

A note on guaranteed stable recovery of sparse signal in compressed sensing via the RIP of order s

Abstract

In this paper, we shall continue a study of the CS-recovery of signals studied in [10]. Under the assumption that a $m \times n$ matrix A obeys the RIP of order s we shall research a good condition of CS-recovery by using a certain function p on a good location $T_0 = \{1, 2, \dots, s\}$.

Keywords: Compressed sensing, Restricted isometry constants, Restricted isometry property, Sparse approximation, Sparse signal recovery.

1 Introduction

This paper introduces the theory of compressed sensing(CS). For a signal $x \in \mathbf{R}^n$, let $\|x\|_0$ be the l_0 -norm of x , which is defined to be the number of nonzero coordinates, $\|x\|_1$ be the l_1 -norm of x and $\|x\|_2$ be the l_2 -norm of x . Let x be a sparse or nearly sparse vector. Compressed sensing aims to recover a high-dimensional signal (for example: images signal, voice signal, code signal...etc.) from only a few samples or linear measurements. The efficient recovery of sparse signals has been a very active field in applied mathematics, statistics, machine learning and signal processing. Formally, one considers the following model:

$$y = Ax + z, \quad (1.1)$$

where A is a $m \times n$ matrix($m < n$) and z is an unknown noise term.

Our goal is to reconstruct an unknown signal x based on A and y given. Then we consider reconstructing x as the solution x^* to the optimization problem

$$\min_x \|x\|_1, \quad \text{subject to } \|y - Ax\|_2 \leq \varepsilon, \quad (1.2)$$

where ε is an upper bound on the the size of the noisy contribution.

In fact, a crucial issue is to research good conditions under which the inequality

$$\|x - x^*\|_2 \leq C_0 \|x - x_T\|_1 + C_1 \varepsilon, \quad (1.3)$$

for suitable constants C_0 and C_1 , where T is any location of $\{1, 2, \dots, n\}$ with number $|T| = s$ of elements of T and x_T is the restriction of x to indices in T . One of the most generally known condition for CS theory is the restricted isometry property(RIP) introduced by [6]. When we discuss our proposed results, it is an important notion. The RIP needs that subsets of columns of A for all locations in $\{1, 2, \dots, n\}$ behave nearly orthonormal system. In detail, a matrix A satisfies the RIP of order s if there exists a constant δ with $0 < \delta < 1$ such that

$$(1 - \delta)\|a\|_2^2 \leq \|Aa\|_2^2 \leq (1 + \delta)\|a\|_2^2 \quad (1.4)$$

for all s -sparse vectors a . A vector is said to be an s -sparse vector if it has at most s nonzero entries. The minimum δ satisfying the above restrictions is said to be the restricted isometry constant and is denoted by δ_s .

Many researchers has been shown that the l_1 optimization can recover an unknown signal in noiseless cases and in noisy cases under various sufficient conditions on δ_s or δ_{2s} when A obeys the RIP. For example, E.J. Candès and T. Tao have proved that if $\delta_{2s} < \sqrt{2} - 1$, then an unknown signal can be recovered [6]. Later, S. Foucart and M. Lai have improved the bound to $\delta_{2s} < 0.4531$ [8]. Others, $\delta_{2s} < 0.4652$ is used in [7], $\delta_{2s} < 0.4721$ for cases such that s is a multiple of 4 or s is very large in [2], $\delta_{2s} < 0.4734$ for the case such that s is very large in [7] and $\delta_s < 0.307$ in [2]. In a resent paper, Q. Mo and S. Li have improved the sufficient condition to $\delta_{2s} < 0.4931$ for general case and $\delta_{2s} < 0.6569$ for the special case such that $n \leq 4s$ [11]. T. Cai and A. Zhang have improved the sufficient condition to $\delta_s < 0.333$ for general case [3]. T. Cai and A. Zhang have improved the sufficient condition to δ_k in case of $k \geq \frac{4}{3}s$, in particular, $\delta_{2s} < 0.707$ [4]. By using a rescaling method, H. Inoue has obtained the sufficient conditions of $\tilde{\delta}_s < 0.5$ and $\tilde{\delta}_{2s} < 0.828$ in [9].

