

Solution of Heat and Wave – Like Equations by Combine Adomain Decomposition Method and Sumudu Transform

Abstract

In this work, we combine form sumudu transform method and A domain decomposition method to find exact or approximate solutions of heat and wave – like equations. This method is called the Adomain decomposition sumudu transform method (ADSTM). The proposed scheme finds the solution without any restrictive assumption and avoids the round off errors. The result tells that the proposed method is very efficient, simple and can be applied to other problems.

Keywords: Adomain decomposition method, sumudu transform, heat and wave – like equations and Adomain polynomials.

1. Introduction

Heat and wave – like models are the integral part of applied mathematics and arise in various physical phenomena. Several techniques such that characteristic, modified variation iteration, Adomain decomposition method and Hè s polynomials have been used for solving these problems see ([6-10]).

In spared and motivated by the ongoing research in this area, in the present paper we introduce a new method called a domain decomposition sumudu transform method (ADSTM) for solving the nonlinear equations. It is worth mentioning that the proposed method is an elegant combination of the sumudu transform method and the Adomain decomposition method is which was first introduced by Adomain see ([11-18]).

In the present article, we apply (ADSTM) for solving heat and wave – like equations.

2. Sumudu transform

Watugala (see [5]) introduce a new integral transform, named the sumudu transform and applied it to the ordinary differential equations and partial differential equations in control engineering problems. Sumudu transform is defined over the set of function

$$A = \left\{ f(t): \exists M, \tau_1, \tau_2 > 0, |f(t)| < M e^{\frac{|t|}{\tau_j}}, \text{ if } t \in (-1)^j \times [0, \infty) \right\}$$

By the following formula

$$\bar{f}(u) = S[f(t)] = \int_0^{\infty} f(ut) e^{-t} dt, u \in (-\tau_1, \tau_2) \quad (1)$$

Some of properties were established see ([1-3]), similarly, this transform was applied to the one – dimensional neutron equation see [4].

3. Adomian Decomposition Sumudu Transform Method (ADSTM)

To mention the basic idea of this method, we consider a general nonlinear non-homogeneous partial differential equation with initial conditions of the form

$$\begin{aligned} DU(x,t) + RU(x,t) + NU(x,t) &= g(x,t) \\ U(x,0) &= h(x), U_t(x,0) = f(x) \end{aligned} \quad (2)$$

Where D is the second order linear differential operator $D = \frac{\partial^2}{\partial t^2}$, R is linear differential operator of less order than D , N represent the general nonlinear operator and $g(x,t)$ is the source term.

Taking the sumudu transform on both sides of Eq. (2), we get

$$S[DU(x,t)] + S[RU(x,t)] + S[N(x,t)] = S[g(x,t)] \quad (3)$$

Using the differentiation property of the sumudu transform and given initial conditions, we have

$$S[U(x,t)] = u^2 S[g(x,t)] + h(x) + u f(x) - u^2 S[RU(x,t) + NU(x,t)]. \quad (4)$$

Now, applying the inverse of sumudu transform to both sides of Eq. (4) we get

$$U(x,t) = G(x,t) - S^{-1} \left[u^2 S[RU(x,t) + NU(x,t)] \right] \quad (5)$$

Where $G(x,t)$ represents the term arising from the source term and the prescribed initial conditions. Now, we apply the Adomain decomposition method

$$U(x,t) = \sum_{n=0}^{\infty} U_n(x,t) \quad (6)$$

And the nonlinear term can be decomposed as

$$NU(x,t) = \sum_{n=0}^{\infty} A_n(U) \quad (7)$$

For some Adomain polynomials $A_n(U)$ that are given by

$$A_n(U_0, U_1, U_2, \dots, U_n) = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[N \left(\sum_{n=0}^{\infty} \lambda^n U_n \right) \right]_{\lambda=0}, \quad n = 0, 1, 2, \dots \quad (8)$$

Substituting Eq. (6) and Eq. (7) in Eq. (5), we get

$$\sum_{n=0}^{\infty} U_n(x,t) = G(x,t) - S^{-1} \left[u^2 S \left[R \sum_{n=0}^{\infty} U_n(x,t) + \sum_{n=0}^{\infty} A_n(U) \right] \right] \quad (9)$$

So that the recursive relation is given by

$$\begin{aligned} U_0(x, t) &= G(x, t), \\ U_{k+1}(x, t) &= -S^{-1} \left[u^2 S \left[R U_k + A_k \right] \right]. \quad k \geq 0. \end{aligned} \quad (10)$$

4. NUMERICAL APPLICATION

In this section, we apply the Adomain decomposition sumudu transform method (ADSTM) for solving heat and wave – like equations.

