

Short Research Article

Bayesian Optimal Filtering in **Dynamic Linear Models**: An Empirical Example from Econometric Analysis of Economic Time Series Data

Olushina Olawale Awe^{*1} and Adedayo O. Adepoju²

¹*Department of Mathematics (Statistics Unit)*

Obafemi Awolowo University, Ile-Ife, Nigeria.

²*Department of Statistics,
University of Ibadan, Nigeria.*

Abstract

This paper reviews a methodology for optimal data cleaning and filtering of economic time series data using the Kalman filter to estimate the parameters of a specified state space model which describes the economic phenomena under study. The Kalman filter is a recursive algorithm and as such is ideal for usage on the economic data. We find that the Kalman filter is both predictive and adaptive, as it looks forward with an estimate of the covariance and mean of the time series one step into the future and unlike a neural network, it does not require stationary data, hence usage of the Kalman filter is proposed to detect the anomalies in the trend of economic time series data.

Key Words: Bayesian Inference, Kalman Filter, High Frequency Data, Dynamic Linear Model.

1 Introduction

State space models originated in engineering in the early sixties (Kalman, 1960; Kalman and Bucy, 1961). Their use have since been extended to a large number of applied fields such as econometrics, time series analysis, genetics, spatio-temporal analysis, hydrology, climatology, speech recognition, signal processing, population dynamics etc. **A state space model allows the researcher to model an observed (multiple) time series y_t , $t = 1, 2, \dots, T$ which is driven** by a stochastic process. Many econometric models (time series models, in particular) can be written in state space form (Hamilton, 1994; Chib et al., 2002; Petris et al., 2009). The obvious flexibility of the state space approach has contributed to their immense popularity in econometric time series analysis in recent years (Harvey,

^{*}Corresponding author: E-mail: olawaleawe@yahoo.co.uk

1990; Box et al., 2013). Dynamic linear models (DLMs) are a particular class of state space models that allow many of the relevant inferences to be carried out exactly using the Kalman

filter, especially in the case of a completely specified model like the one considered here. They provide a very rich class of models for the analysis and forecasting of time series data (Petrís et al., 2009).

In this paper, we describe a selection of basic filtering procedures for estimating state space models and apply them to some economic time series data of the Nigerian economy. The rest of this paper is organized as follows: In section 2, we present a brief review of literature on application of state space models to time series and econometric analysis. Section 3 is on the model and Kalman filter method used in this paper, while section 4 borders on an application to the Nigerian economy, and section 5 concludes the paper.

2 Application of State Space Models in Time Series Econometrics

In recent years, there has been an increasing interest in the application of state- space model and its variants in econometrics and time series analysis, partly because of its flexibility and widely because of the development of modern Monte Carlo methods. Economic theory and practice is often concerned with latent or unobserved dynamic processes, hence the state space form of a dynamic system with unobserved components is a very powerful and flexible instrument because it builds on the dependence structure of a Markov chain to define more complex models for the dynamic observations (Petrís et al., 2009).

A state space model is in principle, any model that includes an observation process Y_t and a state process θ_t . It consists of a state equation and an observation equation which can be used for modeling univariate and multivariate time series in the presence of non-stationarity, structural changes, irregular patterns. The linear econometric state space model encompasses two models which are widely used in time series analysis and economics. First, the observation equation resembles a linear regression model with the distinction that the coefficients are stochastic dynamic processes rather than fixed parameters. Second, the state vector follows a first order vector autoregressive model, which in turn is a generalisation of the scalar Autoregressive Moving Average (ARMA) models. All univariate ARMA models can be written as a linear state space model (Harvey, 1989). We are motivated to apply this method to modeling economic indicators because, in many real life applications, the driving

force behind the evolution of economic variables are not always observable. Usually, it is assumed that there is an unobservable Markov chain θ_t called the 'state' and that Y_t is an imprecise measurement of θ_t .

In econometric applications, we think of the state θ_t as an auxiliary time series that is assumed to follow a Markov process which facilitates the task of specifying the probability distribution of the observable time series Y_t . State-space models consider a time series as the output of a dynamic system perturbed by random disturbances. They have an elegant and powerful probabilistic structure, offering a flexible framework for a very wide range of applications. Computations can be implemented by recursive algorithms (Ng and Young, 1990; Young, 1988). While the state equation formulates the dynamics of the state variables, the observation equation relates the observed variables to the unobserved state vector. The state vector can contain trend, seasonal, cycle and regression components plus an error term, and has to be estimated from the data, hence the maximum likelihood estimates of the parameters can be obtained by applying the Kalman filter.

