

UNIQUENESS OF q -DIFFERENCE ON SHARING ONE VALUE

ABSTRACT. In this paper, we investigate uniqueness problems of q -difference transcendental meromorphic functions with zero order sharing one value. We obtain some results on q -difference, which extend many previous results.

1. INTRODUCTION AND MAIN RESULTS

In this paper, a meromorphic function always means meromorphic in the whole complex plane except at possible poles. If no poles occur, it reduces to an entire function. Let q be non-zero complex constant in what follows, and q -difference of $f(z)$ be defined by $f(qz)$. We assume the reader is familiar with the standard notations and results such as the proximity function $m(r, f)$, counting function $N(r, f)$, characteristic function $T(r, f)$, the elementary Nevanlinna theory, see, e.g., [4, 6, 13]. We denote by $S(r, f)$ any quantity satisfying $S(r, f) = o(T(r, f))$, as $r \rightarrow \infty$ possibly outside a set of log logarithmic density 0.

We say that f, g are meromorphic and share a value a IM (ignoring multiplicities) when $f - a$ and $g - a$ have the same zeros. If $f - a$ and $g - a$ have the same zeros with the same multiplicities, then we say that f and g share a CM (counting multiplicities).

In 1959, Hayman[3] proved that $f^n f'$ takes every non-zero complex value infinitely often if $n \geq 3$, f is a transcendental meromorphic function, and n is a positive integer. In particular, he proved the following famous theorem.

Theorem A. *If f is a transcendental meromorphic function, $n(\geq 5)$ is a positive integer, and $a(\neq 0)$ is a constant, then $f'(z) - af(z)^n$ takes every finite complex value b infinitely often. If f is a transcendental entire function, then $n \geq 3$ or $n \geq 2$, when $b = 0$ also holds.*

Liu[9] considered the difference counterpart of Theorem A, and proved the following result.

2010 *Mathematics Subject Classification.* 30D35.

Key words and phrases. uniqueness; meromorphic functions; q -difference; share value; zero order.

This research was supported by the NNSF of China (No. 11201014, 11171013 and 11126036), and the YWF-14-SXXY-008 of Beihang University. This research was also supported by the youth talent program of Beijing No. 29201443.

Theorem B. *If f is a transcendental meromorphic function with finite order, period is not c , and a is a non-zero constant, $n \geq 8$, then $f(z)^n + a(f(z+c) - f(z)) - s(z)$ has infinitely many zeros.*

In this paper, on the basis of Theorems A and B, we study uniqueness problems of two q -difference polynomials sharing one value and obtain the following results.

Theorem 1.1. *Let f and g be two transcendental meromorphic functions with zero order, $a \in \mathbb{C} \setminus \{0\}$, $b \in \mathbb{C}$ and let n be a positive integer satisfying $n \geq 14$. Assume that the functions*

$$\phi_f := \frac{-af(qz) + b}{f^n} \quad \text{and} \quad \phi_g := \frac{-ag(qz) + b}{g^n} \quad (1.1)$$

share the value 1 CM. Then

$$\phi_f = \phi_g \quad (1.2)$$

or

$$\phi_f \cdot \phi_g = 1. \quad (1.3)$$

Theorem 1.2. *Let f and g be two transcendental entire functions with zero order, $a \in \mathbb{C} \setminus \{0\}$, $b \in \mathbb{C}$ and let n be a positive integer satisfying $n \geq 8$. Assume that the functions ϕ_f and ϕ_g defined as in (1.1) share the value 1 CM. If $b \neq 0$, then (1.2) holds. If $b = 0$, then $f(z) = c_1 e^{p(z)}$ and $g(z) = c_2 e^{-p(z)}$, where c_1, c_2 are two constants satisfying $(c_1 c_2)^{n-1} = a^2$, and $p(z)$ is a nonconstant polynomial.*

When considering two functions sharing one value IM, we have similar results reading as follows.

