A generalized Vietoris mapping

by

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ABSTRACT. In this article the properties and applications of generalized Vietoris mappings are presented. The important application of generalized Vietoris mappings is the application to the coincidence theory (see Theorem 4.4).

1 Introduction

In mathematical literature a Vietoris mapping is very well known, it plays a large role in many mathematical fields. It is an integral part of strongly admissible mappings whose definition is based on the so-called selected pairs (p, q), where p is a Vietoris mapping (see [1]). If, in a definition of some version of strongly admissible mappings (multi-functions, see [2]), a continuous mapping q is substituted with a Vietoris mapping, then it results in generalized Vietoris mappings. In this article the properties and applications of these mappings are presented.

2 Preliminaries

Throughout this paper all topological spaces are assumed to be metrizable. A continuous mapping $f : X \to Y$ is called proper if for every compact set $K \subset Y$ the set $f^{-1}(K)$ is nonempty and compact. Let X and Y be two spaces and assume that for every $x \in X$ a non-empty and compact subset $\varphi(x)$ of Y is given. In such a case we say

that $\varphi : X \multimap Y$ is a multi-valued mapping. For a multi-valued mapping $\varphi : X \multimap Y$ and a subset $A \subset Y$, we let:

$$\varphi_s^{-1}(A) = \{x \in X;\ \varphi(x) \subset A\},$$

$$\varphi_b^{-1}(A) = \{x \in X;\ \varphi(x) \cap A \neq \emptyset\}.$$

If for every open $U \subset Y$ the set $\varphi_s^{-1}(U)$ is open, then φ is called an upper semi-continuous mapping; we shall write that φ is u.s.c. Let H_* be the *Čech* homology functor with compact carriers and coefficients in the field of rational numbers $\mathbb{Q}$ from the category of Hausdorff topological spaces and continuous maps to the category of a graded vector space and linear maps of degree zero. Thus $H_*(X) = \{H_q(X)\}$ is a graded vector space, $H_q(X)$ being a q-dimensional *Čech* homology group with compact carriers of X. For a continuous map $f : X \to Y$, $H_*(f)$ is the induced linear map $f_* = \{f_q\}$ where $f_q : H_q(X) \to H_q(Y)$ ([1]). A space X is acyclic if:

(i) X is non-empty,

(ii) $H_q(X) = 0$ for every $q \geq 1$ and

(iii) $H_0(X) \approx \mathbb{Q}$.

Let $u : E \to E$ be an endomorphism of an arbitrary vector space. Let us put $N(u) = \{x \in E : u^n(x) = 0 \quad for\ some\ n\}$, where u^n is the nth iterate of u and $\widetilde{E} = E/N(u)$. Since $u(N(u)) \subset N(u)$, we have the induced endomorphism $\widetilde{u} : \widetilde{E} \to \widetilde{E}$ defined by $\widetilde{u}([x]) = [u(x)]$. We call u admissible provided $dim\widetilde{E} < \infty$.

Let $u = \{u_q\} : E \to E$ be an endomorphism of degree zero of a graded vector space $E = \{E_q\}$. We call u a Leray endomorphism if

(i) all u_q are admissible,

(ii) almost all $\widetilde{E_q}$ are trivial. For such u, we define the (generalized) Lefschetz number $\Lambda(u)$ of u by putting

$$\Lambda(u) = \sum_q (-1)^q tr(\widetilde{u_q}),$$

where $tr(\widetilde{u_q})$ is the ordinary trace of $\widetilde{u_q}$ (comp. [1]). The following important property of the Leray endomorphism is a consequence of the well-known formula $tr(u \circ v) = tr(v \circ u)$ for the ordinary trace.

Proposition 2.1. *(see [1]) Assume that, in the category of graded vector spaces, the following diagram commutes*

$$\begin{tikzcd}
E' \arrow[r, "u"] & E'' \\
E' \arrow[u, "u'"] \arrow[r, "u"] & E'' \arrow[u, "u''"'] \arrow[ul, "v"']
\end{tikzcd}$$

Then, if u' or u'' is a Leray endomorphism, so is the other; and, in that case,

$$\Lambda(u') = \Lambda(u'').$$

A proper map $p : X \to Y$ is called Vietoris provided for every $y \in Y$ the set $p^{-1}(y)$ is acyclic. The symbol $D(X, Y)$ will denote the set of all diagrams of the form

$$X \xleftarrow{p} Z \xrightarrow{q} Y,$$

where $p : Z \to X$ denotes a Vietoris map and $q : Z \to Y$ denotes a continuous map. Each such diagram will be denoted by (p, q). We recall that the composition of two Vietoris mappings is a Vietoris mapping and if $p : X \to Y$ is a Vietoris map then $p_* : H_*(X) \to H_*(Y)$ is an isomorphism (see [1]).

