

A short note on weak estimation of Sharp function

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Abstract

In paper [1] Ahmad et al investigated the use of sharp function, known from functional analysis, in image processing. The sharp function gives a measure of variations of a function and can be used as an edge detector [2]. We extend the classical notion of sharp function to prove the classical Lebesgue differentiation theorem and Marcinkiewicz theorem for the sublinear operator $T(x, y)$.

Keywords: Maximal function, bounded mean oscillations (BMO), sharp function, distribution, sublinear operator.

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1 Introduction

The sharp function is a well known functional analytic concept to measure the oscillatory behavior of functions. It goes back to the maximal function which was introduced in Hardy and Littlewood [5] to solve a problem in the theory of complex variable. Based on this idea John and Nirenberg [6] introduced the concept of bounded mean oscillation (BMO) functions. Fefferman and Stein [4] introduced the sharp function (denoted by $f^\#$) and found that a function $f \in BMO$ is equivalent with $f^\# \in L^\infty$. For more details we refer to Fefferman [3], Kurtz [7] and Wojtaszczyk [8]. The idea of applying the sharp operator to measure the oscillation and classification of images was first proposed by Ahmad and Siddiqi [1] where it was used to find a suitable compression technique.

In this paper, we use sharp function to prove the Marcinkiewicz Interpolation theorem for the weak sublinear operator $T(x, y)$ and for generalization of the concept of distribution. This paper also includes a proof of Lebesgue differentiation theorem.

2 Maximal Function

In this section we give a short introduction to the maximal function and its background. There is a rich theory behind it, and we are going to point out some of the main results connected to it. The Hardy-Littlewood maximal function was developed to solve a problem in the theory of functions of complex

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variable. Further it can also be used in the proof of Lebesgue differentiation theorem, Fatou's theorem and in the theory of singular integral operators.

Definition 2.1. Let $\mathbb{R}^n$ be the n -dimensional Euclidean space and $f(x)$ be a real valued measurable function on $\mathbb{R}^n$. For such a function f on $\mathbb{R}^n$ its Hardy-Littlewood maximal function is defined by the formula

$$Mf(x) = \sup_{Q: x \in Q} \left\{ \frac{1}{\lambda(Q)} \int_Q |f(y)| dy : Q \subset \mathbb{R}^n, x \in Q \right\}, \quad (2.1)$$

where the supremum ranges over all finite cubes Q in $\mathbb{R}^n$ and $\lambda(Q)$ is the Lebesgue measure of Q .

Definition 2.2. Let $\mathbb{R}^n$ be the n -dimensional Euclidean space and $f(x)$ be a real valued measurable function on $\mathbb{R}^n$. For such a function f on $\mathbb{R}^n$ its distribution is defined by the formula

$$d_f(t) = \lambda(\{x \in \mathbb{R}^n : |f(x)| > t\}), \text{ for } t \geq 0. \quad (2.2)$$

It is easy to find a function whose maximal function is unbounded.

Example 2.1. For $f(x) = |x|^t$ with $t > 0$, we get $Mf(x) = \infty$, for each $x \in \mathbb{R}$.

Theorem 2.2. [Hardy-Littlewood maximal theorem] For each function $f \in L^1(\mathbb{R}^n)$ we have

$$\lambda(\{x : Mf(x) > t\}) \leq 6^n t^{-1} \|f\|_1, \quad t > 0. \quad (2.3)$$

Proof. If $\|f\|_1 = \infty$, then it is trivial. Without the loss of generality, we can assume that $\|f\|_1 = 1$. For a fixed $t \geq 0$, let $E_t = \{x : Mf(x) > t\}$. Then for each $x \in E_t$ there is a cube Q_x such that $x \in Q_x \subset E_t$, and

$$\frac{1}{\lambda(Q_x)} \left\{ \int_{Q_x} |f(y)| dy \right\} > t. \quad (2.4)$$

Thus,

$$\frac{\|f\|_1}{t} \geq \lambda(Q_x) \text{ or } \frac{1}{t} \geq \lambda(Q_x). \quad (2.5)$$