Theorem 1.3. *Let f and g be two transcendental meromorphic functions with zero order, $a \in \mathbb{C} \setminus \{0\}$, $b \in \mathbb{C}$ and let n be a positive integer satisfying $n \geq 26$. Assume that the functions ϕ_f and ϕ_g defined as in (1.1) share the value 1 IM. Then (1.2) or (1.3) holds.*

Theorem 1.4. *Let f and g be two transcendental entire functions with zero order, $a \in \mathbb{C} \setminus \{0\}$, $b \in \mathbb{C}$ and let n be a positive integer satisfying $n \geq 14$. Assume that the functions ϕ_f and ϕ_g defined as in (1.1) share the value 1 IM. If $b \neq 0$, then (1.2) holds. If $b = 0$, then $f(z) = c_1 e^{p(z)}$ and $g(z) = c_2 e^{-p(z)}$, where c_1, c_2 are two constants satisfying $(c_1 c_2)^{n-1} = a^2$, and $p(z)$ is a nonconstant polynomial.*

2. LEMMAS

In this section, we present some lemmas which play an important role in the following proofs. The following q -shift difference analogue of the logarithmic derivative lemma is very important when considering q -shift difference polynomials.

Lemma 2.1 ([10, Theorem 2.1]). *Let $f(z)$ be a meromorphic function of zero order. Then*

$$m\left(r, \frac{f(qz+c)}{f(z)}\right) = o(T(r, f)),$$

on a set of logarithmic density 1.

The next two lemmas are essential for our proofs, which provide us with estimating the characteristic function and the counting function of $f(qz)$, see the Theorems 1.1 and 1.3 in [16].

Lemma 2.2. *Let $f(z)$ be a nonconstant zero order meromorphic function, and $q \in \mathbb{C} \setminus \{0\}$. Then*

$$T(r, f(qz)) = T(r, f(z)) + S(r, f)$$

on a set of logarithmic density 1.

Lemma 2.3. *Let $f(z)$ be a nonconstant zero order meromorphic function, and $q \in \mathbb{C} \setminus \{0\}$. Then*

$$N(r, f(qz)) = (1 + o(1))N(r, f(z))$$

on a set of lower logarithmic density 1.

Now, we introduce some notations. Let F and G be two nonconstant meromorphic functions, and $a \in \mathbb{C} \cup \infty$. Suppose that z_0 is an a -point of F of order p and an a -point of G of order q . We denote by $N_A(r, a)$ the counting function of those common a -point of F and G satisfying the condition “A”, each point is counted once. For example, $N_{p=1=q}(r, a)$ denotes the counting function of those common simple a -point of F and G , $N_{p>q}(r, a)$ denotes the counting function of those common a -point of F and G with the orders $p > q$. Furthermore, we denote by $N_p(r, \frac{1}{F-a})$ the counting function of the zeros of $F - a$ with the orders less than or equal to p , and by $N_{(p+1)}(r, \frac{1}{F-a})$ the counting function of the zeros of $F - a$ with the orders larger than p , each point is counted once. However, $N_p(r, \frac{1}{f-a})$ denotes the counting function of the zeros of $f - a$ where m -fold zeros are counted m times if $m \leq p$ and p times if $m > p$. Denote

$$N^*(r, a) = N_{(2)}\left(r, \frac{1}{F-a}\right) + N_{(2)}\left(r, \frac{1}{G-a}\right) - N_{p>1, q>1}(r, a) \geq 0. \quad (2.1)$$

The auxiliary function H in the following lemma plays an important role when considering two meromorphic functions that share one finite value.

Lemma 2.4 ([15, Lemma 3]). *Let*

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right), \quad (2.2)$$

where F and G are two nonconstant meromorphic functions. If $H \not\equiv 0$, then

$$N_{p=1=q}(r, 1) \leq N(r, H) + S(r, F) + S(r, G).$$

When considering two nonconstant meromorphic functions F, G that share at least one finite value IM, the following lemma plays a key role. The proof of the lemma in J.L Zhang is submitting, for the convenience of readers, we list here again. In the original paper, $S(r, F)$ denotes any quantity of $S(r, F) = o(T(r, F))$, as $r \rightarrow \infty$ possibly outside a set of finite linear measure. So it holds when $S(r, F) = o(T(r, F))$, as $r \rightarrow \infty$ possibly outside a set of logarithmic density 0.