Definition 2.2. *(see [1]) Let $(p, q) \in D(X, Y)$ and $(r, s) \in D(Y, T)$. The composition of the diagrams*

$$X \xleftarrow{p} Z_1 \xrightarrow{q} Y \xleftarrow{r} Z_2 \xrightarrow{s} T,$$

is called the diagram $(u, v) \in D(X, T)$

$$X \xleftarrow{u} Z_1 \triangle_{qr} Z_2 \xrightarrow{v} T,$$

$$\text{where } Z_1 \triangle_{qr} Z_2 = \{(z_1, z_2) \in Z_1 \times Z_2 : \ q(z_1) = r(z_2)\},$$

$$u = p \circ f_1, \ \ v = s \circ f_2,$$

$$Z_1 \xleftarrow{f_1} Z_1 \triangle_{qr} Z_2 \xrightarrow{f_2} Z_2,$$

$$f_1(z_1, z_2) = z_1 \text{ (Vietoris map)}, \ \ f_2(z_1, z_2) = z_2 \text{ for each } (z_1, z_2) \in Z_1 \triangle_{qr} Z_2.$$

It shall be written

$$(u, v) = (r, s) \circ (p, q).$$

Let $Id_X : X \to X$ be an identity map. In the set of all diagrams $D(X, Y)$, the following relation is introduced:

Definition 2.3. *Let $(p_1, q_1), (p_2, q_2) \in D(X, Y)$.*

$$(p_1, q_1) \sim_m (p_2, q_2)$$

if and only if there exist spaces Z, Z_1 and Z_2, Vietoris maps $p_3 : Z \to Z_1, p_4 : Z \to Z_2$ such that the following diagram is commutative:

$$\begin{array}{ccccc}
X & \xleftarrow{p_1} & Z_1 & \xrightarrow{q_1} & Y \\
\uparrow{\scriptstyle Id_X} & & \uparrow{\scriptstyle p_3} & & \uparrow{\scriptstyle Id_Y} \\
X & \xleftarrow{p} & Z & \xrightarrow{q} & Y \\
\downarrow{\scriptstyle Id_X} & & \downarrow{\scriptstyle p_4} & & \downarrow{\scriptstyle Id_Y} \\
X & \xleftarrow{p_2} & Z_2 & \xrightarrow{q_2} & Y,
\end{array}$$

that is

$$p = p_1 \circ p_3 = p_2 \circ p_4, \ \ q = q_1 \circ p_3 = q_2 \circ p_4.$$

Proposition 2.4. *(see [2]) The relation in the set $D(X,Y)$ introduced in Definition 2.3 is an equivalence relation.*

The set of the equivalence class of the above relation will be denoted by the symbol

$$M_m(X,Y) = D(X,Y)_{/\sim_m}.$$

The elements of the space $M_m(X,Y)$ will be called multi-morphisms and will be denoted by

$$\varphi_m = [(p,q)]_m$$

where

$$X \xleftarrow{p} Z \xrightarrow{q} Y.$$

Proposition 2.5. *(see [2, 3]) Let $[(p,q)]_m = \varphi_m \in M_m(X,Y)$.*
2.5.1 $((p_1,q_1),(p_2,q_2) \in \varphi_m) \Rightarrow$ (for each $x \in X$ $q_1(p_1^{-1}(x)) = q_2(p_2^{-1}(x))$),
2.5.2 $((p_1,q_1),(p_2,q_2) \in \varphi_m) \Rightarrow (q_{1*} \circ p_{1*}^{-1} = q_{2*} \circ p_{2*}^{-1})$,
2.5.3 Let $\psi_m = [(r,s)]_m \in M_m(Y,T)$ and let $\psi_m \circ \varphi_m = [(r,s) \circ (p,q)]_m \in M_m(X,T)$ (see Definition 2.2). Then for any $(p_1,q_1) \in \varphi_m$ and $(r_1,s_1) \in \psi_m$ we have

$$((r_1,s_1) \circ (p_1,q_1)) \in (\psi_m \circ \varphi_m).$$

From Proposition 2.5 (2.5.1) we get the following definition:

Definition 2.6. *For any $\varphi_m \in M_m(X,Y)$, the set $\varphi(x) = q(p^{-1}(x))$ where $\varphi_m = [(p,q)]_m$ is called an image of point x in a multi-morphism φ_m.*

