = 0 \ \ in \ \ \Omega, \\ u|_{\partial\Omega} = 0, u > 0, \end{cases}$$

where f is a decreasing C^1 function such that there exists $c_0 > 0$ such that $f(u) \leq 0$ for $u \geq c_0$ and Ω is a bounded domain in R^n.
If $f(0) > \lambda_1$, then the above equation has a unique positive solution, where λ_1 is the first eigenvalue of $-\Delta$ with homogeneous boundary conditions whose corresponding eigenfunction is denoted by ϕ_1. We denote this unique positive solution as θ_f.

The most important property of this positive solution is that θ_f is increasing as f is increasing.

We specifically note that for $a > \lambda_1$, the unique positive solution of

$$\begin{cases} \Delta u + u(a - u) = 0 \ \text{ in } \ \Omega, \\ u|_{\partial\Omega} = 0, u > 0, \end{cases}$$

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is denoted by $\omega_a \equiv \theta_{a-x}$. Hence, θ_a is increasing as $a > 0$ is increasing.

Consider the system

$$\begin{aligned} \Delta u + f(x,u) &= 0 \text{ in, } \Omega, \\ u &= 0 \text{ on } \partial\Omega, \end{aligned} \tag{4}$$

where $u = (u_1, ..., u_m)$ and $f = (f_1, ..., f_m)$ is quasimonotone increasing, i.e. $f_i(x,u)$ is increasing in u_j for all $j \neq i$.

Lemma 2.3 *([13]) Let w_λ be a family of subsolutions($\alpha \leq \lambda \leq \beta$) to (4), increasing in λ such that*

$$\Delta w_\lambda + f(x, w_\lambda) \geq 0 \ \ in \ \ \Omega, w_\lambda = 0 \ \ on \ \ \partial\Omega.$$

Assume also $u \geq w_\alpha$, w_λ does not satisfy (4) for any λ, and $\frac{\partial w_\lambda}{\partial n}$ changes continuously in λ on $\partial\Omega$. Then $u \geq \sup w_\lambda$.

3 Competing Species

Consider the system for two competing species of anmials

$$\begin{cases} \Delta u(x) + g(u,v) = 0 \\ \Delta v(x) + h(u,v) = 0 \\ u|_{\partial\Omega} = v|_{\partial\Omega} = 0, \end{cases} \text{ in } \Omega, \tag{5}$$

where $g, h \in C^2$ are such that $g_v < 0$ for $u \geq 0$, $h_u < 0$ for $v \geq 0$, $g_{uu} < 0, g_{uv} < 0, h_u < 0, h_{uv} < 0, h_{vv} < 0$, $g(0,v) = h(u,0) = 0$, there exist constants $c_0 > 0, c_1 > 0$ such that $g(u,0) \leq 0$ for $u \geq c_0$ and $h(0,v) \leq 0$ for $v \geq c_1$, and for constants $d > 0$ and $e > 0$, there are $c_d > 0$ and $c_e > 0$ such that $g_u(c_d, d) = h_v(e, c_e) = 0$.

The following theorem provides a sufficient condition for the existence of a positive smooth solution to (5).

Theorem 3.1 *Suppose $g_u(0,0) = hv(0,0)$ and one of the following three sets of conditions holds.*

$$\begin{array}{lll} (1) g_u(0,0) > \lambda_1, & \inf(g_{uu}) < \inf(h_{uv}), & \inf(g_{uv}) > \inf(h_{vv}) \\ (2) g_u(0,0) > \lambda_1, & \inf(g_{uu}) = \inf(h_{uv}), & \inf(g_{uv}) = \inf(h_{vv}) \\ (3) g_u(0,0) > \lambda_1, & \inf(g_{uu}) > \inf(h_{uv}), & \inf(g_{uv}) < \inf(h_{vv}) \end{array}$$

Then (5) has a positive smooth solution.