Lemma 2.5. *Let F and G be two nonconstant meromorphic functions. If F and G share 1 IM, then one of the following three cases holds:*

- (1) $T(r, F) + T(r, G)$
 $\leq 3\{\overline{N}(r, 1/F) + \overline{N}(r, 1/G) + \overline{N}(r, F) + \overline{N}(r, G)\}$
 $+ 2\{N_2(r, 1/F) + N_2(r, 1/G) + N_2(r, F) + N_2(r, G)$
 $- N_{p>1, q>1}(r, 0) - N_{p>1, q>1}(r, \infty)\} + S(r, F) + S(r, G), \quad (2.3)$
- (2) $FG = 1$,
- (3) $F = G$.

Proof. Let H be given as in (2.2). Then

$$\begin{aligned} N(r, H) &\leq N^*(r, 0) + N^*(r, \infty) + N_{p>q}(r, 1) + N_{p<q}(r, 1) \\ &\quad + N_0(r, 1/F') + N_0(r, 1/G'), \end{aligned} \quad (2.4)$$

where $N_0(r, 1/F')$ denotes the counting function corresponding to the zeros of F' which are not the zeros of F and $F - 1$, and correspondingly for G' . If $H \neq 0$, we get from Lemma 2.4 and (2.4) that

$$\begin{aligned} &\overline{N}\left(r, \frac{1}{F-1}\right) + \overline{N}\left(r, \frac{1}{G-1}\right) \\ &= 2N_{p=q=1}(r, 1) + 2N_{p>q}(r, 1) + 2N_{p<q}(r, 1) + 2N_{p=q>1}(r, 1) \\ &\leq N_{p=q=1}(r, 1) + N_{p>q}(r, 1) + N(r, 1/(G-1)) \\ &\leq N_{p=q=1}(r, 1) + N_{p>q}(r, 1) + T(r, G) + O(1) \\ &\leq N^*(r, 0) + N^*(r, \infty) + 2N_{p>q}(r, 1) + N_{p<q}(r, 1) + T(r, G) \\ &\quad + N_0(r, 1/F') + N_0(r, 1/G') + S(r, F) + S(r, G). \end{aligned}$$

From the second fundamental theorem, we have

$$\begin{aligned} T(r, F) &\leq \overline{N}(r, 1/F) + \overline{N}(r, F) + \overline{N}\left(r, \frac{1}{F-1}\right) - N_0(r, F') + S(r, F), \\ T(r, G) &\leq \overline{N}(r, 1/G) + \overline{N}(r, G) + \overline{N}\left(r, \frac{1}{G-1}\right) - N_0(r, G') + S(r, G). \end{aligned}$$

Combining the above three inequalities, we obtain

$$\begin{aligned} T(r, F) + T(r, G) &\leq N_2(r, 1/F) + N_2(r, 1/G) - N_{p>1, q>1}(r, 0) + N_2(r, F) \\ &\quad + N_2(r, G) - N_{p>1, q>1}(r, \infty) + 2N_{p>q}(r, 1) + N_{p<q}(r, 1) \\ &\quad + T(r, G) + S(r, F) + S(r, G), \end{aligned}$$

which

$$\begin{aligned} T(r, F) &\leq N_2(r, 1/F) + N_2(r, 1/G) - N_{p>1, q>1}(r, 0) + N_2(r, F) \\ &\quad + N_2(r, G) - N_{p>1, q>1}(r, \infty) + 2N_{p>q}(r, 1) + N_{p<q}(r, 1) \\ &\quad + S(r, F) + S(r, G). \end{aligned}$$