Let $\varphi_m \in M_m(X,Y)$. The symbol $\varphi : X \to_m Y$ will denote a multi-valued mapping determined by a multi-morphism φ_m (see Definition 2.6). We define (see Proposition 2.5 (2.5.2))

$$\varphi_* = q_* \circ p_*^{-1}, \tag{1}$$

where $(p,q) \in \varphi_m$ and if $\psi : Y \to_m T$ then $\psi \circ \varphi : X \to_m T$ is a multi-valued map determined by $\psi_m \circ \varphi_m$ (see Proposition 2.5 (2.5.3)) and we have (see [2]):

$$(\psi \circ \varphi)_* = \psi_* \circ \varphi_*. \tag{2}$$

Let $f : X \to Y$ be a continuous map and let $Id_X : X \to X$ be an identity map. Then $f_m = [(Id_X, f)]_m \in M_m(X,Y)$ and for each $(p,q) \in f_m$ (see [2])

$$q_* \circ p_*^{-1} = f_*. \tag{3}$$

Let $A \subset X$ be a non-empty set and let $\varphi : X \to_m Y$. Then the map $\varphi_A : A \multimap Y$ given by the formula

$$\varphi_A(x) = \varphi(x) \quad \text{for each } x \in A \tag{4}$$

is determined by a multi-morphism $(\varphi_A)_m = [(\widetilde{p}, \widetilde{q})]_m$ (see [2]), where

$$A \xleftarrow{\widetilde{p}} p^{-1}(A) \xrightarrow{\widetilde{q}} Y$$

and $(\widetilde{p}, \widetilde{q}) \in D(A,Y)$ is a contraction of some $(p,q) \in \varphi_m$. Hence $\varphi_A : A \to_m Y$ is a multi-valued map determined by $(\varphi_A)_m = [(\widetilde{p}, \widetilde{q})]_m$.

Theorem 2.7. *[1] Consider a diagram:*

$$X \xleftarrow{p} Z \xrightarrow{q} X,$$

in which $X \in ANR$, p is Vietoris and q is compact. Then $q_* \circ p_*^{-1}$ is a Leray endomorphism and $\Lambda(q_* \circ p_*^{-1}) \neq 0$ implies that p and q have a coincidence point.

Definition 2.8. *We say that a metrizable space X multi-dominates a metrizable space Y (we write $X\circ \geq \circ Y$) if there exist multi-functions $\varphi : X \to_m Y$ and $\psi : Y \to_m X$ such that the following conditions are satisfied:*
2.8.1 *for each $y \in Y$ and for each $x \in X$ $(x \in \psi(y)) \Rightarrow (y \in \varphi(x))$,*
2.8.2 $\varphi_* \circ \psi_* = Id_{H_*(Y)}$.

Definition 2.9. *We say that a metrizable space X multi-dominates to the right over a metrizable space Y (we write $X \geq \circ Y$) if there exists a continuous function $f : X \to Y$ and a multi-function $\psi : Y \to_m X$ such that for each $y \in Y$ $\psi(y) \subset f^{-1}(y)$.*

Definition 2.10. *We say that a metrizable space X multi-dominates to the left over a metrizable space Y (we write $X\circ \geq Y$) if there exists a multi-function $\varphi : X \to_m Y$ and a continuous map $g : Y \to X$ such that the following conditions are satisfied:*
2.8.1 *for each $y \in Y$ $y \in \varphi(g(y))$,*
2.8.2 $\varphi_* \circ g_* = Id_{H_*(Y)}$.

Remark 2.11. *[4] It can be proven easily that if in Definition 2.8 the mapping φ will be substituted with a single-valued mapping, then it will result in a domination to the right (Definition 2.9), However, if the mapping ψ is substituted with a single-valued mapping, it will result in a domination to the left (Definition 2.10). In the case when both of the mappings are substituted with single-valued mappings, that will give a domination in the sense of Borsuk (see [5]).*

Definition 2.12. *Let X be an ANR and let $X_0 \subset X$ be a closed subset. We say that X_0 is movable in X provided every neighborhood U of X_0 admits a neighborhood U' of X_0, $U' \subset U$, such that for every neighborhood U'' of X_0, $U'' \subset U$, there exists a homotopy $H\colon U' \times [0,1] \to U$ with $H(x,0) = x$ and $H(x,1) \in U''$, for any $x \in U'$.*

Definition 2.13. *Let X be a compact metric space. We say that X is movable provided there exists $Z \in ANR$ and an embedding $e : X \to Z$ such that $e(X)$ is movable in Z.*

Let us notice that the property of being movable is an absolute property, that is if A is a movable set in some ANR X and $j : A \to X'$ is an embedding into an ANR X', then $j(A)$ is movable in X' (see [6]).

Remark 2.14. *We know that movable spaces are of the following types, among others:*

AR, ANR, AANR (in the sense of Clapp), FAR (of trivial shape) and FANR (see [6]).