Let $\alpha_1 = \max\{\lambda(Q_x) : x \in E_t\}$, so $\alpha_1 \leq t^{-1}$. Let us fix a cube Q_1 , call it Q_1 such that $\lambda(Q_1) > \frac{1}{2} \alpha_1$. Consider all cubes Q_x such that $Q_x \cap Q_1 = \emptyset$. If there are no such cubes, we stop. Otherwise we put $\alpha_2 = \max\{\lambda(Q_x) : x \in E_t \text{ and } Q_x \cap Q_1 = \emptyset\}$ and fix such a cube Q_2 , call it Q_2 satisfying $\lambda(Q_2) > \frac{1}{2} \alpha_2$. Continuing in this way we get a sequence of cubes $Q_1, Q_2, \dots$, possibly finite such that

- (i) the cubes Q_j are disjoint,
- (ii) $\lambda(Q_j) > \frac{1}{2} \max\{\lambda(Q_x) : Q_x \cap Q_s = \emptyset \text{ for } s = 1, 2, 3, \dots, j-1\}$,
- (iii) if $Q_x \cap Q_s = \emptyset$ for $s = 1, 2, 3, \dots, j-1$, then $\lambda(Q_x) \leq 2\lambda(Q_j)$.

From (2.4) and (i), we get

$$\lambda\left(\bigcup_{i=1}^n Q_i\right) = \sum_{i=1}^n \lambda(Q_i) \leq \frac{1}{t} \sum_{i=1}^n \int_{Q_i} |f(y)| dy \leq \frac{1}{t} \int_{\bigcup Q_i} |f(y)| dy \leq \frac{\|f\|_1}{t} = \frac{1}{t}. \quad (2.6)$$

It is important to mention here that each Q_x intersects some Q_i . If there exists a Q_x disjoint from all Q_i 's then our process was infinite. So from (2.6), we see that $\lambda(Q_j) \rightarrow \infty$ which contradicts (iii). Now for a given Q_x , let Q_s be the first Q_i that intersects with Q_x . So by (iii), $\lambda(Q_x) \leq 2\lambda(Q_s)$ and hence $Q_x \subset 6 \diamond Q_s$ (by $c \diamond Q$ we mean a cube with the same center as Q whose sides are c times longer than sides of Q , $c > 0$). Thus from equation (2.6), we have

$$\begin{aligned}
 \lambda(\{x : Mf(x) > t\}) &= \lambda(\cup_x Q_x) \\
 &\leq \lambda(\cup_i 6 \circ Q_i) \\
 &\leq \sum_i \lambda(6 \circ Q_i) \\
 &\leq 6^n \sum_i \lambda(Q_i) \leq 6^n \frac{1}{t} = 6^n \frac{\|f\|_1}{t}.
 \end{aligned} \tag{2.7}$$

□

An interesting application of the Maximal Theorem is a version of the Lebesgue differentiation theorem.

Theorem 2.3. Let $f \in L^1(\mathbb{R}^n)$. For almost all $x \in \mathbb{R}^n$ and for every decreasing sequence of cubes $\{Q_j\}_{j=1}^\infty$ such that $\bigcap_{j=1}^\infty Q_j = \{x\}$, we have

$$\lim_{j \rightarrow \infty} \frac{1}{\lambda(Q_j)} \int_{Q_j} f(y) dy = f(x). \tag{2.8}$$

Proof. If $f \in L^1(\mathbb{R}^n) \cap C(\mathbb{R}^n)$, then (2.8) holds for all $x \in \mathbb{R}^n$. Given $f \in L^1(\mathbb{R}^n)$ we take ϵ such that $0 < \epsilon < 1$ and write $f = g + h$ with $g \in L^1(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ and $\|h\|_1 < \epsilon$. Then by the above observation and by the definition of Hardy-Littlewood maximal function, we have,

$$\begin{aligned}
 &\limsup_{j \rightarrow \infty} \left| \frac{1}{\lambda(Q_j)} \int_{Q_j} f(y) dy - f(x) \right| \\
 &= \limsup_{j \rightarrow \infty} \left| \frac{1}{\lambda(Q_j)} \int_{Q_j} (g+h)(y) dy - (g+h)(x) \right| \\
 &= \limsup_{j \rightarrow \infty} \left| \frac{1}{\lambda(Q_j)} \int_{Q_j} g(y) dy - g(x) + \frac{1}{\lambda(Q_j)} \int_{Q_j} h(y) dy - h(x) \right| \\
 &= \limsup_{j \rightarrow \infty} \left| \frac{1}{\lambda(Q_j)} \int_{Q_j} h(y) dy - h(x) \right| \\
 &\leq \limsup_{j \rightarrow \infty} \left\{ \frac{1}{\lambda(Q_j)} \int_{Q_j} |h(y)| dy \right\} + |h(x)| \\
 &\leq |h(x)| + Mh(x).
 \end{aligned}$$