Similarly,

$$\begin{aligned} T(r, G) &\leq N_2(r, 1/F) + N_2(r, 1/G) - N_{p>1, q>1}(r, 0) + N_2(r, F) \\ &\quad + N_2(r, G) - N_{p>1, q>1}(r, \infty) + N_{p>q}(r, 1) + 2N_{p<q}(r, 1) \\ &\quad + S(r, F) + S(r, G). \end{aligned}$$

Then

$$\begin{aligned} T(r, F) + T(r, G) &\leq 2\{N_2(r, 1/F) + N_2(r, 1/G) - N_{p>1, q>1}(r, 0) + N_2(r, F) \\ &\quad + N_2(r, G) - N_{p>1, q>1}(r, \infty)\} + 3\{N_{p>q}(r, 1) + N_{p<q}(r, 1)\} \\ &\quad + S(r, F) + S(r, G), \end{aligned} \quad (2.5)$$

Noting that

$$\begin{aligned} N_{p>q}(r, 1) &\leq N(r, F/F') \leq N(r, F'/F) + S(r, F) \\ &\leq \overline{N}(r, 1/F) + \overline{N}(r, F) + S(r, F), \end{aligned} \quad (2.6)$$

$$\begin{aligned} N_{p<q}(r, 1) &\leq N(r, G/G') \leq N(r, G'/G) + S(r, G) \\ &\leq \overline{N}(r, 1/G) + \overline{N}(r, G) + S(r, G). \end{aligned} \quad (2.7)$$

The conclusion (1) follows by inserting the last two inequalities into (2.5).

If $H \neq 0$, by integration, we get from (2.2) that

$$\frac{1}{F-1} = \frac{A}{G-1} + B, \quad (2.8)$$

where $A(\neq 0)$ and B are constants. From (2.8) we have

$$F = \frac{(1+B)G + A - B - 1}{BG + A - B}, \quad G = \frac{(B-A)F + A - B - 1}{BF - (B+1)}. \quad (2.9)$$

We discuss the following three cases.

Case 1. Suppose that $B \neq 0, -1$. If $B \neq A$, we get from (2.9) that $\overline{N}(r, 1/(F - \frac{B+1}{B})) = \overline{N}(r, G)$, $\overline{N}(r, 1/(G + \frac{A-B}{B})) = \overline{N}(r, F)$. From the second fundamental theorem, we have

$$\begin{aligned} T(r, F) &\leq \overline{N}(r, F) + \overline{N}(r, 1/F) + \overline{N}(r, 1/(F - \frac{B+1}{B})) + S(r, F) \\ &\leq \overline{N}(r, F) + \overline{N}(r, 1/F) + \overline{N}(r, G) + S(r, F), \end{aligned}$$

and

$$\begin{aligned} T(r, G) &\leq \overline{N}(r, G) + \overline{N}(r, 1/G) + \overline{N}(r, 1/(G + \frac{A-B}{B})) + S(r, G) \\ &\leq \overline{N}(r, G) + \overline{N}(r, 1/G) + \overline{N}(r, F) + S(r, G). \end{aligned}$$

Thus

$$T(r, F) + T(r, G) \leq \overline{N}(r, 1/F) + \overline{N}(r, 1/G) + 2\overline{N}(r, F) + 2\overline{N}(r, G) + S(r, G) + S(r, F).$$

The conclusion (1) holds. If $B = A$, rewrite (2.9) as

$$F = \frac{(1+B)G-1}{BG}, \quad G = \frac{-1}{BF-(B+1)}.$$

Noting that $\bar{N}(r, 1/(F - \frac{B+1}{B})) = \bar{N}(r, G)$ and $\bar{N}(r, 1/(G - \frac{1}{B+1})) = \bar{N}(r, F)$. The conclusion (1) holds by the second fundamental theorem and the same arguments as above.