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Proof. By the Theorem 1.1, if one of the above three sets of conditions holds, then there is a positive smooth solution (u, v) to

$$\begin{array}{rl} \Delta u + u[g_u(0,0) - (-\inf(g_{uu}))u - (-\inf(g_{uv}))v] = 0 & \\ \Delta v + v[h_v(0,0) - (-\inf(h_{uv}))u - (-\inf(h_{vv}))v] = 0 & \text{in } \Omega \\ u|_{\partial\Omega} = v|_{\partial\Omega} = 0. & \end{array}$$

But, by the Mean Value Theorem,

$$\begin{array}{rl} & \Delta u + g(u,v) \\ = & \Delta u + g(u,v) - g(0,v) \\ = & \Delta u + u g_u(\tilde{u}, v) \\ = & \Delta u + u[g_u(0,0) + g_u(\tilde{u},v) - g_u(0,0)] \\ = & \Delta u + u[g_u(0,0) + g_u(\tilde{u},v) - g_u(0,v) + g_u(0,v) - g_u(0,0)] \\ = & \Delta u + u[g_u(0,0) + g_{uu}(\bar{u},v)\tilde{u} + g_{uv}(0,\bar{v})v] \\ \geq & \Delta u + u[g_u(0,0) + g_{uu}(\bar{u},v)u + g_{uv}(0,\bar{v})v] \\ \geq & \Delta u + u[g_u(0,0) + \inf(g_{uu})u + \inf(g_{uv})v] \\ = & \Delta u + u[g_u(0,0) - (-\inf(g_{uu}))u - (-\inf(g_{uv}))v] \\ = & 0, \end{array}$$

for some $0 \leq \bar{u} \leq \tilde{u} \leq u, 0 \leq \bar{v} \leq v$, and

$$\begin{array}{rl} & \Delta v + h(u,v) \\ = & \Delta v + h(u,v) - h(u,0) \\ = & \Delta v + v h_v(u, \check{v}) \\ = & \Delta v + v[h_v(0,0) + h_v(u,\check{v}) - h_v(0,0)] \\ = & \Delta v + v[h_v(0,0) + h_v(u,\check{v}) - h_v(u,0) + h_v(u,0) - h_v(0,0)] \\ = & \Delta v + v[h_v(0,0) + h_{vv}(u,\hat{v})\check{v} + h_{uv}(\hat{u},0)u] \\ \geq & \Delta v + v[h_v(0,0) + h_{vv}(u,\hat{v})v + h_{uv}(\hat{u},0)u] \\ \geq & \Delta v + v[h_v(0,0) + \inf(h_{vv})v + \inf(h_{uv})u] \\ = & \Delta v + v[h_v(0,0) - (-\inf(h_{uv}))u - (-\inf(h_{vv}))v] \\ = & 0 \end{array}$$

for some $0 \leq \hat{v} \leq \check{v} \leq v$. Hence, (u, v) is a subsolution to (5).
But by the conditions of g, h, any large positive constant M satisfying $u < M, v < M$ in Ω is a supersolution to (5).
Therefore, by the Lemma 2.1, (5) has a positive smooth solution.

The next theorem establishes a necessary condition for the existence of a positive smooth solution to (5).

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Theorem 3.2 *Suppose $g_u(0,0) = h_v(0,0)$. If (5) has a positive smooth solution, then $g_u(0,0) > \lambda_1$ and one of the following six sets of conditions holds.*

(1)$g_{uu} \equiv h_{uv}$ *are constants,* $\inf(h_{vv}) \leq \sup(g_{uv}), \sup(h_{vv}) \geq \inf(g_{uv})$
(2) $\inf(h_{uv}) = \sup(g_{uu}), \sup(h_{uv}) > \inf(g_{uu}), \inf(h_{vv}) \leq \sup(g_{uv})$
(3) $\inf(h_{uv}) > \sup(g_{uu}), \inf(h_{vv}) < \sup(g_{uv})$
(4) $\inf(h_{uv}) < \sup(g_{uu}), \sup(h_{uv}) = \inf(g_{uu}), \sup(h_{vv}) \geq \inf(g_{uv})$
(5) $\inf(h_{uv}) < \sup(g_{uu}), \sup(h_{uv}) < \inf(g_{uu}), \sup(h_{vv}) > \inf(g_{uv})$
(6) $\inf(h_{uv}) < \sup(g_{uu}), \sup(h_{uv}) > \inf(g_{uu})$