Recently, In [10] we have researched good conditions for the recovery of sparse signals by investigating the difference between the l_∞ -norm of $h \equiv x^* - x$ and the mean $\frac{|h_1| + |h_2| + \dots + |h_s|}{s}$ of $\{|h_1|, \dots, |h_s|\}$. In more details, we considered a function p on $T_0 \equiv \{1, 2, \dots, s\}$ defined by

$$p(r) = \frac{|h_1| + |h_2| + \dots + |h_r|}{|h_1| + |h_2| + \dots + |h_s|}, \quad r = 1, 2, \dots, s,$$

where the index of h is sorted by $|h_1| \geq |h_2| \geq \dots \geq |h_n|$ and have shown that for $c > 1$ and $\frac{c}{s} < p(1)$ if A obeys the RIP of order $\frac{2s}{c}$ and $\delta_{\frac{2s}{c}} < \frac{1}{1 + \sqrt{\frac{2-p(r_c)}{p(r_c)}}}$, then we have stable recovery of approximately

sparse signals, where r_c is a natural number such that $\frac{c}{s}(r_c - 1) < p(r_c) < \frac{c}{s}r_c$, $2 \leq r_c < \frac{s}{c}$. But, the function p on T_0 and r_c depend on x . Furthermore r_c is not easily searched. In this paper, in order to compensate for these defects, we decompose $K_\varepsilon(y, A) \equiv \{x \in \mathbf{R}^n; \|y - Ax\|_2 \leq \varepsilon\}$ into the following subsets $\{M_0, M_1, \dots, M_7\}$:

$$\begin{aligned} M_0 &= \left\{ x \in K_\varepsilon(y, A); \quad p_x\left(\frac{1}{5}s\right) \leq \frac{2}{5} \right\}, \\ M_1 &= \left\{ x \in K_\varepsilon(y, A); \quad p_x\left(\frac{1}{5}s\right) > \frac{2}{5} \text{ and } p_x\left(\frac{1}{4}s\right) \leq \frac{1}{2} \right\}, \\ &\vdots \\ M_k &= \left\{ x \in K_\varepsilon(y, A); \quad p_x\left(\frac{k+3}{20}s\right) > \frac{k+3}{10} \text{ and } p_x\left(\frac{k+4}{20}s\right) \leq \frac{k+4}{10} \right\}, \quad 2 \leq k \leq 6, \\ M_7 &= \left\{ x \in K_\varepsilon(y, A); \quad p_x\left(\frac{1}{2}s\right) = 1 \right\} \end{aligned}$$

by deviding $T_0 = \{1, 2, \dots, s\}$ into $T_0 \cap [1, \frac{s}{5}]$, $T_0 \cap (\frac{k+3}{20}, \frac{k+4}{20}]$ ($k = 1, \dots, 6$) and $T_0 \cap (\frac{1}{2}s, s]$, and we show for any $x \in M_k$ ($k = 1, 2, \dots, 7$) that if A obeys the RIP of order s and $\delta_s < \frac{1}{1 + \sqrt{\frac{20}{k+3} - 1}}$, then the inequality (1.3) holds. We also state in Section 2 the existence of CS-solution.