Example 4.1 consider the following one – dimensional initial value boundary problem

$$U_t = \frac{1}{2} x^2 U_{xx}, \quad 0 < x < 1, \quad t > 0 \quad (11)$$

With boundary conditions

$$U(0, t) = 0, \quad U(1, t) = e^t \quad (12)$$

And the initial condition

$$U(x, 0) = x^2 \quad (13)$$

Taking the sumudu transform on both sides of Eq. (11) and substituting the initial condition we obtain

$$S[U(x, t)] = x^2 + \frac{1}{2} x^2 u S[U_{xx}]. \quad (14)$$

Applying the inverse sumudu transform implies that

$$U(x, t) = x^2 + \frac{1}{2} x^2 S^{-1} \left[u S[U_{xx}] \right]. \quad (15)$$

Using the Adomain decomposition method as a series, we get

$$\sum_{n=0}^{\infty} U_n(x, t) = x^2 + \frac{1}{2} x^2 S^{-1} \left[u S \left(\sum_{n=0}^{\infty} (U_n)_{xx}(x, t) \right) \right] \quad (16)$$

This leads to the recursive relation

$$\begin{aligned} U_0(x, t) &= x^2, \\ U_{k+1}(x, t) &= \frac{1}{2} x^2 S^{-1} \left[u S \left[(U_k)_{xx} \right] \right] \end{aligned} \quad (17)$$

This gives

$$\begin{aligned}
 U_0(x, t) &= x^2, \\
 U_1(x, t) &= \frac{1}{2} x^2 S^{-1} \left[u S \left[(U_0)_{xx} \right] \right] = x^2 t, \\
 U_2(x, t) &= \frac{1}{2} x^2 S^{-1} \left[u S \left[(U_1)_{xx} \right] \right] = x^2 \frac{t^2}{2!}, \\
 U_3(x, t) &= \frac{1}{2} x^2 S^{-1} \left[u S \left[(U_2)_{xx} \right] \right] = x^2 \frac{t^3}{3!},
 \end{aligned} \tag{18}$$

And so on. The solution in a series form is given by

$$U(x, t) = x^2 \left[1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots \right]. \tag{19}$$

The closed form solution is given by

$$U(x, t) = x^2 e^t. \tag{20}$$

Example 4.2 consider the following two – dimensional initial value boundary problem

$$U_t = \frac{1}{2} (y^2 U_{xx} + x^2 U_{yy}), \quad 0 < x, y < 1, \quad t > 0 \tag{21}$$

With boundary conditions

$$\begin{aligned}
 U_x(0, y, t) &= 0, \quad U_x(1, y, t) = 2 \sinh t \\
 U_y(x, 0, t) &= 0, \quad U_y(x, 1, t) = 2 \cosh t
 \end{aligned} \tag{22}$$

And the initial condition

$$U(x, y, 0) = y^2 \tag{23}$$

In a similar way as above, we have

$$\begin{aligned}
 \sum_{n=0}^{\infty} U_n(x, y, t) &= y^2 + \frac{1}{2} y^2 S^{-1} \left[u S \left[\left(\sum_{n=0}^{\infty} (U_n)_{xx}(x, y, t) \right) \right] \right] + \\
 &\quad + \frac{1}{2} x^2 S^{-1} \left[u S \left[\left(\sum_{n=0}^{\infty} (U_n)_{yy}(x, y, t) \right) \right] \right]
 \end{aligned} \tag{24}$$

This gives

$$\begin{aligned}
 U_0(x, y, t) &= y^2, \\
 U_1(x, y, t) &= \frac{1}{2} y^2 S^{-1} \left[u S \left[(U_0)_{xx} \right] + \frac{1}{2} x^2 S^{-1} \left[u S \left[(U_0)_{yy} \right] \right] = x^2 t, \\
 U_2(x, y, t) &= \frac{1}{2} y^2 S^{-1} \left[u S \left[(U_1)_{xx} \right] + \frac{1}{2} x^2 S^{-1} \left[u S \left[(U_1)_{yy} \right] \right] = y^2 \frac{t^2}{2!}, \\
 U_3(x, y, t) &= x^2 \frac{t^3}{3!}, \\
 U_4(x, y, t) &= y^2 \frac{t^4}{4!},
 \end{aligned} \tag{25}$$

And so on. The solution in a series form is given by

$$U(x, t) = y^2 \left(1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \dots \right) + x^2 \left(t + \frac{t^3}{3!} + \frac{t^5}{5!} + \dots \right), \tag{26}$$

The closed form solution is given by

$$U(x, t) = y^2 \cosh t + x^2 \sinh t. \tag{27}$$

Example 4.3 let us consider the following three-dimensional inhomogeneous initial boundary value problem

$$U_t = x^4 y^4 z^4 + \frac{1}{36} (x^2 U_{xx} + y^2 U_{yy} + z^2 U_{zz}), \quad 0 < x, y, z < 1, \quad t > 0 \tag{28}$$