Proposition 2.15. *[2] Let $(p_1, q_1), (p_2, q_2) \in D(X, Y)$, where*

$$X \xleftarrow{p_1} Z_1 \xrightarrow{q_1} Y, \qquad X \xleftarrow{p_2} Z_2 \xrightarrow{q_2} Y.$$

Then there exists a space Z and Vietoris maps $p_3 : Z \to Z_1$, $p_4 : Z \to Z_2$ such that $p_1 \circ p_3 = p_2 \circ p_4 = p$ and

$$(p_1, q_1) \sim_m (p, q_1 \circ p_3) \ \text{ and } \ (p_2, q_2) \sim_m (p, q_2 \circ p_4).$$

3 Generalized Vietoris mappings and its properties

In this section, maps determined by multi-morphisms $\varphi_m \in M_m(X,Y)$ will be denoted $\varphi : X \to_m Y$ and will be called multi-functions, while for the single-valued mappings the letters $f, g, h, ...$ are reserved.

Definition 3.1. *A multi-function $\varphi : X \to_m Y$ is called a generalized Vietoris mapping ($\varphi \in GV$) if there exists a space Z and Vietoris mappings $p_1 : Z \to X$ and $p_2 : Z \to Y$ such that $(p_1, p_2) \in \varphi_m$.*

Definition 3.2. *Let $\varphi : X \to_m Y$, $\varphi \in GV$. The generalized Vietoris mapping $\psi : Y \to_m X$ is called an inverse mapping to the mapping φ if there exists $(p_1, p_2) \in \varphi_m$ such that $(p_2, p_1) \in \psi_m$. An inverse mapping to the mapping φ will be denoted by $\overleftarrow{\varphi}$.*

Let $p : X \to Y$ be a Vietoris map. The symbol $\varphi_p : Y \multimap X$ will denote the mapping given by the formula

$$\varphi_p(y) = p^{-1}(y) \text{ for every } y \in Y. \tag{5}$$

Let $\varphi : X \to_m Y$ $\varphi \in GV$, then $\varphi_* : H_*(X) \to H_*(Y)$ is an isomorphism given by the formula (see (1)):

$$\varphi_* = p_{2*} \circ p_{1*}^{-1}.$$

Some properties of generalized Vietoris mappings will be necessary. It shall be reminded that the composition of two Vietoris mappings is a Vietoris mapping (see [1]).

Proposition 3.3. *Let $\varphi : X \to_m Y$, $\psi : Y \to_m T$, $\varphi, \psi \in GV$.*
3.3.1 $(\psi \circ \varphi) \in GV$.
3.3.2 $\overleftarrow{(\overleftarrow{\varphi})} = \varphi$.
3.3.3 $\overleftarrow{(\psi \circ \varphi)} = \overleftarrow{\varphi} \circ \overleftarrow{\psi}$.
3.3.4 $\varphi(X) = Y$.
3.3.5 *For each $x \in X$ $x \in \overleftarrow{\varphi}(\varphi(x))$.*
3.3.6 *For each $y \in Y$ $y \in \varphi(\overleftarrow{\varphi}(y))$.*
3.3.7 *For each $x \in X$ and $y \in Y$ $(y \in \varphi(x)) \Leftrightarrow (x \in \overleftarrow{\varphi}(y))$.*
3.3.8 $\varphi = p \in GV$, *since* $(Id_X, p) \in \varphi_m$.
3.3.9 $\varphi = \varphi_p \in GV$, *since* $(p, Id_X) \in \varphi_m$.
3.3.10 *Let $\eta : X_1 \to_m Y_1$, $\eta \in GV$. Then $\varphi \times \eta : X \times X_1 \to_m Y \times Y_1$, $(\varphi \times \eta) \in GV$ and*

$$\overleftarrow{\varphi \times \eta} = \overleftarrow{\varphi} \times \overleftarrow{\eta}.$$

3.3.11 $\varphi_* : H_*(X) \to H_*(Y)$ *is an isomorphism and*

$$\overleftarrow{\varphi}_* \circ \varphi_* = Id_{H_*(X)} \text{ and } \varphi_* \circ \overleftarrow{\varphi}_* = Id_{H_*(Y)}.$$

3.3.12 *Let $h : X \to Y$ be a homeomorphism (in particular if $X = Y$ and $h = Id_X$). Then $h \in GV$.*
3.3.13 *Let $q_i : Z_i \to Y$, $i = 1, 2$ be continuous maps and let $\varphi : Z_1 \to_m Z_2$ $\varphi \in GV$. Then*

$$(q_2 \circ \varphi = q_1) \Leftrightarrow (q_1 \circ \overleftarrow{\varphi} = q_2).$$