2 CS-solution

Let a $m \times n$ matrix A ($m < n$) and a data $\mathbf{y} \in \mathbf{R}^m$ be given. We define closed convex subsets of $\mathbf{R}^n$ by

$$\begin{aligned} K_0(\mathbf{y}, A) &= \{\mathbf{x} \in \mathbf{R}^n; \mathbf{y} = A\mathbf{x}\}, \\ K_\varepsilon(\mathbf{y}, A) &= \{\mathbf{x} \in \mathbf{R}^n; \|\mathbf{y} - A\mathbf{x}\|_2 \leq \varepsilon\}, \quad \varepsilon > 0. \end{aligned}$$

When $K_0(\mathbf{y}, A) \neq \emptyset$, that is, $\mathbf{y} \in A\mathbf{R}^n$, then $K_0(\mathbf{y}, A)$ and $K_\varepsilon(\mathbf{y}, A)$ are

$$K_0(\mathbf{y}, A) = \mathbf{x}_0 + \ker A$$

for some vector $\mathbf{x}_0 \in K_0(\mathbf{y}, A)$, where $\ker A \equiv \{\mathbf{x} \in \mathbf{R}^n; A\mathbf{x} = \mathbf{0}\}$. For example, if the rank $r(A)$ of A equals m , then AA^* is invertible and $A(A^*(AA^*)^{-1}\mathbf{y}) = \mathbf{y}$. Hence, $A^*(AA^*)^{-1}\mathbf{y} \in K_0(\mathbf{y}, A)$. Let $\mathbf{y} \notin A\mathbf{R}^n$. Since $A\mathbf{R}^n$ is a closed subspace of $\mathbf{R}^m$, there exists a unique vector $\mathbf{y}_0 \in A\mathbf{R}^n$ such that $\|\mathbf{y} - \mathbf{y}_0\|_2 = \min\{\|\mathbf{y} - A\mathbf{x}\|_2; \mathbf{x} \in \mathbf{R}^n\}$. Then $\mathbf{y}_0$ is a vector in $A\mathbf{R}^n$ such that $\mathbf{y} - \mathbf{y}_0$ is a vector in the orthogonal complement $(A\mathbf{R}^n)^\perp$ of $A\mathbf{R}^n$. It is clear that $K_\varepsilon(\mathbf{y}, A) \neq \emptyset$ if and only if $\|\mathbf{y} - \mathbf{y}_0\|_2 \leq \varepsilon$. In this paper, we assume that $K_0(\mathbf{y}, A) \neq \emptyset$ in noiseless cases and $K_\varepsilon(\mathbf{y}, A) \neq \emptyset$ in noise cases. We show the existence of CS-solutions.

For any $t > 0$ we put

$$D_t = \{\mathbf{x} \in \mathbf{R}^n; \|\mathbf{x}\|_1 \leq t\}.$$

Then AD_t is a closed convex subset of $A\mathbf{R}^n$ such that $A(\partial D_t) = \partial AD_t$, where ∂K is a boundary of a set K . Assume that $\mathbf{y}_0 \notin AD_t$. Then there exists a vector $\mathbf{x}_t$ in ∂D_t such that $\|\mathbf{y} - A\mathbf{x}_t\|_2 = \min\{\|\mathbf{y} - A\mathbf{x}\|_2; \mathbf{x} \in D_t\}$. Since

$$\|\mathbf{y} - A\mathbf{x}_t\|_2^2 = \|\mathbf{y} - \mathbf{y}_0\|_2^2 + \|\mathbf{y}_0 - A\mathbf{x}_t\|_2^2,$$

we have

$$\|\mathbf{y} - A\mathbf{x}_t\|_2 = \min\{\|\mathbf{y} - A\mathbf{x}\|_2; \mathbf{x} \in D_t\},$$

which implies that there exists a vector $\mathbf{x}_t^*$ in $(\mathbf{x}_t + \ker A) \cap D_t$ such that

$$\|\mathbf{x}_t^*\|_1 \leq \|\mathbf{x}_t + \mathbf{x}\|_1, \quad \forall \mathbf{x} \in \ker A.$$

Thus we have the following:

Proposition 2.1. Suppose that $K_\varepsilon(\mathbf{y}, A) \neq \emptyset$. Then there exists a positive number t_0 such that

$$\|\mathbf{y}_0 - A\mathbf{x}_{t_0}\|_2^2 = \varepsilon^2 - \|\mathbf{y} - \mathbf{y}_0\|_2^2$$

and the vector $\mathbf{x}_{t_0}^*$ determined by $\mathbf{x}_{t_0}$ equals the CS-solution $\mathbf{x}^*$. In particular, in noiseless cases, $\mathbf{x}^* = \mathbf{x}_{t_0}^*$, where t_0 is a positive number satisfying $\mathbf{y}_0 = A\mathbf{x}_{t_0}$.