With boundary conditions

$$\begin{aligned}
 U(0, y, z, t) &= 0, & U(1, y, z, t) &= x^4 z^4 (e^t - 1), \\
 U(x, 0, z, t) &= 0, & U(x, 1, z, t) &= x^4 z^4 (e^t - 1), \\
 U(x, y, 0, t) &= 0, & U(x, y, 1, t) &= x^4 y^4 (e^t - 1),
 \end{aligned} \tag{29}$$

And the initial condition

$$U(x, y, z, 0) = 0. \tag{30}$$

In a similar way as above, we have

$$\begin{aligned}
 \sum_{n=0}^{\infty} U_n(x, y, z, t) &= x^4 y^4 z^4 t + \frac{1}{36} x^2 S^{-1} \left[u S \left[\left(\sum_{n=0}^{\infty} (U_n)_{xx} (x, y, z, t) \right) \right] \right] + \frac{1}{36} y^2 S^{-1} \left[u S \left[\left(\sum_{n=0}^{\infty} (U_n)_{yy} (x, y, z, t) \right) \right] \right] \\
 &+ \frac{1}{36} z^2 S^{-1} \left[u S \left[\left(\sum_{n=0}^{\infty} (U_n)_{zz} (x, y, z, t) \right) \right] \right]
 \end{aligned} \tag{31}$$

This gives

$$\begin{aligned}
 U_0(x, y, z, t) &= x^4 y^4 z^4 t, \\
 U_1(x, y, z, t) &= x^4 y^4 z^4 \frac{t^2}{2!}, \\
 U_2(x, y, z, t) &= x^4 y^4 z^4 \frac{t^3}{3!}, \\
 U_3(x, y, z, t) &= x^4 y^4 z^4 \frac{t^4}{4!},
 \end{aligned} \tag{32}$$

And so on. The solution in a series form is given by

$$U(x, y, z, t) = x^4 y^4 z^4 \left(t + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots \right), \tag{33}$$

The closed form solution is given by

$$U(x, y, z, t) = x^4 y^4 z^4 (e^t - 1). \tag{34}$$

Example 4.4 Let us consider the initial – boundary value problem

$$U_{tt} = \frac{1}{2} x^2 U_{xx}, \quad 0 < x < 1, \quad t > 0 \tag{35}$$

With boundary conditions

$$U(0, t) = 0, \quad U(1, t) = 1 + \sinh t, \tag{36}$$

And the initial condition

$$U(x, 0) = x, \quad U_t(x, 0) = x^2 \tag{37}$$

In a similar way as above, we have

$$\sum_{n=0}^{\infty} U_n(x, t) = x + x^2 t + \frac{1}{2} x^2 S^{-1} \left[u^2 s \left[\left(\sum_{n=0}^{\infty} (U_n)_{xx}(x, t) \right) \right] \right]. \tag{38}$$

This gives

$$\begin{aligned}
 U_0(x, t) &= x^2 t + x, \\
 U_1(x, t) &= x^2 \frac{t^3}{3!}, \\
 U_2(x, t) &= x^2 \frac{t^5}{5!}, \\
 U_3(x, t) &= x^2 \frac{t^7}{7!},
 \end{aligned} \tag{39}$$

And so on. The solution in a series form is given by

$$U(x,t) = x + x^2 \left(t + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots \right), \quad (40)$$

The closed form solution is given by

$$U(x,t) = x + x^2 \sinh t. \quad (41)$$

Example 4.5 Consider the following two-dimensional initial boundary value problem which describes the wave-like models

$$U_{tt} = \frac{1}{12} (x^2 U_{xx} + y^2 U_{yy}), \quad 0 < x, y < 1, \quad t > 0 \quad (42)$$

Subject the Neumann boundary conditions

$$\begin{aligned} U_x(0, y, t) &= 0, \quad U_x(1, y, t) = 4 \cosh t, \\ U_y(x, 0, t) &= 0, \quad U_y(x, 1, t) = 4 \sinh t, \end{aligned} \quad (43)$$

And the initial condition

$$U(x, y, 0) = x^4, \quad U_t(x, y, 0) = y^4. \quad (44)$$

In a similar way as above, we have

$$\begin{aligned} \sum_{n=0}^{\infty} U_n(x, y, t) &= x^4 + y^4 t + \frac{1}{12} x^2 S^{-1} \left[u^2 S \left[\left(\sum_{n=0}^{\infty} (U_n)_{xx}(x, y, t) \right) \right] \right] + \\ &\frac{1}{12} y^2 S^{-1} \left[u^2 S \left[\left(\sum_{n=0}^{\infty} (U_n)_{yy}(x, y, t) \right) \right] \right]. \end{aligned} \quad (45)$$