In the Box and Jenkins methodology, observations are first made stationary (Box and Jenkins, 1976; Box et al., 2013). This is accomplished by using various data transformations like taking logarithms and time differences before fitting ARMA models to the transformed data. Wide availability of computer software and a well developed mathematical theory has contributed to the enduring popularity of this approach. The state space methodology, however, offers a number of important advantages over the popular class of ARMA models in so many ways. First is the fact that the state space method does not require the data to be stationary, which eliminates a problem which the Box-Jenkins theory offers minimum guidance. Another fact is that state space methods can accommodate a multivariate framework and they are able to deal with irregular patterns, interventions, sudden jumps and regime shifts in data. The problems of estimation and forecasting in state space models are solved by recursively computing the conditional distribution of the quantities of interest, given the available information (Koopman, 1997; Evensen, 2003), in this sense, they are quite naturally treated from a Bayesian approach (Zellner, 1996). This paper tends to contribute to the theoretical and empirical literature on the topic by demonstrating the ease at which Kalman filter can be used to estimate economic data using an example from analysis of Nigerian economic data.

3 Methodology and Model Specification

3.1 Linear- Gaussian State Space Model

Consider a linear-Gaussian state space model for an $m \times 1$ -dimensional time series y_t consisting of a measurement equation relating the observed time series to a $p \times 1$ -dimensional unobserved state vector θ_t and a Markovian transition equation that describes the evolution of the state vector over time (Harvey and Koopman, 1993). The model takes the following general form.

$$y_t = F_t \theta_t + v_t \quad v_t \sim N(0, V_t) \quad (1)$$

$$\theta_t = G_t \theta_{t-1} + w_t \quad w_t \sim N_p(0, W_t) \quad (2)$$

$$\theta_0 \sim N_p(m_0, C_0) \quad (3)$$

where y_t is a vector of dimension $m \times 1$

Equation (1) is known as the observation equation while equation (2) is a first order Markov process called the evolution equation. G_t , F_t , are known matrices of order $p \times p$ and $m \times p$ respectively that determine how the observation and state equations evolve in time (West and Harrison, 1997).

The matrices F_t , V_t , G_t and W_t are known as the system matrices and contain non-random elements. If they do not depend deterministically on t , the state space system is time invariant, otherwise they are time varying.

v_t and w_t are assumed to be independent both within and between, and independent of θ_0 . The initial state vector θ_0 is assumed to be normally distributed with parameters m_0 and C_0 as shown in (3) where $E(v_t \theta_0') = 0$, and $E(w_t \theta_0') = 0$ for $t = 1, \dots, T$. If some or all the elements of θ_t are assumed to be covariance stationary, the corresponding elements of m_0 and C_0 can be solved analytically from the elements of the system matrices. However, for non-stationary elements of θ , it is customary to set the corresponding elements of C_0 to a very large positive number (West and Harrison, 1997).

3.2 Optimal State Estimation and Prediction Using Kalman Filter

In a state space model, the unobserved state vector θ is the signal and the measurement error w_t is the noise. The objective of filtering is to track the dynamic evolution of unobservable state variables using (typically)

noisy measurements of observables (Tsay, 2005). This requires the computation of integrals over the unobserved states, which in general must be approximated numerically. Given observed data $y_1, \dots, y_2$, the goals of state space estimation using the Kalman filter is twofold:

- (1) Optimal signal extraction (estimation) and
- (2) Optimal h-step ahead prediction of states and data.

Basically, there are two types of state estimation, namely (a) Filtering and

(b) Smoothing. Filtering implies getting optimal estimates of θ_t given information available at time t , $D_t = (y_1, \dots, y_t)$, so that $E(\theta_t|D_t)$ is the filtered estimate of θ . On the other hand, smoothing implies getting optimal estimates of θ_t given information available at time T , $D_T = (y_1, \dots, y_T)$, so that $E(\theta_t|D_T)$ is the smoothed estimate of θ .