3.3.14 *φ is a closed mapping.*

Proof. 3.3.1 Let $(p_1, p_2) \in \varphi_m$ and $(p_3, p_4) \in \psi_m$. We have the following diagram:

$$X \xleftarrow{p_1} Z_1 \xrightarrow{p_2} Y \xleftarrow{p_3} Z_2 \xrightarrow{p_4} T$$

and maps

$$Z_1 \xleftarrow{f_1} Z_1 \triangle_{p_2 p_3} Z_2 \xrightarrow{f_2} Z_2,$$

given by the formula (see Definition 2.2) $f_1(z_1, z_2) = z_1$, $f_2(z_1, z_2) = z_2$ for each $(z_1, z_2) \in Z_1 \triangle_{p_2 p_3} Z_2$. We observe that f_1 and f_2 are Vietoris maps. Let $p = p_1 \circ f_1$, $p' = p_4 \circ f_2$. It is clear that $(p, p') \in (\psi \circ \varphi)_m$ and the proof is complete.

3.3.7 Let $(p_1, p_2) \in \varphi_m$ and let $y \in \varphi(x) = p_2(p_1^{-1}(x))$ for some $x \in X$. Then there exists $z \in p_1^{-1}(x)$ such that $p_2(z) = y$. Hence $z \in p_2^{-1}(y)$ and $x \in p_1(p_2^{-1}(y)) = \overleftarrow{\varphi}(y)$. The proof of an inverse implication is similar.

3.3.10 Let $(p_1, p_2) \in \varphi_m$ and $(p_3, p_4) \in \eta_m$. We have the following diagrams:

$$X \xleftarrow{p_1} Z_1 \xrightarrow{p_2} Y, \quad X_1 \xleftarrow{p_3} Z_2 \xrightarrow{p_4} Y_1.$$

We define Vietoris maps:

$$X \times X_1 \xleftarrow{p} Z_1 \times Z_2 \xrightarrow{p'} Y \times Y_1$$

given by the formula:

$$p(z_1, z_2) = (p_1(z_1), p_3(z_2)), \quad p'(z_1, z_2) = (p_2(z_1), p_4(z_2)) \text{ for each } (z_1, z_2) \in Z_1 \times Z_2.$$

It is obvious that $(p, p') \in (\varphi \times \eta)_m$ and $\overleftarrow{\varphi \times \eta} = \overleftarrow{\varphi} \times \overleftarrow{\eta}$.

3.3.13 Let $q_2 \circ \varphi = q_1$ and let $y \in q_1(\overleftarrow{\varphi}(z_2))$ for some $z_2 \in Z_2$. There exists $z_1 \in \overleftarrow{\varphi}(z_2)$ such that $y = q_1(z_1)$. From (3.3.7) $z_2 \in \varphi(z_1)$ and from the assumption we have $q_2(z_2) = q_1(z_1)$. Hence $y = q_2(z_2)$. The proof of an inverse implication is similar.

3.3.14 We know that a Vietoris map is closed and continuous (see [1]). Hence φ is a closed mapping.

The proofs of other properties are obvious. □

The symbol

$$X \approx_{GV} Y \tag{6}$$

means that there is a map $\varphi : X \to_m Y$ such that $\varphi \in GV$. We observe that from Proposition 3.3 (3.3.1, 3.3.12) and from Definition 3.2 the relation $X \approx_{GV} Y$ is an equivalence relation in the class of metric spaces.

4 The applications

A few examples of the applications of generalized Vietoris mappings will be given. Firstly, the mappings of GV will be applied to the theory of coincidence and essential mappings.

Definition 4.1. *Let $(p_1, q_1), (p_2, q_2) \in D(X, Y)$, where*

$$X \xleftarrow{p_1} Z_1 \xrightarrow{q_1} Y, \qquad X \xleftarrow{p_2} Z_2 \xrightarrow{q_2} Y.$$

We say that the diagrams (p_1, q_1) and (p_2, q_2) have a coincidence point

$$(\textit{we write } (p_1, q_1) \sim_{z_0} (p_2, q_2))$$

if there exists a metrizable space Z, Vietoris maps $p_3 : Z \to Z_1$, $p_4 : Z \to Z_2$ and a point $z_0 \in Z$ such that $p_1 \circ p_3 = p_2 \circ p_4$ and

$$q_1(p_3(z_0)) = q_2(p_4(z_0)).$$

Definition 4.2. *Let $\varphi, \psi : X \to_m Y$. We say that the multi-functions φ and ψ have a coincidence point if there exist diagrams $(p_1, q_1) \in \varphi_m$ and $(p_2, q_2) \in \psi_m$ such that*

$$(p_1, q_1) \sim_{z_0} (p_2, q_2).$$

Firstly, we shall prove the following fact:

Proposition 4.3. *Let $\varphi, \psi : X \to_m Y$. The multi-functions φ and ψ have a coincidence point if and only if there exists a point $x_0 \in X$ such that*

$$\varphi(x_0) \cap \psi(x_0) \neq \emptyset.$$

Proof. Assume that the multi-functions φ and ψ have a coincidence point. From the Definition 4.2 we get $(p_1, q_1) \in \varphi_m$ and $(p_2, q_2) \in \psi_m$ such that (p_1, q_1) and (p_2, q_2) have a coincidence point, where

$$X \xleftarrow{p_1} Z_1 \xrightarrow{q_1} Y, \qquad X \xleftarrow{p_2} Z_2 \xrightarrow{q_2} Y.$$

In turn, by Definition 4.1 there exists a metrizable space Z, Vietoris maps $p_3 : Z \to Z_1$, $p_4 : Z \to Z_2$ and a point $z_0 \in Z$ such that $p_1 \circ p_3 = p_2 \circ p_4$ and $q_1(p_3(z_0)) = q_2(p_4(z_0))$. Let $z_1 = p_3(z_0) \in Z_1$, $z_2 = p_4(z_0) \in Z_2$ and let $x_0 = p_1(z_1) = p_2(z_2) \in X$. Then $q_1(z_1) = q_2(z_2)$ and hence

$$q_1(p_1^{-1}(x_0)) \cap q_2(p_2^{-1}(x_0)) \neq \emptyset.$$

Thus from Proposition 2.5 (2.5.1) $\varphi(x_0) \cap \psi(x_0) \neq \emptyset$ and the proof, in one direction, is complete. Assume now that there exists a point $x_0 \in X$ such that $\varphi(x_0) \cap \psi(x_0) \neq \emptyset$. Then from Proposition 2.5 (2.5.1) and Definition 2.6 there exist $(p_1, q_1) \in \varphi_m$ and $(p_2, q_2) \in \psi_m$ such that $q_1(p_1^{-1}(x_0)) \cap q_2(p_2^{-1}(x_0)) \neq \emptyset$, where

$$X \xleftarrow{p_1} Z_1 \xrightarrow{q_1} Y, \qquad X \xleftarrow{p_2} Z_2 \xrightarrow{q_2} Y.$$

There exist points $z_1^0 \in Z_1$, $z_2^0 \in Z_2$ such that

$$p_1(z_1^0) = p_2(z_2^0) = x_0 \text{ and } q_1(z_1^0) = q_2(z_2^0).$$

Let $Z = Z_1 \triangle_{p_1 p_2} Z_2$, $f_1 : Z \to Z_1$, $f_2 : Z \to Z_2$ be projections (see Definition 2.2). It is obvious that f_1 and f_2 are Vietoris maps and $p_1 \circ f_1 = p_2 \circ f_2$. Let $z_0 = (z_1^0, z_2^0)$. Then $z_0 \in Z$ and

$$q_1(f_1((z_0)) = q_1(f_1((z_1^0, z_2^0)) = q_1(z_1^0) = q_2(z_2^0) = q_2(f_2(z_1^0, z_2^0)) = q_2(f_2(z_0))$$

and the proof is complete. □

Note that the Definition 4.2, Proposition 4.3 and the Proposition 2.5 (2.5.1) show that if the multi-functions φ and ψ have a point of coincidence, then for each diagram $(p_1, q_1) \in \varphi_m$ and $(p_2, q_2) \in \psi_m$

$$(p_1, q_1) \sim_{z_0} (p_2, q_2) \text{ (see Definition 4.1).}$$

We say that a multi-function $\psi : X \to_m Y$ is compact if for some $(p, q) \in \psi_m$ q is a compact mapping.

Theorem 4.4. *Let $\Delta : X \to_m Y$, $\Delta \in GV$ and let $Y \in ANR$. Assume that $\psi : X \to_m Y$ is a compact multi-function. Then*

$$(\psi \circ \overleftarrow{\Delta})_* : H_*(Y) \to H_*(Y)$$

is a Leray endomorphism and if $\Lambda((\psi \circ \overleftarrow{\Delta})_*) \neq 0$ then ψ and Δ have a point of coincidence.