Case 2. Suppose that $B = 0$. From (2.9) we have

$$F = \frac{(G+A-1)}{A}, \quad G = AF - (A-1). \quad (2.10)$$

If $A \neq 1$, by (2.10) we obtain $\bar{N}(r, \frac{1}{G+A-1}) = \bar{N}(r, 1/F)$ and $\bar{N}(r, \frac{1}{F-\frac{A-1}{A}}) = \bar{N}(r, 1/G)$. The conclusion (1) follows again by the second fundamental theorem. If $A = 1$, then $F = G$, which is the conclusion (3).

Case 3. Suppose that $B = -1$. (2.9) yields

$$F = \frac{(-A)}{G-A+B}, \quad G = \frac{(A+1)F-A}{F}. \quad (2.11)$$

If $A \neq -1$, we obtain the conclusion (1) by the same reasoning discussed in Case 2. If $A = -1$, we get $FG = 1$ from (2.11). $\square$

The following corresponding result is about two meromorphic functions sharing 1 CM.

Lemma 2.6 ([12, Lemma 3]). *Let F and G be two nonconstant meromorphic functions. If F and G share 1 CM, then one of the following three cases holds:*

- (1) $\max\{T(r, F), T(r, G)\} \leq N_2(r, 1/F) + N_2(r, 1/G) + N_2(r, F) + N_2(r, G) + S(r, F) + S(r, G),$
- (2) $FG = 1,$
- (3) $F = G,$

where $N_2(r, 1/F)$ denotes the counting function of zeros of F such that simple zeros are counted once and multiple zeros twice.

3. PROOFS OF THE THEOREMS

In this section, the proofs of our results are given.

Proof of Theorem 1.4. The first main theorem and Lemma 2.2 give

$$\begin{aligned} nT(r, f) &= T(r, 1/f^n) + O(1) \\ &\leq T\left(r, \frac{-af(qz)+b}{f^n}\right) + T\left(r, \frac{1}{-af(qz)+b}\right) + O(1) \\ &\leq T(r, \phi_f) + T(r, f) + S(r, f), \end{aligned} \quad (3.1)$$

which

$$(n-1)T(r, f) \leq T(r, \phi_f) + S(r, f). \quad (3.2)$$

Similarly,

$$(n-1)T(r, g) \leq T(r, \phi_g) + S(r, g). \quad (3.3)$$

Then ϕ_f is nonconstant, so is ϕ_g . From the conditions of Theorem 1.3, we know that ϕ_f and ϕ_g share 1 IM. By (1.1) and Lemma 2.3, we get

$$N_2(r, 1/\phi_f) \leq N_2\left(r, \frac{1}{-af(qz) + b}\right) + 2\overline{N}(r, f),$$

and

$$\begin{aligned} N_2(r, \phi_f) &\leq N(r, f(qz)) + 2\overline{N}(r, 1/f) + S(r, f) \\ &\leq N(r, f) + 2\overline{N}(r, 1/f) + S(r, f). \end{aligned}$$

Similarly, we get

$$\begin{aligned} N_2(r, 1/\phi_g) &\leq N_2\left(r, \frac{1}{-ag(qz) + b}\right) + 2\overline{N}(r, g), \\ N_2(r, \phi_g) &\leq N(r, g) + 2\overline{N}(r, 1/g) + S(r, g). \end{aligned}$$

Suppose that (2.3) in Lemma 2.5 holds. The above two inequalities, Lemmas 2.2 and 2.3 yield

$$\begin{aligned} T(r, \phi_f) + T(r, \phi_g) &\leq 3\{\overline{N}(r, 1/\phi_f) + \overline{N}(r, \phi_f) + \overline{N}(r, 1/\phi_g) + \overline{N}(r, \phi_g)\} \\ &\quad + 2\{N_2(r, 1/\phi_f) + N_2(r, 1/\phi_g) + N_2(r, \phi_f) + N_2(r, \phi_g)\} \\ &\quad + S(r, \phi_f) + S(r, \phi_g) \\ &\leq 3\left\{\overline{N}\left(r, \frac{1}{f(qz) - b/a}\right) + 2\overline{N}(r, f) + \overline{N}\left(r, \frac{1}{g(qz) - b/a}\right)\right. \\ &\quad + 2\overline{N}(r, g) + \overline{N}(r, 1/f) + \overline{N}(r, 1/g)\} + 2\left\{N_2\left(r, \frac{1}{f(qz) - b/a}\right)\right. \\ &\quad + N_2\left(r, \frac{1}{g(qz) - b/a}\right) + 2\overline{N}(r, f) + 2\overline{N}(r, g) + 2\overline{N}(r, 1/f) \\ &\quad + 2\overline{N}(r, 1/g) + N(r, f) + N(r, g)\} + S(r, f) + S(r, g) \\ &\leq 24[T(r, f) + T(r, g)] + S(r, f) + S(r, g). \end{aligned} \quad (3.4)$$