Now by equations (2.2) and (2.3), we get

$$\lambda(\{x \in \mathbb{R}^n : |h(x)| > \sqrt{\epsilon}\}) \leq \frac{\|h\|_1}{\sqrt{\epsilon}} \leq \sqrt{\epsilon}$$

and

$$\lambda(\{x : Mh(x) > \sqrt{\epsilon}\}) \leq 6^n \frac{\|h\|_1}{\sqrt{\epsilon}} \leq 6^n \sqrt{\epsilon}$$

respectively. This shows that outside the set $\{x \in \mathbb{R}^n : Mh(x) > \sqrt{\epsilon}\} \cup \{x \in \mathbb{R}^n : |h(x)| > \sqrt{\epsilon}\}$ which has measure at most $C\sqrt{\epsilon}$, we have

$$\limsup_{j \rightarrow \infty} \left| \frac{1}{\lambda(Q_j)} \int_{Q_j} f(y) dy - f(x) \right| \leq 2\sqrt{\epsilon}.$$

Since ϵ was arbitrary small, we obtain

$$\lim_{j \rightarrow \infty} \frac{1}{\lambda(Q_j)} \int_{Q_j} f(y) dy = f(x).$$

□

The Hardy-Littlewood maximal operator M is a significant nonlinear operator used in real and harmonic analysis. It takes a locally integrable function $f: \mathbb{R}^n \rightarrow \mathbb{C}$ and returns another function Mf that at each $x \in \mathbb{R}^n$ gives the maximal average value that f can have on cubes containing the point $x \in Q_x$. Hardy-Littlewood maximal inequality states that M is bounded as a sublinear operator from $L^p(\mathbb{R}^n)$ to itself for $p > 1$.

Definition 2.3. A measurable function f on $\mathbb{R}^n$ has bounded p -mean oscillation, $1 \leq p < \infty$, if

$$\|f\|_{BMO_p} = \sup_Q \left\{ \frac{1}{\lambda(Q)} \int_Q |f(x) - f_Q|^p dx \right\}^{1/p} < \infty$$

where the sup ranges over all finite cubes Q in $\mathbb{R}^n$ and $f_Q = \frac{1}{\lambda(Q)} \int_Q f(x) dx$ is the mean value of the function f on the cube Q .

The set of all functions of bounded p -mean oscillation is denoted by $BMO_p(\mathbb{R}^n)$. $\|f\|_{BMO_p}$ is "almost" a norm since it has the following properties

- (i) $\|f + g\|_{BMO_p} \leq \|f\|_{BMO_p} + \|g\|_{BMO_p}$
- (ii) $\|\alpha f\|_{BMO_p} \leq |\alpha| \|f\|_{BMO_p}$
- (iii) $\|f\|_{BMO_p} = 0$ iff $f = \text{constant a.e.}$,

where f and g are the measurable functions on $\mathbb{R}^n$ and α is some scalar quantity.

Example 2.4. The function

$$f(x) = \begin{cases} 1 & \text{if } x \in [0, 1] \\ 0 & \text{if } x \notin [0, 1] \end{cases}$$

is in $BMO_1(\mathbb{R})$.

Let $Q = [0, 1]$, then

$$\frac{1}{1-0} \int_0^1 |f(x) - f_Q| dx = \int_0^1 |f(x) - 1| dx = 0 < \infty.$$

Every bounded function has a bounded p -mean oscillations.
That is, $L^\infty(\mathbb{R}^n) \subset BMO_1(\mathbb{R}^n)$, $1 \leq p < \infty$.