3 Recovery of CS

Take an arbitrary $\mathbf{x} \in K_\varepsilon(\mathbf{y}, A)$. We denote by $\mathbf{x}^T$ a vector obtained by changing coefficients of $\mathbf{x}$ as follows;

$$|h_1| \geq |h_2| \geq \cdots \geq |h_n|,$$

where $\mathbf{h} = (h_1, h_2, \dots, h_n) \equiv \mathbf{x}^* - \mathbf{x}^T$. Let $T_0 = \{1, 2, \dots, s\}$ and we define a function $p_x(r)$ on T_0 depending on x by

$$p_x(r) = \frac{|h_1| + |h_2| + \dots + |h_r|}{\|\mathbf{h}_{T_0}\|_1}, \quad r \in T_0.$$

By deviding $T_0 = \{1, 2, \dots, s\}$ into $T_0 \cap [1, \frac{s}{5}]$, $T_0 \cap (\frac{k+3}{20}s, \frac{k+4}{20}s]$ ($k = 1, \dots, 6$) and $T_0 \cap (\frac{1}{2}s, s]$, we decomposed $K_\varepsilon(\mathbf{y}, A)$ into the following subsets $\{M_0, M_1, \dots, M_7\}$;

$$\begin{aligned} M_0 &= \left\{ \mathbf{x} \in K_\varepsilon(\mathbf{y}, A); \quad p_x\left(\frac{1}{5}s\right) \leq \frac{2}{5} \right\}, \\ M_1 &= \left\{ \mathbf{x} \in K_\varepsilon(\mathbf{y}, A); \quad p_x\left(\frac{1}{5}s\right) > \frac{2}{5} \text{ and } p_x\left(\frac{1}{4}s\right) \leq \frac{1}{2} \right\}, \\ &\vdots \\ M_k &= \left\{ \mathbf{x} \in K_\varepsilon(\mathbf{y}, A); \quad p_x\left(\frac{k+3}{20}s\right) > \frac{k+3}{10} \text{ and } p_x\left(\frac{k+4}{20}s\right) \leq \frac{k+4}{10} \right\}, \quad 2 \leq k \leq 6, \\ M_7 &= \left\{ \mathbf{x} \in K_\varepsilon(\mathbf{y}, A); \quad p_x\left(\frac{1}{2}s\right) = 1 \right\}. \end{aligned}$$

Then, $K_\varepsilon(\mathbf{y}, A) = \bigcup_{k=0}^7 M_k$ and $M_i \cap M_j = \emptyset (i \neq j)$.