Combining (3.2) and (3.3) with the last inequality, we obtain

$$(n-25)[T(r, f) + T(r, g)] \leq S(r, f) + S(r, g).$$

which is a contradiction since $n \geq 26$. Lemma 2.5 gives $\phi_f = \phi_g$ or $\phi_f \cdot \phi_g = 1$. Therefore, Theorem 1.3 is proved. $\square$

Proof of Theorem 1.4. Let ϕ_f and ϕ_g be defined as in (1.1). Suppose that (2.3) holds. By the same arguments as in the proof of Theorem 1.3, noting that f and g are entire functions, (3.2)-(3.4) turn out to be

$$\begin{aligned} (n-1)T(r, f) &\leq T(r, \phi_f) + S(r, f), \\ (n-1)T(r, g) &\leq T(r, \phi_g) + S(r, g), \end{aligned}$$

and

$$T(r, \phi_f) + T(r, \phi_g) \leq 12[T(r, f) + T(r, g)] + S(r, f) + S(r, g).$$

Combining the last three inequalities, we get

$$(n - 13)[T(r, f) + T(r, g)] \leq S(r, f) + S(r, g),$$

which contradicting with $n \geq 14$. Then from Lemma 2.5, we have $\phi_f = \phi_g$ or $\phi_f \cdot \phi_g = 1$.

If $\phi_f = \phi_g$ holds, (1.2) follows by Lemma 2.5.

Next, we assume that $\phi_f \cdot \phi_g = 1$ holds. We know from (1.1) that

$$f^n g^n = (af(qz) - b)(ag(qz) - b). \quad (3.5)$$

We consider two cases in the following.

Case 1. If $b = 0$. By the above equation, we get

$$f^n g^n = a^2 f(qz)g(qz).$$

Let $fg = h$. If h is not a constant, then by the above equation, we have

$$h^n = a^2 h(qz). \quad (3.6)$$

Thus, from the first main theorem, we obtain

$$\begin{aligned} nT(r, h) &= T(r, h^n) = T(r, ah(qz)) + O(1) \\ &\leq T(r, h) + S(r, h). \end{aligned}$$

Since $n \geq 2$, we know that h is a constant, i.e. fg is a constant, so f and g have no zeros. Hence, according to the Decomposition Theorem, we suppose that $f(z) = c_1 e^{p(z)}$, $g(z) = c_2 e^{-p(z)}$, where $p(z)$ is a nonconstant polynomial. And by (3.6), we get $h^{n-1} = a^2$, thus $(c_1 c_2)^{n-1} = a^2$.

Case 2. If $b \neq 0$. The equation 3.5 gives that

$$\begin{aligned} N(r, 1/f) + N(r, 1/g) &= \frac{1}{n} N\left(r, \frac{1}{f(qz) - b/a}\right) + \frac{1}{n} N\left(r, \frac{1}{g(qz) - b/a}\right), \\ \overline{N}\left(r, \frac{1}{f(qz) - b/a}\right) + \overline{N}\left(r, \frac{1}{g(qz) - b/a}\right) &\leq 2\overline{N}\left(r, \frac{1}{f}\right) + 2\overline{N}\left(r, \frac{1}{g}\right), \end{aligned}$$

and the second main theorem gives that

$$T(r, f(qz)) \leq \overline{N}\left(r, \frac{1}{f(qz)}\right) + \overline{N}\left(r, \frac{1}{f(qz) - b/a}\right) + S(r, f).$$