3.3 The Kalman Filter

A filter is a device which separates entities into their constituting components. The process of finding the best estimate from noisy data is known as filtering out the noise. It does not just clean up the data measurements but also projects these measurements into the state estimate. The Kalman filter was developed in 1960 by Rudolf E. Kalman (Kalman, 1960) as an algorithm used for aerospace guidance applications and to solve state space models in the linear case. It is an optimal estimator which infers parameters of interest from indirect, and sometimes inaccurate and uncertain observations. It is an optimal process because if all noise is gaussian, the Kalman filter minimizes the mean square error of the estimated parameters. The Kalman filter calculates the mean and variance of the unobserved state θ_t , given the observations. It provides an exact and complete solution to the Bayesian filtering problem which is very economical in computer calculation and storage requirements. It is a recursive algorithm i.e the current best estimate is updated whenever a new observation is obtained. It is an optimal process because if all noise is Gaussian, the Kalman filter minimizes the mean square error of the estimated parameters. The variance of the measurements equation needs to be known before implementing a Kalman filter.

Given the initial state and covariance, we have sufficient information to find the optimal state estimate using the Kalman filter equations. The Kalman filter has originally been applied by engineers and physicists to estimate the state of a noisy system. The classic Kalman filter application is the example of tracking an orbiting satellites whose exact position and speed, which are not directly measurable at any point of time, but can

be estimated using available data . A discussion of engineering-type applications to non-linear models is provided in Julier and Uhlmann (1997); Wan and Van Der Merwe (2000).

As mentioned earlier, the Kalman filter is a set of recursion equations for determining the optimal estimates of the state vector θ_t , given information available at time t , D_t . The filter consists of two sets of equations: (i) Prediction Equations (ii) Updating Equations

The filter prediction and update steps require a few basic matrix calculations of which only the conditional means and variances of the filtering and prediction density need to be stored in each step of the iteration. To describe the filter, let

$$m_t = E(\theta_t | D_t)$$

be the optimal estimator of θ_t based on D_t and let

$$C_t = E((\theta_t - m_t)(\theta_t - m_t)^T | D_t)$$

be the mean square error matrix of m_t .

Prediction Equations

Let $\theta_{t-1} | y_{1:t-1} \sim N(m_{t-1}, C_{t-1})$, where $y_{1:t-1}$ denotes all observations up to time $t-1$. Then the one-step-ahead predictive density $\theta_t | y_{1:t-1}$ is Gaussian with parameters:

$$\begin{aligned} E(\theta_t | y_{1:t-1}) &= m_{t-1} \equiv A_t \\ \text{Var}(\theta_t | y_{1:t-1}) &= C_{t-1} + W_t \equiv R_t \end{aligned}$$

The one-step-ahead predictive density of $Y_t | y_{1:t-1}$ is Gaussian with parameters:

$$\begin{aligned} f_t &= E(Y_t | y_{1:t-1}) = X_t A_t \\ Q_t &= \text{Var}(Y_t | y_{1:t-1}) = X_t R_t X_t' + V \end{aligned}$$

The filtering density of θ_t given $y_{1:t}$ is Gaussian with parameters:

$$\begin{aligned} m_t &= E(\theta_t | y_{1:t}) = A_t + R_t X_t' Q_t^{-1} e_t \\ C_t &= \text{Var}(\theta_t | y_{1:t}) = R_t - R_t X_t' Q_t^{-1} X_t R_t \end{aligned}$$

where $e_t = Y_t - f_t$ is the forecast error.

Given m_{t-1} and C_{t-1} at time $t - 1$, the optimal predictor of θ_t and its associated MSE matrix are

$$\begin{aligned} m_{t/t-1} &= E(\theta_t | D_{t-1}) = G_t m_{t-1} \\ C_{t/t-1} &= E((\theta_t - m_{t-1})(\theta_t - m_{t-1})' | D_{t-1}) \\ &= G_t C_{t-1} G_t' + W_t \end{aligned}$$

.

Updating Equations

When new observations y_t become available, the optimal predictor $m_{t|t-1}$ and its MSE matrix are updated using

$$\begin{aligned} m_t &= m_{t|t-1} + C_{t|t-1} F_t' Q_t^{-1} (y_t - F_t m_{t|t-1}) \\ &= m_{t|t-1} + C_{t|t-1} F_t' Q_t^{-1} v_t \\ C_t &= C_{t|t-1} - C_{t|t-1} F_t' Q_t^{-1} F_t C_{t|t-1} \\ &= C_{t|t-1} - K_t v_t \end{aligned}$$

where

$$Q_t = F_t C_{t|t-1} F_t' + V_t$$

and

$$K_t = C_{t|t-1} F_t' Q_t^{-1}$$

is the Kalman gain matrix which gives the weight on new information $e_t = y_t - F_t m_{t|t-1}$ in the updating equation for m_t .