Proof. Suppose (u, v) is a positive smooth solution to (5).
By the Mean Value Theorem, there are $\tilde{u}, \bar{u}, \tilde{v}$ with $0 \leq \bar{u} \leq \tilde{u} \leq u, 0 \leq \bar{v} \leq v$ such that

$$\begin{aligned} g(u,v) - g(0,v) &= g_u(\tilde{u}, v)u, \\ g_u(\tilde{u}, 0) - g_u(0,0) &= g_{uu}(\bar{u}, 0)\tilde{u}, \\ g_u(\tilde{u}, v) - g_u(\tilde{u}, 0) &= g_{uv}(\tilde{u}, \bar{v})v. \end{aligned}$$

Hence, by the Green's Identity,

$$\begin{aligned} & \textstyle\int_\Omega u\phi_1[\lambda_1 - g_u(0,0) - g_{uu}(\bar{u},0)\tilde{u} - g_{uv}(\tilde{u},\bar{v})v]dx \\ = & \textstyle\int_\Omega u\phi_1[\lambda_1 - g_u(0,0) + g_u(0,0) - g_u(\tilde{u},0) + g_u(\tilde{u},0) - g_u(\tilde{u},v)]dx \\ = & \textstyle\int_\Omega u\phi_1[\lambda_1 - g_u(\tilde{u},v)]dx \\ = & \textstyle\int_\Omega -\phi_1 u g_u(\tilde{u},v)] + u\lambda_1\phi_1 dx \\ = & \textstyle\int_\Omega \phi_1[g(0,v) - g(u,v)] + u\lambda_1\phi_1 dx \\ = & \textstyle\int_\Omega \phi_1[-g(u,v)] + u\lambda_1\phi_1 dx \\ = & \textstyle\int_\Omega \phi_1\Delta u - u\Delta\phi_1 dx \\ = & 0. \end{aligned}$$

But, since $-g_{uu}(\bar{u},0)\tilde{u} - g_{uv}(\tilde{u},\bar{v})v > 0$ in Ω, $g_u(0,0) > \lambda_1$.
By the Mean Value Theorem again, there are u_1, v_1 with $0 \leq u_1 \leq u, 0 \leq v_1 \leq v$ such that

$$\begin{aligned} g(u,v) - g(0,v) &= g_u(u_1, v)u, \\ h(u,v) - h(u,0) &= h_v(u, v_1)v. \end{aligned}$$

Hence,

$$\begin{aligned} & \Delta u + u g_u(u,v) \\ \leq & \Delta u + u g_u(u_1,v) \\ = & \Delta u + g(u,v) - g(0,v) \\ = & \Delta u + g(u,v) \\ = & 0, \end{aligned}$$

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and

$$\begin{aligned} & \Delta v + v h_v(u,v) \\ \leq\ & \Delta v + v h_v(u,v_1) \\ =\ & \Delta v + h(u,v) - h(u,0) \\ =\ & \Delta v + h(u,v) \\ =\ & 0, \end{aligned}$$

so, (u,v) is a supersolution to

$$\begin{cases} \Delta u(x) + u g_u(u,v) = 0 \\ \Delta v(x) + v h_v(u,v) = 0 \\ u|_{\partial\Omega} = v|_{\partial\Omega} = 0. \end{cases} \quad \text{in } \Omega,$$