Example 2.5. For $f(x) = |x|^\alpha$ with $\alpha \in (-1, 0)$, we find maximal function $Mf(x)$ of $f(x)$ as

$$Mf(x) = \sup_{x \in I} \left\{ \frac{1}{|I|} \int_I |f(t)| dt \right\}.$$

We know that $Mf(x) = \max\{Mf_+(x), Mf_-(x)\}$. Assume that $x > 0$, then clearly,

$$Mf(x) = Mf_-(x) = \sup_{y < x} \frac{1}{x-y} \int_y^x |f(t)| dt = \sup_{y < x} \frac{1}{x-y} \int_y^x |t|^\alpha dt.$$

Further, to obtain sup we consider $y < 0$, then we have

$$Mf(x) = \sup_{y < 0} \frac{1}{x-y} \left\{ \frac{|t|^{\alpha+1}}{\alpha+1} \right\}_y.$$

Since $y < 0$, so that choose $y = -s$, where $s > 0$ then

$$\begin{aligned} Mf(x) &= \sup_{s > 0} \frac{1}{x+s} \left\{ \frac{x^{\alpha+1} + s^{\alpha+1}}{\alpha+1} \right\} \\ &= \sup_{s > 0} \frac{1}{x(1+(s/x))} \left\{ \frac{x^{\alpha+1}(1+(s/x)^{\alpha+1})}{\alpha+1} \right\} \\ &= \sup_{s > 0} x^\alpha \left\{ \frac{(1+(s/x)^{\alpha+1})}{(1+(s/x))(1+\alpha)} \right\}. \end{aligned}$$

Putting $\frac{s}{x} = t > 0$, we get

$$Mf(x) = x^\alpha \sup_{t > 0} \left\{ \frac{1+t^{\alpha+1}}{(1+t)(1+\alpha)} \right\} = x^\alpha C_\alpha.$$

C_α tends to 0 as t goes to infinity and the supremum is obtained.

3 Sharp Function

Sharp function is a very powerful tool that mediates between L^p spaces and the space of bounded mean oscillations, i.e., BMO_p .

Definition 3.1. For locally integrable function f on $\mathbb{R}^n$ the sharp function is given by

$$f^\#(x) = \sup_{Q: x \in Q} \left\{ \frac{1}{\lambda(Q)} \int_Q |f(y) - f_Q|^p dy \right\}^{1/p} < \infty.$$

Thus, $f \in BMO$ is identical with the statement $f^\# \in L^\infty$. The interest in $f^\#$ lies in the fact that $f^\# \in L^p, p < \infty$ implies that $f \in L^p$.

Proposition 3.1. If $f^\#$ is a function in $L^p(\mathbb{R}^n)$, and $1 \leq p < \infty$, then

$$d_{f^\#}(t) \leq t^{-p} \|f^\#(x)\|_p^p \quad (3.1)$$

and

$$\|f^\#(x)\|_p^p = p \int_0^\infty t^{p-1} d_{f^\#}(t) dt. \quad (3.2)$$

Proof. By the definition of the distribution

$$\begin{aligned} d_{f^\#}(t) &= \lambda(\{x \in \mathbb{R}^n : |f^\#(x)| > t\}) \\ &= \int_{\{x \in \mathbb{R}^n : |f^\#(x)| > t\}} dx \end{aligned} \quad (3.3)$$

or,

$$\begin{aligned} t^p d_{f^\#}(t) &= \int_{\{x \in \mathbb{R}^n : |f^\#(x)| > t\}} t^p dx \\ &\leq \int_{\mathbb{R}^n} |f^\#(x)|^p dx. \end{aligned} \quad (3.4)$$

Since $f^\#$ is integrable on $\mathbb{R}^n$, then we have

$$t^p d_{f^\#}(t) \leq \|f^\#(x)\|_p^p$$

$$d_{f^\#}(t) \leq t^{-p} \|f^\#(x)\|_p^p.$$

For the remaining part we define the set $A \subset \mathbb{R}^n \times [0, \infty)$ as $A = \{(x, s) : s < |f^\#(x)|^p\}$. Using Fubini's theorem, we get

$$\begin{aligned} \int_{\mathbb{R}^n} |f^\#(x)|^p dx &= \int_{\mathbb{R}^n} \int_0^{|f^\#(x)|^p} 1 ds dx \\ &= \int_{\mathbb{R}^n \times [0, \infty)} 1_A(x, s) ds dx \\ &= \int_0^\infty \lambda(\{x : |f^\#(x)|^p > s\}) ds. \end{aligned}$$