Kalman Smoother

Once all data D_T are observed, the optimal estimates $E(\theta | D_T)$ can be computed using the backwards kalman smoothing recursions

$$E(\theta_t | D_T) = m_{t|T} = m_t + C * _t (m_{t+1|T} - G_{t+1} m_t)$$

and

$$\begin{aligned} E(\theta_t - m | T)(\theta_t - m | T)' | D_T &= C_{t|T} \\ &= C_t + C * _t (C_{t+1|T} - C_{t+1|t}) C * _t' \\ C * _t &= C_t G_{t+1}' C_{t+1|t}^{-1} \end{aligned}$$

The algorithm starts by setting $m_T|T = m_T$ and $C_T|T = C_T$ and then proceeds backwards for $t = T - 1, \dots, 1$. Proofs of this algorithm and detailed procedures can be found in Harvey (1990); West and Harrison (1997), and recent applications to mixed-measurement time series can be found in Cui and Dunson (2014).

4 Empirical Example

4.1 Dynamic Linear Modeling of Some Nigerian Economic Data

Economic indicators are usually complex high frequency data (Fuquene et al., 2013). By way of illustration, in this section, we fit a linear-

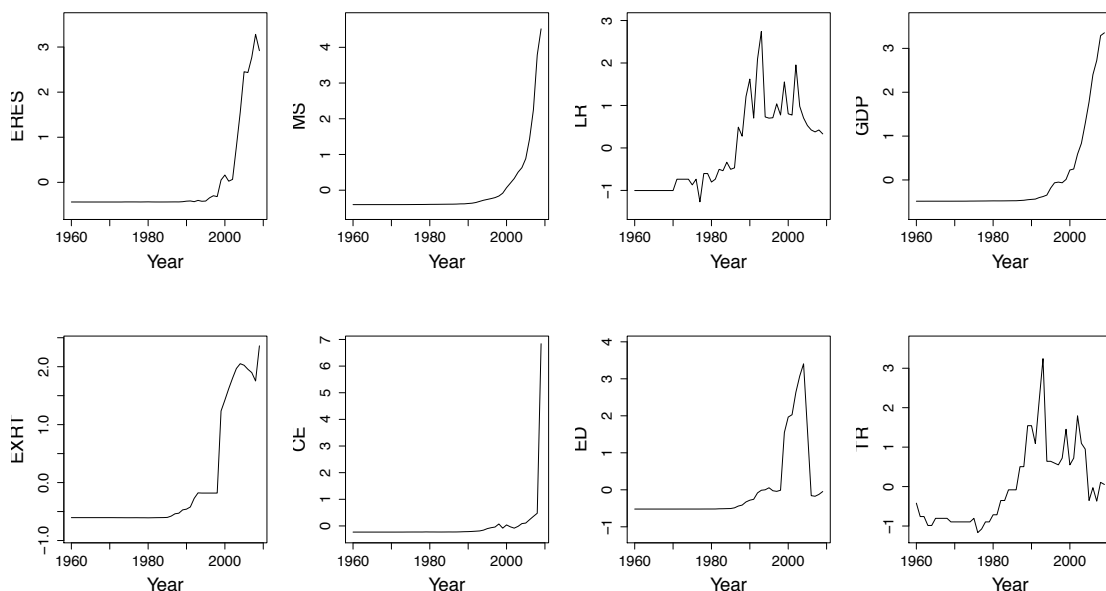


Figure 1: Annual time-series data on External Reserve (ERES), Money Supply (MS), Lending Rate (LR), Gross Domestic Product (GDP), Exchange Rate (EXRT), Capital Expenditure (CE), External Debt (ED) and Treasury bill Rate (TR). These data were standardized prior to analysis.

Gaussian univariate Dynamic Linear Model (DLM) to eight economic indicators of the Nigerian economy (shown in Figure 1).