Let $c > 0, d > 0$ be such that $u(x) \leq c$ and $v(x) \leq d$ for all $x \in \bar{\Omega}$. Then for sufficiently small $m > 0$,

$$\begin{aligned} & \Delta[m\theta_{g_u(\cdot,d)}] + m\theta_{g_u(\cdot,d)} g_u(m\theta_{g_u(\cdot,d)}, m\theta_{h_v(c,\cdot)}) \\ =\ & m[\Delta\theta_{g_u(\cdot,d)} + \theta_{g_u(\cdot,d)} g_u(m\theta_{g_u(\cdot,d)}, m\theta_{h_v(c,\cdot)})] \\ \geq\ & m[\Delta\theta_{g_u(\cdot,d)} + \theta_{g_u(\cdot,d)} g_u(\theta_{g_u(\cdot,d)}, d)] \\ =\ & 0 \end{aligned}$$

and

$$\begin{aligned} & \Delta[m\theta_{h_v(c,\cdot)}] + m\theta_{h_v(c,\cdot)} h_v(m\theta_{g_u(\cdot,d)}, m\theta_{h_v(c,\cdot)}) \\ =\ & m[\Delta\theta_{h_v(c,\cdot)} + \theta_{h_v(c,\cdot)} h_v(m\theta_{g_u(\cdot,d)}, m\theta_{h_v(c,\cdot)})] \\ \geq\ & m[\Delta\theta_{h_v(c,\cdot)} + \theta_{h_v(c,\cdot)} h_v(c, \theta_{h_v(c,\cdot)})] \\ =\ & 0 \end{aligned}$$

So, $(m\theta_{g_u(\cdot,d)}, m\theta_{h_v(c,\cdot)})$ is a subsolution to

$$\begin{cases} \Delta u(x) + u g_u(u,v) = 0 \\ \Delta v(x) + v h_v(u,v) = 0 \\ u|_{\partial\Omega} = v|_{\partial\Omega} = 0. \end{cases} \quad \text{in } \Omega,$$

Therefore, by the Lemma 2.1, there is a positive smooth solution, again denoted by (u,v) to

$$\begin{cases} \Delta u(x) + u g_u(u,v) = 0 \\ \Delta v(x) + v h_v(u,v) = 0 \\ u|_{\partial\Omega} = v|_{\partial\Omega} = 0. \end{cases} \quad \text{in } \Omega,$$

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By the Mean Value Theorem again, there are u_1, u_2, v_1, v_2 with $0 \leq u_1, u_2 \leq u, 0 \leq v_1, v_2 \leq v$ such that

$$\begin{array}{l} g_u(u,v) - g_u(0,v) = g_{uu}(u_1,v)u \\ h_v(u,v) - h_v(0,v) = h_{uv}(u_2,v)u \\ g_u(0,v) - g_u(0,0) = g_{uv}(0,v_1)v \\ h_v(0,v) - h_v(0,0) = h_{vv}(0,v_2)v \end{array}$$

Therefore, by the Green's Identity again,

$$\begin{array}{rl} & \int_\Omega uv([h_{uv}(u_2,v) - g_{uu}(u_1,v)]u + [h_{vv}(0,v_2) - g_{uv}(0,v_1)]v)dx \\ = & \int_\Omega uv[h_{uv}(u_2,v)u + h_{vv}(0,v_2)v - g_{uu}(u_1,v)u - g_{uv}(0,v_1)v]dx \\ = & \int_\Omega uv[h_v(u,v) - h_v(0,v) + h_v(0,v) - h_v(0,0) + g_u(0,v) - g_u(u,v) - g_u(0,v) \\ & +g_u(0,0)]dx \\ = & \int_\Omega uv[h_v(u,v) - g_u(u,v)]dx \\ = & \int_\Omega v\Delta u - u\Delta v dx \\ = & 0, \end{array}$$

and so,

$$\begin{array}{l} \int_\Omega uv([\inf(h_{uv}) - \sup(g_{uu})]u + [\inf(h_{vv}) - \sup(g_{uv})]v)dx \leq 0, \\ \int_\Omega uv([\sup(h_{uv}) - \inf(g_{uu})]u + [\sup(h_{vv}) - \inf(g_{uv})]v)dx \geq 0, \end{array}$$