Putting $s = t^p \Rightarrow ds = pt^{p-1} dt$ in the above integral, we obtain

$$\begin{aligned} \int_{\mathbb{R}^n} |f^\#(x)|^p dx &= \int_0^\infty pt^{p-1} \lambda(\{x : |f^\#(x)|^p > t^p\}) dt \\ &= p \int_0^\infty t^{p-1} \lambda(\{x : |f^\#(x)| > t\}) dt. \end{aligned} \quad (3.5)$$

Therefore,

$$\|f^\#(x)\|_p^p = p \int_0^\infty t^{p-1} d_{f^\#}(t) dt.$$

□

Definition 3.2. An operator T defined on some class of measurable functions and mapping it into measurable functions is called sublinear if

$$|T(f+g)(x)| \leq |T(f)(x)| + |T(g)(x)| \quad \text{a.e.}$$

and

$$|T(\lambda f)(x)| \leq |\lambda| |T(f)(x)|, \quad \text{a.e.},$$

for all admissible functions f, g and all scalars λ .

Definition 3.3. A sublinear operator T from $L^p(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$ is said to be of weak type (p, q) , $1 \leq p < \infty, q < \infty$ if there exists a constant C such that for each $f \in L^p(\mathbb{R}^n)$ and each $t > 0$ we have

$$\lambda(\{x : |T(f)(x)| > t\}) \leq \left(\frac{C \|f\|_p}{t} \right)^q.$$

It is clear that each linear operator is sublinear. It immediately follows from (3.1) that if a sublinear operator T satisfies

$$\|T(f)\|_1 \leq C \|f\|_1,$$

then T is of weak type $(1, 1)$.

Theorem 3.1. [Marcinkiewicz] Suppose T is a sublinear operator defined on $L^1(\mathbb{R}^n) + L^\infty(\mathbb{R}^n)$ which is of weak type $(1, 1)$ and for some C satisfies

$$\|Tf^\#\|_\infty \leq C \|f^\#\|_\infty, \quad (3.6)$$

then for each $p, 1 < p < \infty$, there exists a constant $C(p)$ such that

$$\|Tf^\#\|_p \leq C(p) \|f^\#\|_p, \quad \text{where } C(p) = \frac{2^p p}{(p-1)}.$$

Proof. We are given that T is a sublinear operator defined on $L^1(\mathbb{R}^n) + L^\infty(\mathbb{R}^n)$ and is of weak type $(1,1)$ then by the definition of T we can say without the loss of generality that for $C = 1$ and $t > 0$,

$$\lambda(\{x : |Tf^\#(x)| > t\}) \leq \frac{\|f^\#\|_1}{t}.$$

And also T satisfies (3.6), i.e.,

$$\|Tf^\#\|_\infty \leq C\|f^\#\|_\infty$$

or,

$$\|Tf^\#\|_\infty \leq \|f^\#\|_\infty.$$

Now for $f^\# \in L^p(\mathbb{R}^n)$ and given $t > 0$, we write $f^\#(x) = f_t(x) + f^t(x)$, where $f_t(x) \in L^1(\mathbb{R}^n)$, $f^t(x) \in L^\infty(\mathbb{R}^n)$ and

$$f^t(x) = \begin{cases} f^\#(x) & x \geq t \\ 0 & x < t, \end{cases}$$

$$f_t(x) = \begin{cases} f^\#(x) & x \leq t \\ 0 & x > t. \end{cases}$$

Thus $f_t(x) = \mathbf{1}_{\{x : |f^\#(x)| \leq t\}} f^\#(x)$. Since $f^\#$ is a measurable function and T is a sublinear operator, therefore $Tf^\#$ is also a measurable function. Then by countable subadditivity, we have

$$\{x : |Tf^\#(x)| > t\} \subset \left\{x : |Tf_{t/2}(x)| > \frac{t}{2}\right\} \cup \left\{x : |Tf^{t/2}(x)| > \frac{t}{2}\right\}.$$

So from (3.2), we obtain

$$\begin{aligned} \|Tf^\#\|_p^p &= \int_{\mathbb{R}^n} |Tf^\#(x)|^p dx \\ &= p \int_0^\infty t^{p-1} \lambda(\{x : |Tf^\#(x)| > t\}) dt \end{aligned} \quad (3.7)$$

or,

$$\|Tf^\#\|_p^p \leq p \int_0^\infty t^{p-1} \lambda\left(\left\{x : |Tf_{t/2}(x)| > \frac{t}{2}\right\}\right) dt + p \int_0^\infty t^{p-1} \lambda\left(\left\{x : |Tf^{t/2}(x)| > \frac{t}{2}\right\}\right) dt.$$