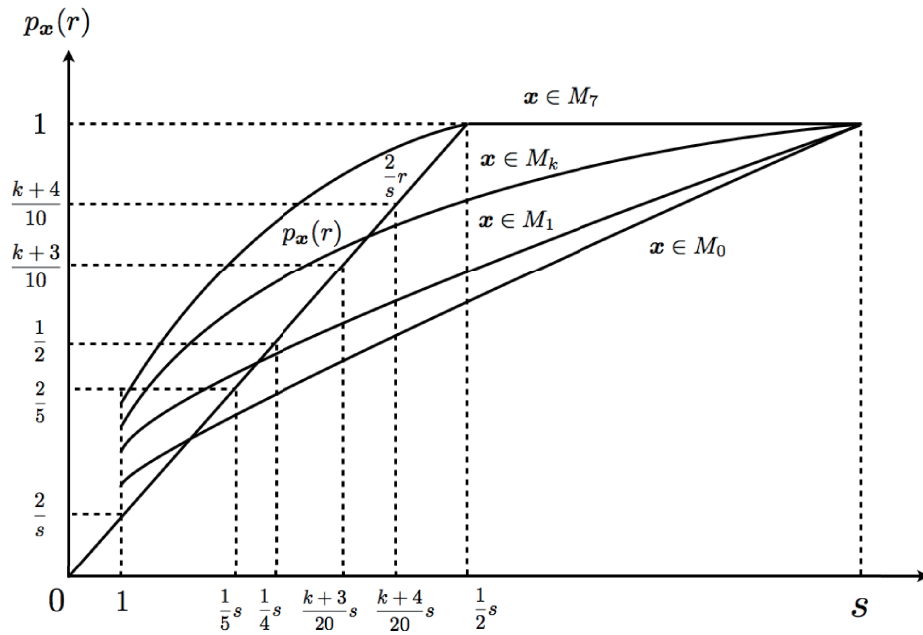


Figure 1:

Using the function $p_x(r)$ on T_0 , we obtain a similar result to that of ([10] Theorem 2.1):

Theorem 3.1. Take an arbitrary $x \in M_k$ ($k = 1, 2, \dots, 7$). Assume that A obeys the RIP of order s and $\delta_s < \frac{1}{1 + \sqrt{\frac{20}{k+3} - 1}}$. Then,

$$\|x^* - x\|_2 \leq C_0^{(k)} \|x - x_s\|_1 + C_1^{(k)} \varepsilon,$$

where x_s is a vector consisting of the s -large entries of x in magnitude and

$$C_0^{(k)} = \frac{4\sqrt{\frac{20}{k+3} - 1} \cdot \delta_s}{1 - \left(1 + \sqrt{\frac{20}{k+3} - 1}\right) \delta_s},$$

$$C_1^{(k)} = \frac{2\sqrt{1 + \delta_s} \sqrt{s}}{\sqrt{\frac{k+3}{20}} \left(1 - \left(1 + \sqrt{\frac{20}{k+3} - 1}\right) \delta_s\right)}.$$

Proof. Take an arbitrary $x \in M_k$. Let r_k be a natural number such that

$$\frac{k+3}{20}s < r_k \leq \frac{k+4}{20}s \quad \text{and} \quad \frac{2}{s}(r_k - 1) < p_x(r_k) \leq \frac{2}{s}r_k. \quad (3.1)$$

Then,

$$\frac{k+3}{10} < p_x(r_k) \leq \frac{k+4}{10}. \quad (3.2)$$

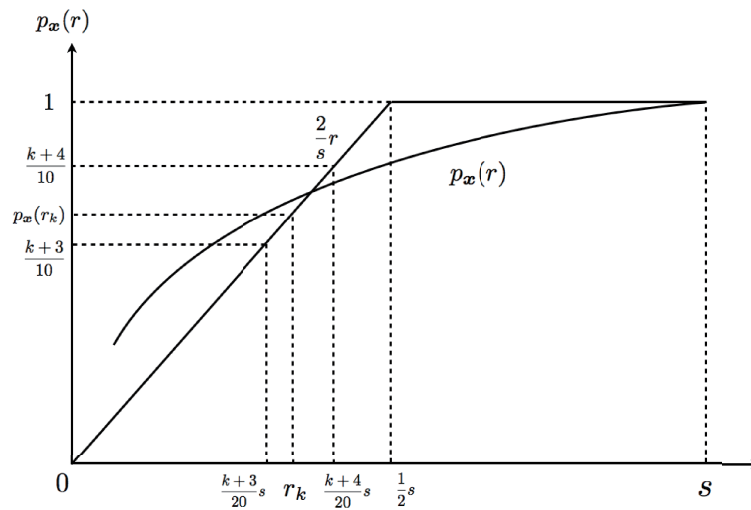


Figure 2:

We put

$$\alpha = \frac{\|h_{T_0}\|_1 + 2\|x - x_s\|_1}{s}.$$

Let $T_1 = \{1, 2, \dots, r_2\}$ and $T_2 = \{r_2 + 1, \dots, n\}$ be a decomposition of $\{1, 2, \dots, n\}$. By (3.1) and (3.2) we have

$$\|\mathbf{h}_{T_2}\|_\infty \leq \frac{p(r_2)}{r_2} \|\mathbf{h}_{T_0}\|_1 \leq 2\alpha. \quad (3.3)$$

By the definition of CS optimization (1.2), we have

$$\|\mathbf{h}_{T_0^c}\|_1 \leq \|\mathbf{h}_{T_0}\|_1 + 2\|\mathbf{x} - \mathbf{x}_s\|_1. \quad (3.4)$$

Hence it follows from (3.3) and (3.4) that

$$\begin{aligned} \|\mathbf{h}_{T_2}\|_1 &= \|\mathbf{h}_{T_0^c}\|_1 + \|\mathbf{h}_{T_0 \cap T_2}\|_1 \\ &\leq \alpha s + (1 - p_{\mathbf{x}}(r_k)) \|\mathbf{h}_{T_0}\|_1 \\ &\leq (2 - p_{\mathbf{x}}(r_k)) \alpha s \\ &\leq 2\alpha \left(1 - \frac{k+3}{20}\right) s, \end{aligned}$$

which implies by ([10] Lemma 1.1) that there exist $\{\lambda_i\}_{1 \leq i \leq N}$ and $\{\mathbf{u}_i\}_{1 \leq i \leq N}$ such that

$$\mathbf{h}_{T_2} = \sum_{i=1}^N \lambda_i \mathbf{u}_i,$$

where

$$\begin{aligned} 0 \leq \lambda_i \leq 1, \quad \sum_{i=1}^N \lambda_i &= 1, \\ \text{supp } \mathbf{u}_i \subset T_2, \quad |\text{supp } \mathbf{u}_i| &\leq \left(1 - \frac{k+3}{20}\right) s \\ \|\mathbf{u}_i\|_\infty &\leq 2\alpha. \end{aligned} \quad (3.5)$$

Hence we have

$$\begin{aligned} \|\mathbf{u}_i\|_2 &\leq \|\mathbf{u}_i\|_\infty \sqrt{|\text{supp } \mathbf{u}_i|} \\ &\leq 2\alpha \sqrt{s} \sqrt{1 - \frac{k+3}{20}}, \\ |T_1| + |\text{supp } \mathbf{u}_i| &\leq r_k + \left(1 - \frac{k+3}{20}\right) s \leq s \end{aligned}$$

and

$$\begin{aligned} \alpha s &= \|\mathbf{h}_{T_0}\|_1 + 2\|\mathbf{x} - \mathbf{x}_s\|_1 \\ &= \frac{1}{p_{\mathbf{x}}(r_k)} \|\mathbf{h}_{T_1}\|_1 + 2\|\mathbf{x} - \mathbf{x}_s\|_1 \\ &\leq \frac{\sqrt{r_k}}{p_{\mathbf{x}}(r_k)} \|\mathbf{h}_{T_1}\|_2 + 2\|\mathbf{x} - \mathbf{x}_s\|_1 \\ &\leq \frac{\sqrt{s}}{2\sqrt{\frac{k+3}{20}}} \|\mathbf{h}_{T_1}\|_2 + 2\|\mathbf{x} - \mathbf{x}_s\|_1, \end{aligned}$$