which derives

$$\begin{array}{l} (A)\inf(h_{uv}) = \sup(g_{uu}), \inf(h_{vv}) \leq \sup(g_{uv}), \\ (B)\inf(h_{uv}) > \sup(g_{uu}), \inf(h_{vv}) < \sup(g_{uv}), \\ (C)\inf(h_{uv}) < \sup(g_{uu}), \end{array}$$

and

$$\begin{array}{l} (A')\sup(h_{uv}) = \inf(g_{uu}), \sup(h_{vv}) \geq \inf(g_{uv}), \\ (B')\sup(h_{uv}) < \inf(g_{uu}), \sup(h_{vv}) > \inf(g_{uv}), \\ (C')\sup(h_{uv}) > \inf(g_{uu}). \end{array}$$

Combining $(A), (B), (C)$ and $(A'), (B'), (C')$ together, we establish the result of the Theorem.

We easily recognize that combining the Theorems 3.1 and 3.2 generalizes the result of Theorem 1.1 with linear growth rates.

We also prove a nonexistence result. In biological terms, this result confirms that sufficiently large reproduction capacity of the species 1 with fixed reproduction rate of the species 2 results in their extinction.

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Theorem 3.3 *If $g_u(0,0) > \frac{\nu}{\mu} h_v(0,0), -1 \leq g_{uu} < 0$, and $h_{vv} \leq -1$, where $\mu = \min[-\sup(h_{uv}), 1]$ and $\nu = \max[-\inf(g_{uv}), 1]$, then there is no positive solution to (5).*

Proof. Suppose there is a positive solution (u, v) to (5).
As we have proved in the previous theorem, there is a positive solution, again denoted by (u, v), to

$$\begin{cases} \Delta u(x) + u g_u(u,v) = 0 \\ \Delta v(x) + v h_v(u,v) = 0 \\ u|_{\partial\Omega} = v|_{\partial\Omega} = 0. \end{cases} \quad \text{in } \Omega,$$

So, by the Mean Value Theorem, the Green's Identity and the inequality conditions,

$$\begin{array}{ll} & \int_\Omega (g_u(0,0) - h_v(0,0) + [-\sup(h_{uv}) - 1]u + [1 + \inf(g_{uv})]v) uv dx \\ \leq & \int_\Omega (g_u(0,0) - h_v(0,0) + [\inf(g_{uu}) - \sup(h_{uv})]u + [\inf(g_{uv}) \\ & - \sup(h_{vv})]v) uv dx \\ \leq & \int_\Omega [g_u(0,0) - h_v(0,0) - g_u(u,0) - g_u(0,0) + h_v(u,v) + h_v(0,v) \\ & + g_u(u,v) - g_u(u,0) - h_v(0,v) + h_v(0,0)] uv dx \\ = & \int_\Omega [g_u(u,v) - h_v(u,v)] uv dx \\ = & \int_\Omega (-v\Delta u + u\Delta v) dx \\ = & 0. \end{array} \tag{6}$$

But, if $g_u(0,0) > \frac{\nu}{\mu} h_v(0,0)$, then since $g_u(0,0) \geq u$ and $h_v(0,0) \geq v$,

$$g_u(0,0) - h_v(0,0) + [-\sup(h_{uv}) - 1]u + [1 + \inf(g_{uv})]v \geq \mu g_u(0,0) - \nu h_v(0,0) > 0,$$

which contradicts to (6).

4 Cooperating Species

Consider the system for two cooperating species of animals

$$\begin{cases} \Delta u(x) + g(u(x), v(x)) = 0 \\ \Delta v(x) + h(u(x), v(x)) = 0 \\ u|_{\partial\Omega} = v|_{\partial\Omega} = 0, \end{cases} \quad \text{in } \Omega, \tag{7}$$

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where $g, h \in C^2$ are such that $g_{uu} < 0, g_{uv} > 0, h_{uv} > 0, h_{vv} < 0$, and $g(0, v) = h(u, 0) = 0$.