But we suppose that T satisfies (3.6) with $C = 1$, and this implies that $\|Tf_{t/2}\|_\infty \leq \frac{t}{2}$ and thus, the second integral is zero. Then by the definition of sublinear operator and (3.2) we get,

$$\begin{aligned} \|Tf^\#\|_p^p &\leq p \int_0^\infty t^{p-1} \frac{2}{t} \|f_{t/2}\|_1 dt \\ &= 2p \int_0^\infty t^{p-2} \int_{\{x : |f^\#(x)| > \frac{t}{2}\}} |f^\#(x)| dx dt \\ &= 2p \int_{\mathbb{R}^n} |f^\#(x)| \int_0^{2|f^\#(x)|} t^{p-2} dt dx \\ &= 2p \int_{\mathbb{R}^n} |f^\#(x)| \frac{1}{p-1} (2|f^\#(x)|)^{p-1} dx \\ &\leq \frac{2^p p}{(p-1)} \int_{\mathbb{R}^n} |f^\#(x)|^p dx \\ &\leq C(p) \|f^\#\|_p^p \quad 1 \leq p < \infty. \end{aligned}$$

□

The relationship between weak (p, q) inequalities and the almost everywhere convergence is given by the following result.

Theorem 3.2. Let $\{T_t\}$ be a family of linear operators on $L^p(\mathbb{R}^n)$; $1 \leq p < \infty$ and define

$$T^* f^\#(x) = \sup_t |T_t f^\#(x)|.$$

If T^* is weak (p, q) then the set

$$\{f^\# \in L^p(\mathbb{R}^n) : \lim_{t \rightarrow t_0} T_t f^\#(x) = f^\#(x) \text{ a.e.}\}$$

is closed in $L^p(\mathbb{R}^n)$.

Proof. Let $\{f_n\}$ be a sequence of functions which converges to $f^\#$ in $L^p(\mathbb{R}^n)$ such that $T f^\#(x)$ converges to $f^\#(x)$ a.e.

Now, $\mu(\{f^\# \in L^p(\mathbb{R}^n) : \limsup_{t \rightarrow t_0} |T_t f^\#(x) - f^\#(x)| > \lambda\})$

$$\begin{aligned} &= \mu(\{f^\# \in L^p(\mathbb{R}^n) : \limsup_{t \rightarrow t_0} |T_t f^\#(x) - f_n(x) + f_n(x) - f^\#(x)| > \lambda\}) \\ &\leq \mu(\{f^\# \in L^p(\mathbb{R}^n) : \limsup_{t \rightarrow t_0} |T_t(f^\# - f_n)(x)| > \frac{\lambda}{2}\}) \\ &\quad + \mu(\{f^\# \in L^p(\mathbb{R}^n) : \limsup_{t \rightarrow t_0} |f^\# - f_n|(x) > \frac{\lambda}{2}\}) \\ &\leq \mu(\{f^\# \in L^p(\mathbb{R}^n) : |T_t^*(f^\# - f_n)(x)| > \frac{\lambda}{2}\}) \\ &\quad + \mu(\{f^\# \in L^p(\mathbb{R}^n) : |f^\#(x) - f_n(x)| > \frac{\lambda}{2}\}) \\ &\leq \left(\frac{2C(p)}{\lambda} \|f^\#(x) - f_n(x)\|_p \right)^q + \left(\frac{2}{\lambda} \|f^\#(x) - f_n(x)\|_p \right)^p. \end{aligned}$$

Since T^* is weak (p, q) and $f_n \rightarrow f^\#$ as $n \rightarrow \infty$, therefore

$$\begin{aligned} &\mu(\{f^\# \in L^p(\mathbb{R}^n) : \limsup_{t \rightarrow t_0} |T_t f^\#(x) - f^\#(x)| > 0\}) \\ &\leq \sum_{k=1}^{\infty} \mu(\{f^\# \in L^p(\mathbb{R}^n) : \limsup_{t \rightarrow t_0} |T_t f^\#(x) - f^\#(x)| > \frac{1}{k}\}) \\ &= 0. \end{aligned}$$

□

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