which implies since A obeys the RIP of order s that

$$\begin{aligned}
 (1 - \delta_s) \|\mathbf{h}_{T_1}\|_2^2 &\leq \|A\mathbf{h}_{T_1}\|_2^2 \\
 &\leq |\langle A\mathbf{h}_{T_1}, A\mathbf{h} \rangle| + |\langle A\mathbf{h}_{T_1}, A\mathbf{h}_{T_2} \rangle| \\
 &\leq \sqrt{1 + \delta_s} \|\mathbf{h}_{T_1}\|_2 \cdot 2\varepsilon + \sum_{i=1}^N \lambda_i |\langle A\mathbf{h}_{T_1}, A\mathbf{u}_i \rangle| \\
 &\leq 2\sqrt{1 + \delta_s} \varepsilon \|\mathbf{h}_{T_1}\|_2 + \sum_{i=1}^N \lambda_i \delta_s \|\mathbf{h}_{T_1}\|_2 \|\mathbf{u}_i\|_2 \\
 &\leq 2\sqrt{1 + \delta_s} \varepsilon \|\mathbf{h}_{T_1}\|_2 \\
 &\quad + \delta_s \|\mathbf{h}_{T_1}\|_2 \left(\frac{1}{2\sqrt{\frac{k+3}{20}}} \|\mathbf{h}_{T_1}\|_2 + \frac{2}{\sqrt{s}} \|\mathbf{x} - \mathbf{x}_s\|_1 \right) 2\sqrt{1 - \frac{k+3}{20}} \\
 &= 2\sqrt{1 + \delta_s} \varepsilon \|\mathbf{h}_{T_1}\|_2 + \delta_s \sqrt{\frac{20}{k+3} - 1} \|\mathbf{h}_{T_1}\|_2^2 \\
 &\quad + \frac{4\delta_s}{\sqrt{s}} \sqrt{1 - \frac{k+3}{20}} \|\mathbf{x} - \mathbf{x}_s\|_1 \|\mathbf{h}_{T_1}\|_2.
 \end{aligned}$$

Since

$$\left(1 + \sqrt{\frac{20}{k+3} - 1} \right) \delta_s < 1,$$

we have

$$\|\mathbf{h}_{T_1}\|_2 \leq \frac{2\sqrt{1 + \delta_s} \varepsilon + \frac{4\delta_s}{\sqrt{s}} \sqrt{1 - \frac{k+3}{20}} \|\mathbf{x} - \mathbf{x}_s\|_1}{1 - \left(1 + \sqrt{\frac{20}{k+3} - 1} \right) \delta_s},$$

which implies that

$$\begin{aligned}
 \|\mathbf{x} - \mathbf{x}^*\|_2 &\leq \|\mathbf{x} - \mathbf{x}^*\|_1 \\
 &= \|\mathbf{h}_{T_0}\|_1 + \|\mathbf{h}_{T_0^c}\|_1 \\
 &\leq 2\|\mathbf{h}_{T_0}\|_1 + 2\|\mathbf{x} - \mathbf{x}_s\|_1 \\
 &\leq \frac{2\sqrt{r_k}}{p_{\mathbf{x}}(r_k)} \|\mathbf{h}_{T_1}\|_2 + 2\|\mathbf{x} - \mathbf{x}_s\|_1 \\
 &\leq \frac{\sqrt{s}}{\sqrt{\frac{k+3}{20}}} \left(\frac{2\sqrt{1 + \delta_s} \varepsilon + \frac{4}{\sqrt{s}} \sqrt{1 - \frac{k+3}{20}} \delta_s \|\mathbf{x} - \mathbf{x}_s\|_1}{1 - \left(1 + \sqrt{\frac{20}{k+3} - 1} \right) \delta_s} \right) \\
 &\quad + 2\|\mathbf{x} - \mathbf{x}_s\|_1 \\
 &= \frac{2\sqrt{1 + \delta_s} \sqrt{s}}{\sqrt{\frac{k+3}{20}} \left(1 - \left(1 + \sqrt{\frac{20}{k+3} - 1} \right) \delta_s \right)} \varepsilon \\
 &\quad + \frac{4\sqrt{\frac{20}{k+3} - 1} \cdot \delta_s}{1 - \left(1 + \sqrt{\frac{20}{k+3} - 1} \right) \delta_s} \|\mathbf{x} - \mathbf{x}_s\|_1.
 \end{aligned}$$

This completes the proof.