The following Theorem proves a necessary condition for the existence of a positive solution to (7).

Theorem 4.1 *If $g_u(0,0) > \lambda_1, h_v(0,0) > \lambda_1$, $\inf(h_{vv}) \geq -1$, and $\inf(h_{uv}) > 0$, then the existence of a positive solution to (7) implies*

$$\inf(g_{uv})\inf(h_{uv}) + \inf(g_{uu}) < 0.$$

Proof. Suppose $\inf(g_{uv})\inf(h_{uv})+\inf(g_{uu}) \geq 0$. Consider a family $(u_\lambda, v_\lambda) = (\lambda\phi_1, \lambda\inf(h_{uv})\phi_1)$ with any $\lambda > 0$.
Then by the assumption and Mean Value Theorem,

$$\begin{aligned}
& \Delta u_\lambda + g(u_\lambda, v_\lambda) \\
= \; & -\lambda\lambda_1\phi_1 + g(\lambda\phi_1, \lambda\inf(h_{uv})\phi_1) \\
= \; & -\lambda\lambda_1\phi_1 + g(\lambda\phi_1, \lambda\inf(h_{uv})\phi_1) - g(0, \lambda\inf(h_{uv})\phi_1) \\
= \; & -\lambda\lambda_1\phi_1 + \lambda\phi_1 g_u(\tilde{u}, \lambda\inf(h_{uv})\phi_1) \\
= \; & \lambda\phi_1(-\lambda_1 + g_u(\tilde{u}, \lambda\inf(h_{uv})\phi_1) - g_u(\tilde{u}, 0) + g_u(\tilde{u}, 0) - g_u(0,0) + g_u(0,0)) \\
\geq \; & \lambda\phi_1(g_u(0,0) - \lambda_1 + \inf(g_{uv})\inf(h_{uv})\lambda\phi_1 + \inf(g_{uu})\tilde{u}) \\
\geq \; & \lambda\phi_1(g_u(0,0) - \lambda_1 + \inf(g_{uv})\inf(h_{uv})\lambda\phi_1 + \inf(g_{uu})\lambda\phi_1) \\
> \; & 0,
\end{aligned}$$

for some $0 \leq \tilde{u} \leq \lambda\phi_1$ and

$$\begin{aligned}
& \Delta v_\lambda + h(u_\lambda, v_\lambda) \\
= \; & -\lambda\inf(h_{uv})\lambda_1\phi_1 + h(\lambda\phi_1, \lambda\inf(h_{uv})\phi_1) \\
= \; & -\lambda\inf(h_{uv})\lambda_1\phi_1 + h(\lambda\phi_1, \lambda\inf(h_{uv})\phi_1) - h(\lambda\phi_1, 0) \\
= \; & -\lambda\inf(h_{uv})\lambda_1\phi_1 + \lambda\inf(h_{uv}\phi_1 h_v(\lambda\phi_1, \tilde{v}) \\
= \; & \lambda\inf(h_{uv})\phi_1[-\lambda_1 + h_v(\lambda\phi_1, \tilde{v}) - h_v(\lambda\phi_1, 0) + h_v(\lambda\phi_1, 0) - h_v(0,0) + h_v(0,0)] \\
= \; & \lambda\inf(h_{uv})\phi_1[h_v(0,0) - \lambda_1 + \inf(h_{vv})\tilde{v} + \inf(h_{uv})\lambda\phi_1] \\
\geq \; & \lambda\inf(h_{uv})\phi_1[h_v(0,0) - \lambda_1 + \inf(h_{vv})\inf(h_{uv})\lambda\phi_1 + \inf(h_{uv})\lambda\phi_1] \\
> \; & 0,
\end{aligned}$$

for some $0 \leq \tilde{v} \leq \lambda\inf(h_{uv})\phi_1$.
Therefore, $(u_\lambda, v_\lambda) = (\lambda\phi_1, \lambda\inf(h_{uv})\phi_1)$ with any $\lambda > 0$ is a family of subsolutions to (7).
Furthermore, if (u, v) is a positive solution to (7), then $u > \lambda_0\phi_1$ and

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$v > \lambda_0 \inf(h_{uv})\phi_1$ for sufficiently small $\lambda_0 > 0$, and so by the lemma 2.3, we conclude that $u \geq \lambda\phi_1$ and $v \geq \lambda \inf(h_{uv})\phi_1$ for any $\lambda \geq \lambda_0$.
Hence, there is no positive solution to (7).