The components of solution $U(x, y, t)$ is given by

$$\begin{aligned} U_0(x, y, t) &= x^4 + y^4 t, \\ U_1(x, y, t) &= x^4 \frac{t^2}{2!} + y^4 \frac{t^3}{3!}, \\ U_2(x, y, t) &= x^4 \frac{t^4}{4!} + y^4 \frac{t^5}{5!}, \\ U_3(x, y, t) &= x^4 \frac{t^6}{6!} + y^4 \frac{t^7}{7!}, \end{aligned} \quad (46)$$

And so on. The solution in a series form is given by

$$U(x, y, t) = x^4 \left(1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \frac{t^6}{6!} + \dots \right) + y^4 \left(t + \frac{t^3}{3!} + \frac{t^5}{5!} + \dots \right), \quad (47)$$

The closed form solution is given by

$$U(x, y, t) = x^4 \cosh t + y^4 \sinh t. \quad (48)$$

Example 4.6 Consider the initial – boundary value problem

$$U_{tt} = x^2 + y^2 + z^2 + \frac{1}{2}(x^2 U_{xx} + y^2 U_{yy} + z^2 U_{zz}), \quad 0 < x, y, z < 1, \quad t > 0 \quad (49)$$

Subject to the boundary's conditions

$$\begin{aligned} U(0, y, z, t) &= y^2(e^t - 1) + z^2(e^{-t} - 1), \quad U(1, y, z, t) = (y^2 + 1)(e^t - 1) + z^2(e^{-t} - 1), \\ U(x, 0, z, t) &= x^2(e^t - 1) + z^2(e^{-t} - 1), \quad U(x, 1, z, t) = (x^2 + 1)(e^t - 1) + z^2(e^{-t} - 1), \\ U(x, y, 0, t) &= (x^2 + y^2)(e^t - 1), \quad U(x, y, 1, t) = (x^2 + y^2)(e^t - 1) + (e^{-t} - 1), \end{aligned} \quad (50)$$

And having the initial conditions

$$U(x, y, z, 0) = 0, \quad U_t(x, y, z, t) = x^2 + y^2 - z^2. \quad (51)$$

In a similar way as above, we have

$$\begin{aligned} \sum_{n=0}^{\infty} U_n(x, y, z, t) &= (x^2 + y^2 + z^2) \frac{t^2}{2!} + (x^2 + y^2 - z^2)t + \frac{1}{2}x^2 S^{-1} \left[u^2 S \left[\left(\sum_{n=0}^{\infty} U_n(x, y, z, t) \right)_{xx} \right] \right] + \\ &+ \frac{1}{2}y^2 S^{-1} \left[u^2 S \left[\left(\sum_{n=0}^{\infty} U_n(x, y, z, t) \right)_{yy} \right] \right] + \frac{1}{2}z^2 S^{-1} \left[u^2 S \left[\left(\sum_{n=0}^{\infty} U_n(x, y, z, t) \right)_{zz} \right] \right] \end{aligned} \quad (52)$$

The components of solution $U(x, y, z, t)$ is given by

$$\begin{aligned} U_0(x, y, z, t) &= (x^2 + y^2 + z^2) \frac{t^2}{2!} + (x^2 + y^2 - z^2)t, \\ U_1(x, y, z, t) &= (x^2 + y^2 + z^2) \frac{t^4}{4!} + (x^2 + y^2 - z^2) \frac{t^3}{3!}, \\ U_2(x, y, z, t) &= (x^2 + y^2 + z^2) \frac{t^6}{6!} + (x^2 + y^2 - z^2) \frac{t^5}{5!}, \end{aligned} \quad (53)$$

$$U_3(x, y, z, t) = (x^2 + y^2 + z^2) \frac{t^8}{8!} + (x^2 + y^2 - z^2) \frac{t^7}{7!},$$

And so on. The solution in a series form is given by

$$U(x, y, z, t) = (x^2 + y^2) \left(t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \dots \right) + z^2 \left(-t + \frac{t^2}{2!} - \frac{t^3}{3!} + \dots \right), \quad (54)$$

The closed form solution is given by

$$U(x, y, z, t) = (x^2 + y^2)e^t + z^2 e^{-t} - (x^2 + y^2 + z^2). \quad (55)$$

5. Conclusion

In the present paper, we have proposed the Adomian decomposition sumudu transform (ADSTM) for solving heat and wave – like equations.

The method gives the results reveal that the proposed method is very efficient, simple and can be applied to other applications.

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