We state concretely the following case:

(i) Take an arbitrary $\mathbf{x} \in M_1$. If $\delta_s < \frac{1}{3}$, then

$$\|\mathbf{x}^* - \mathbf{x}\|_2 \leq \frac{8\delta_s}{1 - 3\delta_s} \|\mathbf{x} - \mathbf{x}_s\|_1 + \frac{2\sqrt{5}\sqrt{1 + \delta_s}\sqrt{s}}{1 - 3\delta_s} \varepsilon.$$

(ii) Take an arbitrary $\mathbf{x} \in M_2$. If $\delta_s < \frac{\sqrt{3}-1}{2} \approx 0.366$, then

$$\|\mathbf{x}^* - \mathbf{x}\|_2 \leq \frac{4\sqrt{3}\delta_s}{1 - (1 + \sqrt{3})\delta_s} \|\mathbf{x} - \mathbf{x}_s\|_1 + \frac{4\sqrt{1 + \delta_s}\sqrt{s}}{1 - (1 + \sqrt{3})\delta_s} \varepsilon.$$

(iii) Take an arbitrary $\mathbf{x} \in M_7$. If $\delta_s < \frac{1}{2}$, then

$$\|\mathbf{x}^* - \mathbf{x}\|_2 \leq \frac{4\delta_s}{1 - 2\delta_s} \|\mathbf{x} - \mathbf{x}_s\|_1 + \frac{2\sqrt{2}\sqrt{1 + \delta_s}\sqrt{s}}{1 - 2\delta_s} \varepsilon.$$

Though we have decomposed $K_\varepsilon(\mathbf{y}, A)$ into $M_k (k = 0, 1, \dots, 7)$ in this paper, we may consider the other decompositions of $K_\varepsilon(\mathbf{y}, A)$.

References

- [1] T. Cai, L. Wang and G. Xu, Shifting inequality and recovery of sparse signals, *IEEE Transactions on Signal Processing*. 2010;50:1300–1308.
- [2] T. Cai, L. Wang and G. Xu, New bounds for restricted isometry constants, *IEEE Transactions on Information Theory*. 2010;56:4388–4394.
- [3] T. Cai and A. Zhang, Sharp RIP Bound for Sparse Signal and Low-Rank Matrix Recovery, *Applied and Computational Harmonic Analysis*. 2013;35:74–93.
- [4] T. Cai and A. Zhang, Sparse representation of a polytope and recovery of sparse signals and low-rank matrices, *IEEE Transactions on Information Theory*. 2014;60:122–132.
- [5] E.J. Candès, The restricted isometry property and its implications for compressed sensing, *Compte Rendus de l'Academie des Sci*. 2008;346:589–592 Serie I.
- [6] E.J. Candès and T. Tao, Decoding by linear programming, *IEEE Transactions on Information Theory*. 2005;51:4203–4215.
- [7] S. Foucart, A note on guaranteed sparse recovery via l_1 -minimization, *Applied and Computational Harmonic Analysis*. 2010;29:97–103.
- [8] S. Foucart and M. Lai, Sparsest solutions of underdetermined linear systems via l_q -minimization for $0 < q \leq 1$, *Applied and Computational Harmonic Analysis*. 2009;26:395–407.
- [9] H. Inoue, Sufficient conditions for CS-recovery, *British Journal of Mathematics & Computer Science*. 2014;4(2):184–198.
- [10] H. Inoue, Stable Recovery of Sparse Signal in Compressed Sensing via the RIP of Order less than s , *British Journal of Mathematics & Computer Science*. 2014;6(2):91–101.
- [11] Q. Mo and S. Li, New bounds on the restricted isometry constant δ_{2k} , *Applied and Computational Harmonic Analysis*. 2011;31:460–468.