We now establish a sufficient condition for the existence of a positive solution to (7).

Theorem 4.2 *If $g_u(0,0) > \lambda_1, h_v(0,0) > \lambda_1$ and the system of equations*

$$\begin{aligned} g(u,v) &= 0 \\ h(u,v) &= 0 \end{aligned}$$

has a solution (A,B) with $A > 0, B > 0$, then (7) has a positive solution.

Proof. Let $\underline{u} = \alpha\phi_1, \underline{v} = \beta\phi_1$, where $\alpha, \beta > 0$.
Then since $g_u(0,0) > \lambda_1$ and $h_v(0,0) > \lambda_1$, by the Mean Value Theorem, for small enough $\alpha, \beta > 0$,

$$\begin{aligned} & \Delta\underline{u} + g(\underline{u},\underline{v}) \\ = \; & \Delta\underline{u} + g(\underline{u},\underline{v}) - g(0,\underline{v}) \\ = \; & -\alpha\lambda_1\phi_1 + \alpha\phi_1 g_u(\tilde{u}, \beta\phi_1) \\ = \; & \alpha\phi_1[-\lambda_1 + g_u(\tilde{u}, \beta\phi_1)] \\ = \; & \alpha\phi_1[-\lambda_1 + g_u(0,0) + g_u(\tilde{u}, \beta\phi_1) - g_u(0, \beta\phi_1) + g_u(0, \beta\phi_1) - g_u(0,0)] \\ \geq \; & \alpha\phi_1[-\lambda_1 + g_u(0,0) + \inf(g_{uu})\tilde{u} + \inf(g_{uv})\beta\phi_1] \\ \geq \; & 0, \end{aligned}$$

for some $\tilde{u}$ with $0 \leq \tilde{u} \leq \underline{u}$ and

$$\begin{aligned} & \Delta\underline{v} + h(\underline{u},\underline{v}) \\ = \; & \Delta\underline{v} + h(\underline{u},\underline{v}) - h(\underline{u},0) \\ = \; & -\beta\lambda_1\phi_1 + \beta\phi_1 h_v(\alpha\phi_1, \tilde{v}) \\ = \; & \beta\phi_1[-\lambda_1 + h_v(\alpha\phi_1, \tilde{v})] \\ = \; & \beta\phi_1[-\lambda_1 + h_v(0,0) + h_v(\alpha\phi_1, \tilde{v}) - h_v(0, \tilde{v}) + h_v(0, \tilde{v}) - h_v(0,0)] \\ \geq \; & \beta\phi_1[-\lambda_1 + h_v(0,0) + \inf(h_{uv})\alpha\phi_1 + \inf(h_{vv})\tilde{v}] \\ \geq \; & 0, \end{aligned}$$

for some $\tilde{v}$ with $0 \leq \tilde{v} \leq \underline{v}$,
and so, $(\underline{u},\underline{v}) = (\alpha\phi_1, \beta\phi_1)$ is a subsolution to (7) for sufficiently small $\alpha, \beta > 0$.
But, by the condition for g and h, there is a supersolution (M,N) with

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$M > 0, N > 0$ to (7).
We conclude by the Lemma 2.1 that there is a positive solution (u, v) to (7) with $\alpha\phi_1 \leq u \leq M, \beta\phi_1 \leq v \leq N$.

We easily recognize that combining the Theorems 4.1 and 4.2 generalizes the result of Theorem 1.2 with linear growth rates.

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