We consider a univariate variant of the DLM presented in equations

(1) and (2) where:

$$\theta_t = \begin{pmatrix} \mu_t \\ \beta_t \end{pmatrix}, G_t = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad W_t = \begin{pmatrix} 0 & 0 \\ 0 & \sigma_w^2 \end{pmatrix}, F_t = \begin{pmatrix} 1 & 0 \end{pmatrix}.$$

Then:

$$\begin{aligned} y_t &= \theta_t + v & v &\sim N(0, \sigma_v^2) \\ \theta_t &= \theta_{t-1} + \beta_{t-1} \\ \beta_t &= \beta_{t-1} + w & w &\sim N(0, \sigma_w^2) \end{aligned}$$

where y_t is the value of the time series we are trying to model with respect to θ_t . θ can be thought of as a static location parameter and β can be thought of as a time-varying slope parameter.

This model is fully specified with $C_0 = 100$, $m_0 = 0$, $\sigma_w^2 = 2$ and $\sigma_v^2 = 1000$. We compute the filtering and predictive densities of the eight economic time series using the recursive Kalman filter algorithm, starting from $\theta_0|D_0 \sim N(m_0, C_0)$, then computing $p(\theta_1|D_1)$ and proceeding recursively. This analysis was implemented using the "dlm" package in the R software presented in Petris et al. (2009). We obtained convergence of all parameters with 10000 iterations with a burn-in period of 2000. The plots of the filtered states and one-step-ahead forecasts for the eight series considered are shown in Figure 2.

We can see from Figure 2 how the integrated random walk model performs on the economic data under study. Overall, our posterior analysis indicates that the trend and volatility of the Gross Domestic Product (GDP) and Money Supply (MS) are quite smooth when compared to the more volatile series of lending rate and treasury bill rate which experienced substantial changes during the recent economic episode of Nigeria. Note the conspicuous spikes at the tail end in the forecasts of exchange rate, external debt, and external reserves. The one-step ahead forecasts of external reserves and external debt suggests that the external reserve increases in the following year while external debt reduces further. Analyzing causes of different responses across variables may require a more structural econometric model of the economy (Doh and Connolly, 2013). However, the kind of analysis carried out in this paper can be a starting point for such a project.

5 Concluding Remarks

In this paper, the Kalman filter methodology has been successfully tested on macroeconomic dataset. It relies solely on optimal filtering and smooth-

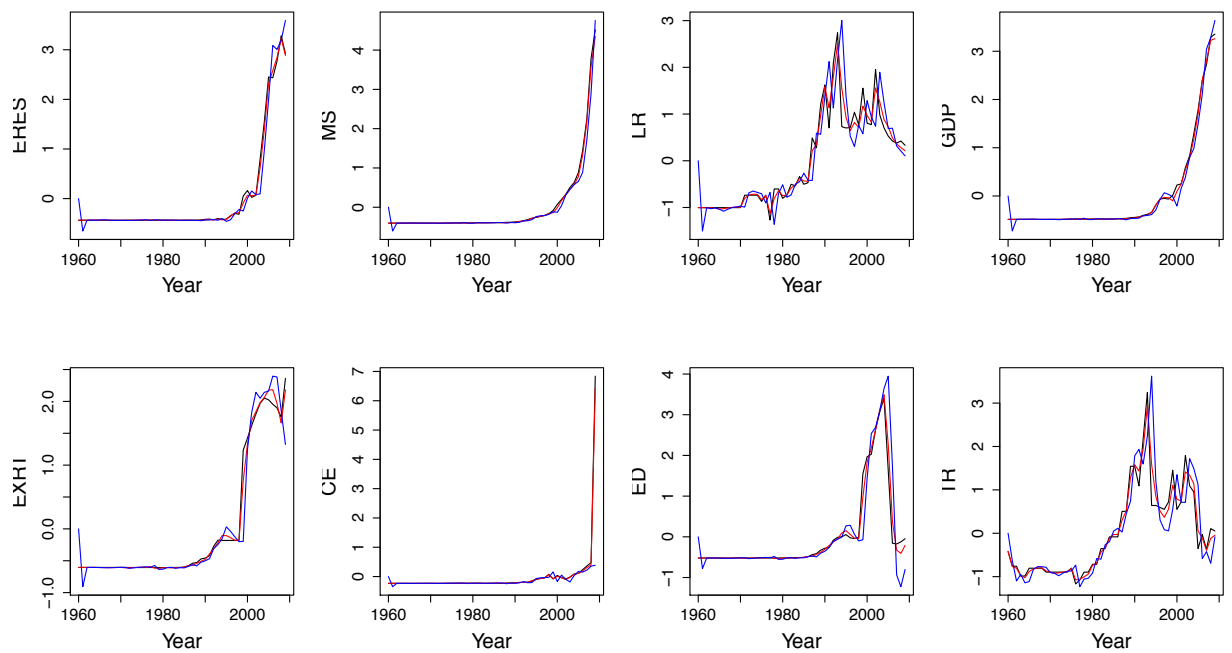


Figure 2: Filtered states, & forecasts. Black line represents the data, red line represents the filtered states, and the blue line represents the one-step-ahead state forecasts.