Similarly,

$$T(r, g(qz)) \leq \overline{N}\left(r, \frac{1}{g(qz)}\right) + \overline{N}\left(r, \frac{1}{g(qz) - b/a}\right) + S(r, g).$$

The above four inequalities and Lemma 2.2 provide us that

$$\begin{aligned}
 T(r, f) + T(r, g) &\leq \overline{N}\left(r, \frac{1}{f(qz)}\right) + \overline{N}\left(r, \frac{1}{f(qz) - b/a}\right) + \overline{N}\left(r, \frac{1}{g(qz)}\right) \\
 &\quad + \overline{N}\left(r, \frac{1}{g(qz) - b/a}\right) + S(r, f) + S(r, g) \\
 &\leq 3(\overline{N}(r, 1/f) + N(r, 1/g)) + S(r, f) + S(r, g) \\
 &\leq \frac{3}{n}\left(N\left(r, \frac{1}{f(z+c) - b/a}\right) + N\left(r, \frac{1}{g(z+c) - b/a}\right)\right) \\
 &\quad + S(r, f) + S(r, g) \\
 &\leq \frac{3}{n}(T(r, f) + T(r, g)) + S(r, f) + S(r, g),
 \end{aligned}$$

contradicting with $n \geq 14$. This completes the proof of Theorem 1.4. $\square$

Proof of Theorem 1.1. By the same arguments as in the proof of Theorem 1.3, noting that ϕ_f and ϕ_g share 1 CM, using Lemma 2.6 instead of Lemma 2.5, (3.4) turns out to be

$$\begin{aligned}
 T(r, \phi_f) &\leq N_2(r, 1/\phi_f) + N_2(r, \phi_f) + N_2(r, 1/\phi_g) + N_2(r, \phi_g) + S(r, \phi_f) + S(r, \phi_g) \\
 &\leq N_2\left(r, \frac{1}{-af(qz) + b}\right) + 2\overline{N}(r, f) + N(r, f) + N_2\left(r, \frac{1}{-ag(qz) + b}\right) \\
 &\quad + 2\overline{N}(r, 1/f) + 2\overline{N}(r, g) + N(r, g) + 2\overline{N}(r, 1/g) + S(r, f) + S(r, g) \\
 &\leq 6[T(r, f) + T(r, g)] + S(r, f) + S(r, g). \tag{3.7}
 \end{aligned}$$

Similarly,

$$T(r, \phi_g) \leq 6[T(r, f) + T(r, g)] + S(r, f) + S(r, g).$$

Combining the above two inequalities with (3.2), (3.3), we get

$$(n - 13)[T(r, f) + T(r, g)] \leq S(r, f) + S(r, g),$$

which is a contradiction since $n \geq 14$. Lemma 2.6 gives $\phi_f = \phi_g$ or $\phi_f \cdot \phi_g = 1$. Therefore, Theorem 1.1 is thus proved. $\square$

Proof of Theorem 1.2. Let ϕ_f and ϕ_g be defined as in (1.1). By the same arguments as in the proof of Theorem 1.3, noting that f and g are entire functions, (3.7) becomes

$$T(r, \phi_f) \leq 3[T(r, f) + T(r, g)] + S(r, f) + S(r, g).$$

We insert the above inequality into (3.2), resulting in

$$(n - 1)T(r, f) \leq 3[T(r, f) + T(r, g)] + S(r, f) + S(r, g).$$

The same inequality holds for $T(r, g)$, then

$$(n - 7)[T(r, f) + T(r, g)] \leq S(r, f) + S(r, g),$$

which contradicting with $n \geq 8$. Then from Lemma 2.4, we have $\phi_f = \phi_g$ or $\phi_f \cdot \phi_g = 1$, which means that (1.2) or (1.3) holds.

The following proof is similar with that of Theorem 1.2, we omit it here. $\square$

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