Proof. From the Proposition 2.5 (2.5.3) and from the assumption $\psi \circ \overleftarrow{\Delta} : Y \to_m Y$ is a compact multi-function and $Y \in ANR$, so $(\psi \circ \overleftarrow{\Delta})_*$ is a Leray endomorphism. Assume that $\Lambda((\psi \circ \overleftarrow{\Delta})_*) \neq 0$. Let $(p_1, p_2) \in \Delta_m$, $(p_3, q_3) \in \psi_m$, where

$$X \xleftarrow{p_1} Z_1 \xrightarrow{p_2} Y, \qquad X \xleftarrow{p_3} Z_2 \xrightarrow{q_3} Y.$$

From Proposition 2.15 we get a metrizable space Z, Vietoris maps $p_4 : Z \to Z_1$, $p_5 : Z \to Z_2$ such that $p = p_1 \circ p_4 = p_3 \circ p_5$ and

$$(p, p_2 \circ p_4) \in \Delta_m, \ \ (p, q_3 \circ p_5) \in \psi_m.$$

It is clear that

$$(p_2 \circ p_4, p) \in \overleftarrow{\Delta}_m.$$

We have

$$\Lambda((\psi \circ \overleftarrow{\Delta})_*) = \Lambda(\psi_* \circ \overleftarrow{\Delta}_*) = \text{ (see (1), (2))}$$
$$= \Lambda(((q_3 \circ p_5)_* \circ p_*^{-1}) \circ (p_* \circ (p_2 \circ p_4)_*^{-1}) = \Lambda(((q_3 \circ p_5)_* \circ (p_2 \circ p_4)_*^{-1}).$$

Hence

$$\Lambda(((q_3 \circ p_5)_* \circ (p_2 \circ p_4)_*^{-1}) \neq 0$$

and from Theorem 2.7 the maps $q_3 \circ p_5$ and $p_2 \circ p_4$ have a coincidence point and from Definition 4.2 and Proposition 4.3 there exists a point $x_0 \in X$ such that

$$\Delta(x_0) \cap \psi(x_0) \neq \emptyset$$

and the proof is complete. □

In particular from Theorem 4.4 we get the following fact:

Proposition 4.5. *Let $\Delta : X \to_m Y$, $\Delta \in GV$ and let $Y \in AR$. Assume that $\psi : X \to_m Y$ is a compact multi-function. Then ψ and Δ have a coincidence point.*

Theorems on coincidence points can be applied in the theorem of essential mappings. Essential mappings are used to solve differential inclusions (see [7]). The definition of an essential mapping should be recalled. The symbol clA will denote the closure of a set A in a space X, while $bdA = clA \cap cl(X \backslash A)$ will denote a border of a set A. Let E_1, E_2 be linear metric spaces and let $U \subset E_1$ be an open set. Assume that $\varphi : clU \to_m E_2$ is a multi-function such that $\varphi_b^{-1}\{0\} \subset U$. We say that φ is an essential map if for each a compact multi-function $\psi : clU \to_m E_2$ such that for each $x \in bdU$ $\psi(x) = \{0\}$ there exists a point $x_0 \in U$ such that $\varphi(x_0) \cap \psi(x_0) \neq \emptyset$ (see [7]). The following fact does not require a proof (see Proposition 4.5):

Proposition 4.6. *Let $\Delta : clU \to_m E_2$, $\Delta \in GV$ and such that $\Delta_b^{-1}\{0\} \subset U$. Then Δ is an essential map.*

Because of properties (see Proposition 3.3 (3.3.2, 3.3.7, 3.3.11)), the generalized Vietoris mappings can be used in the theory of multi-domination (see Definition 2.8).

Proposition 4.7. *Let $X \approx_{GV} Y$ (see (6)). Then $X\circ \geq \circ Y$ and $Y\circ \geq \circ X$.*

The theories of multi-domination and its properties can be found in the articles [4, 8, 9]. Particularly the article [4] is recommended. Let $\Delta : X \to_m Y$ be as in Proposition 4.7 and let $(p_1, p_2) \in \Delta_m$. If the mapping p_1 is substituted with a homeomorphism, then it results in a multi-domination to the right (see Definition 2.9). Then, if the mapping p_2 is substituted with a homeomorphism, then it results in a multi-domination to the left (see Definition 2.10). With the help of generalized Vietoris mappings, the relation described in Definition 2.3 can be denoted in a simpler way. The following fact can be proven with no difficulty (see Proposition 3.3):

Proposition 4.8. *Let $(p_1, q_1), (p_2, q_2) \in D(X, Y)$, where*

$$X \xleftarrow{p_1} Z_1 \xrightarrow{q_1} Y, \qquad X \xleftarrow{p_2} Z_2 \xrightarrow{q_2} Y.$$

$$((p_1, q_1) \sim_m (p_2, q_2)) \Leftrightarrow (\textit{there exists } \varphi : Z_1 \to_m Z_2,\ \varphi \in GV \textit{ such that } p_2 \circ \varphi = p_1 \textit{ and } q_2 \circ \varphi = q_1).$$