ing of the data measurements. From our experience, we find that the Kalman filter is not a complex algorithm, but it can be difficult to adjust all the required initial conditions and parameters, which demand either an expert user or a statistical method to adjust them. With optimal tuning of the parameters, very good results can be achieved. Some advantages to the Kalman filter are that it is predictive and adaptive, as it looks forward with an estimate of the covariance and mean of the time series one step into the future and unlike a Neural Network and the Box-Jenkins method, it does not require stationarity of the data. As future work, we propose to use the Kalman filtering results (its prediction and corresponding variance) for detection of outliers in high frequency and more volatile economic data.

References

- Box, G. E. and G. M. Jenkins (1976). *Time series analysis: forecasting and control, revised ed.* Holden-Day.
- Box, G. E., G. M. Jenkins, and G. C. Reinsel (2013). *Time series analysis: forecasting and control.* John Wiley & Sons.
- Chib, S., F. Nardari, and N. Shephard (2002). Markov chain monte carlo methods for stochastic volatility models. *Journal of Econometrics* 108(2), 281–316.
- Cui, K. and D. B. Dunson (2014). Generalized dynamic factor models for mixed-measurement time series. *Journal of Computational and Graphical Statistics* 23(1), 169–191.
- Doh, T. and M. Connolly (2013). *The state space representation and estimation of a time-varying parameter VAR with stochastic volatility.* Springer.
- Evensen, G. (2003). The ensemble kalman filter: Theoretical formulation and practical implementation. *Ocean dynamics* 53(4), 343–367.
- Fuquene, J., M. Alvarez, and L. Pericchi (2013). A robust bayesian dynamic linear model to detect abrupt changes in an economic time series: The case of puerto rico. *arXiv preprint arXiv:1303.6073*.
- Hamilton, J. D. (1994). *Time series analysis*, Volume 2. Princeton university press Princeton.
- Harvey, A. and S. J. Koopman (1993). Structural time series models. *Encyclopedia of Biostatistics*.

-
- Harvey, A. C. (1989). The kalman filter and forecasting structural time series models.
- Harvey, A. C. (1990). *Forecasting, structural time series models and the Kalman filter*. Cambridge university press.
- Julier, S. J. and J. K. Uhlmann (1997). A new extension of the kalman filter to nonlinear systems. In *Int. symp. aerospace/defense sensing, simul. and controls*, Volume 3, pp. 3–2. Orlando, FL.
- Kalman, R. E. (1960). A new approach to linear filtering and prediction problems. *Journal of Fluids Engineering* 82(1), 35–45.
- Kalman, R. E. and R. S. Bucy (1961). New results in linear filtering and prediction theory. *Journal of Fluids Engineering* 83(1), 95–108.
- Koopman, S. J. (1997). Exact initial kalman filtering and smoothing for nonstationary time series models. *Journal of the American Statistical Association* 92(440), 1630–1638.
- Ng, C. and P. C. Young (1990). Recursive estimation and forecasting of non-stationary time series. *Journal of Forecasting* 9(2), 173–204.
- Petris, G., S. Petrone, and P. Campagnoli (2009). *Dynamic linear models with R*. Springer.
- Tsay, R. S. (2005). *Analysis of financial time series*, Volume 543. John Wiley & Sons.
- Wan, E. A. and R. Van Der Merwe (2000). The unscented kalman filter for nonlinear estimation. In *Adaptive Systems for Signal Processing, Communications, and Control Symposium 2000. AS-SPCC. The IEEE 2000*, pp. 153–158. IEEE.
- West, M. and P. Harrison (1997). *Bayesian Forecasting and Dynamic Models* (2nd ed.). New York: Springer-Verlag.
- Young, P. (1988). Recursive extrapolation, interpolation and smoothing of non-stationary time-series. *Identification and System Parameter Estimation*, 33–44.
- Zellner, A. (1996). *An introduction to Bayesian inference in econometrics*.

Acknowledgement

We would like to thank the referees and the editor for their helpful comments and suggestions which led to the improvement of an earlier version of this manuscript.