Proof. Let $(p_1, q_1) \sim_m (p_2, q_2)$, where

$$X \xleftarrow{p_1} Z_1 \xrightarrow{q_1} Y, \qquad X \xleftarrow{p_2} Z_2 \xrightarrow{q_2} Y.$$

Then from Definition 2.3 there exist spaces Z, Z_1 and Z_2, Vietoris maps $p_3 : Z \to Z_1, p_4 : Z \to Z_2$ such that the following diagram is commutative:

$$\begin{array}{ccccc}
X & \xleftarrow{p_1} & Z_1 & \xrightarrow{q_1} & Y \\
\uparrow{\scriptstyle Id_X} & & \uparrow{\scriptstyle p_3} & & \uparrow{\scriptstyle Id_Y} \\
X & \xleftarrow{p} & Z & \xrightarrow{q} & Y \\
\downarrow{\scriptstyle Id_X} & & \downarrow{\scriptstyle p_4} & & \downarrow{\scriptstyle Id_Y} \\
X & \xleftarrow{p_2} & Z_2 & \xrightarrow{q_2} & Y,
\end{array}$$

that is

$$p = p_1 \circ p_3 = p_2 \circ p_4, \;\; q = q_1 \circ p_3 = q_2 \circ p_4.$$

Let $\varphi : Z_1 \to_m Z_2$ be a multi-function determined by $\varphi_m = [(p_3, p_4)]_m \in M_m(Z_1, Z_2)$ then $\varphi \in GV$ and $p_2 \circ \varphi = p_1$ and $q_2 \circ \varphi = q_1$. Assume now that there exists $\varphi : Z_1 \to_m Z_2$ $\varphi \in GV$ such that $p_2 \circ \varphi = p_1$ and $q_2 \circ \varphi = q_1$. Let $(p_3, p_4) \in \varphi_m$ then we have

$$p_1 \circ p_3 = p_2 \circ p_4, \;\; q_1 \circ p_3 = q_2 \circ p_4$$

and the proof is complete. □

The generalized Vietoris mappings have many properties similar to the properties of homeomorphisms (see Proposition 3.3, Theorem 4.4). It is clear that they retain homologic properties but unfortunately do not retain some topological properties (e.g. the properties of a fixed point, locally connected, movability of space (see Definition 2.12, Definition 2.13)). It will presented with a few examples.

Example 4.9. *Let T be a compact and contractible space and such that the space $Y = T \times [0,1]$ does not have a fixed point property (see [10]). It is obvious that the space $[0,1]$ has a fixed point property. Let $p : Y \to [0,1]$ be a Vietoris map given by the formula*

$$p(x,t) = t \;\; \textit{for each} \;\; (x,t) \in Y.$$

We define a general Vietoris map $\varphi : [0,1] \to_m Y$ given by the formula:

$$\varphi(t) = p^{-1}(t) \quad \textit{for each} \;\; t \in [0,1] \;\; \textit{(see Proposition 3.3 (3.3.9))}.$$

Example 4.10. *We define a set $S \subset \mathbb{R}^2$ given by the formula:*

$$S = \left(\bigcup_{n=1}^{\infty} \{1/n\} \times [0,1] \right) \cup (\{0\} \times [0,1]) \cup ([0,1] \times \{0\}).$$

Let $T = S \times [0,1]$. We know (see [1, 11]) that T is compact, of trivial shape (acyclic) and is not locally connected. As in Example 4.9, we can define a generalized Vietoris map $\varphi : [0,1] \to_m T$. It is clear that $[0,1]$ is locally connected, but T is not locally connected.

Example 4.11. *Let $p : Q \to X$ be a Vietoris map, where Q is a Hilbert cube and X is not movable (see [12]). We define a generalized Vietoris map $\varphi : Q \to X$ given by the formula:*

$$\varphi(x) = p(x) \;\; \textit{for each} \;\; x \in Q \;\; \textit{(see Proposition 3.3 (3.3.8))}.$$

It is obvious that Q is a movable space.

Finally, it is worth noting that every multi-function $\varphi : X \to_m Y$ determined by a multi-morphism $\varphi_m = [(p,q)]_m$ is a composition of a generalized Vietoris mapping $\varphi_p : X \to Z$ (see (5)) determined by a multi-morphism $(\varphi_p)_m = [(p, Id_Z)]_m$ and the continuous mapping $q : Z \to Y$ determined by the multi-morphism $q_m = [(Id_Z, q)]_m$ where

$$X \xleftarrow{\;p\;} Z \xrightarrow{\;Id_Z\;} Z, \qquad Z \xleftarrow{\;Id_Z\;} Z \xrightarrow{\;q\;} Y.$$

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